SPRINGER CATEGORIES FOR REGULAR CENTRALIZERS IN WELL-GENERATED COMPLEX BRAID GROUPS

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Abstract. In his proof of the $K(\pi, 1)$ conjecture for complex reflection arrangements, Bessis defined Garside categories suitable for studying braid groups of centralizers of Springer regular elements in well-generated complex reflection groups. We provide a detailed study of these categories, which we call Springer categories.

We describe in particular the conjugacy of braided reflections of regular centralizer in terms of the Garside structure of the associated Springer category. In so doing we obtain a pure Garside theoretic proof of a theorem of Digne, Marin and Michel on the center of finite index subgroups in complex braid groups in the case of a regular centralizer in a well-generated group. We also provide a "Hurwitz-like" presentation of Springer categories. To this aim we provide additional insights on noncrossing partitions in the infinite series. Lastly, we use this "Hurwitz-like" presentation, along with a generalized Reidemeister-Schreier method we introduce for groupoids, to deduce nice presentations of the complex braid group $B(G_{31})$.

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Introduction

Let W be an irreducible complex reflection group. In the case where W is well-generated (i.e it can be generated by a number of reflections equal to its rank), Bessis introduced in [Bes15, Section 8] a Garside monoid, whose group of fractions is naturally isomorphic to the complex braid group $B(W)$ of W. It is the the so-called dual braid monoid.

This Garside monoid has then been used to study group theoretic questions on complex braid groups. For instance the center of any finite index subgroup is determined in [DMM11] through the use of various Garside monoids, including the dual braid monoid. More recently, González-Meneses and Marin defined in [GM22] a general notion of parabolic subgroups of a complex braid group. They also studied these subgroups using the dual braid monoid notably.

Unfortunately, this approach is not available when W is not well-generated. This proves especially problematic for the exceptional group G_{31}, which is both badly-generated and of high rank. This lack of a well-studied Garside structure was a hindrance in completing the study program of complex braid groups. For instance the proof of [DMM11, Theorem 1.4] for the complex braid group $B(G_{31})$ requires external arguments involving Krammer representations.

However, Bessis also defined in [Bes15, Section 11] a Garside category suitable for studying the braid group of the centralizer of a Springer regular element in a well-generated group. This approach applies in particular to G_{31}, which appears as the centralizer of a Springer regular element inside of the well-generated group G_{37}. The problem is now to understand these Garside categories. Indeed, the theory of Garside categories, which was in its infancy when Bessis wrote his article, is now much more developed, and allows for the generalization of many (if not all) properties of the dual braid monoid to these categories. This article aims to be a first step in this program.

Let W be a well-generated irreducible complex reflection group, and let $g \in W$ be a regular element for the eigenvalue $\zeta_d := \exp(\frac{2i\pi}{d})$, where d is a positive integer. The group $W_g := C_W(g)$ is again a complex reflection group, and the braid group $B(W_g)$ is isomorphic to the centralizer in $B(W)$ of a so-called regular braid ([Bes15, Theorem 12.4]). The proof of this statement by Bessis relies on heavy topological arguments, which boils down to the construction of a Garside category $\mathcal{C}$, whose enveloping groupoid (obtained by formally inverting all morphisms) is equivalent to $B(W_g)$. The dual braid monoid can be seen as a particular case of this topological construction (associated to the regular element Id). We refer to the category $\mathcal{C}$ as the Springer category (associated to W_g and W) from now on.

On the other hand, the construction of the category $\mathcal{C}$ can be formulated in a pure Garside theoretic manner as a category of periodic elements. We recall this construction in Section 1, before we restate the topological construction in Section 2.

The interaction between these two construction then allows us to describe the braided reflections of the group $B(W_g)$ in terms of the Garside structure of $\mathcal{C}$. In the classical case, the atoms of the dual braid monoid are known to be a set of braided reflections generating the braid group $B(W)$. The same result cannot hold in the case of a category, as an atom of the category $\mathcal{C}$ may have a different source and target, and thus it cannot be identified with an element of $B(W_g)$. However we have a similar result. For every atom a in $\mathcal{C}$, we define two morphisms $a^\flat$ and $a^\#$ which have same target and same source as a, respectively. We then get an element $\lambda(a) := aa^\# a^{\#\#} \cdots a^{\flat\flat} a^\flat \in \mathcal{C}(u, u)$, which we call an atomic loop (associated to a).

Theorem. *(Theorem 3.31) Let u be an object of $\mathcal{C}$. Any atomic loop inside $\mathcal{C}(u,u)$ is a braided reflection of the group $B(W_g) \simeq \mathcal{B}(u,u)$. Conversely, any braided reflection $\sigma \in B(W_g) \simeq \mathcal{B}(u,u)$ is conjugate in $\mathcal{B}$ to some atomic loop.*

In the case where $\mathcal{C}$ is a monoid, the atomic loop associated to an atom is simply the atom itself, and we recover the situation of the classical case. This explicit description of the braided reflections in terms of the Garside structure of $\mathcal{B}$ and $\mathcal{C}$ enables us to study the conjugacy of braided reflections and their powers using Garside theory. The following theorem ensures that the conjugacy of powers of braided reflections is "the same as" that of braided reflections.

Theorem. *(Theorem 3.37) Let $\lambda(s) \in \mathcal{C}(u,u)$ be an atomic loop of some object u, and let $f \in \mathcal{G}$. If there is some endomorphism $z \in \mathcal{C}$ such that $\lambda(s)^n f = fz$ for some $n \geqslant 1$, then $z = \lambda(s')^j$ for some atomic loop $\lambda(s')$ such that $\lambda(s)f = f\lambda(s')$.*

This result was already proven by Digne, Marin and Michel in the case of the dual braid monoid ([DMM11, Proposition 2.2]). As a corollary, we obtain the analogue of [DMM11, Theorem 1.4] in our case:

Corollary. *(Corollary 3.39) Let W be a well-generated irreducible complex reflection group, and let $g \in W$ be a ζ_d-regular element for some integer d (where $\zeta_d := \exp(\frac{2i\pi}{d})$). If $U \subset B(W_g)$ is a finite index subgroup, then we have $Z(U) \subset Z(B(W_g))$.*

This was already proven in [DMM11] for all complex braid groups, although the proof in the case of $B(G_{31})$ relied on computations and the use of the generalized Krammer representation.

Our other point of focus in this article concerns presentations of Springer categories, and a way to deduce from such a presentation a presentation of the associated group $B(W_g)$. First, we have a presentation of the Springer category $\mathcal{C}$ associated to $B(W_g)$

Theorem. *(Theorem 3.27) A Springer category is presented by its atoms, endowed with the Hurwitz relations, that is, all the relations induced by commutative squares of atoms.*

Again, this is known to hold for dual braid monoids ([Bes15, Lemma 8.8]). The proof of this theorem relies on the particular case where the Springer category happens to be a monoid. In this case we show (Theorem 3.10 and Corollary 3.11)) that the Springer category is naturally isomorphic to the dual braid monoid associated to W_g (which is well-generated in this case). As a byproduct we obtain (Corollary 3.22) an isomorphism between the lattice of simples of the dual braid monoid associated to $G(d,1,n)$ and the lattice of noncrossing partitions of type $(d,1,n)$ defined in [BC06, Definition 1.11].

In the case of well-generated groups, following the indication of [Bes15, Remark 8.9], one can use the Hurwitz presentation of the dual braid monoid to prove other presentations of the associated braid group, like the ones of [BMR98, Table 3] or [BM04, Section 2].

Following the idea of [Bes15, Remark 11.29], we propose in Section 4 a generalization of this work to the case of categories, which we apply to the complex braid group $B(G_{31})$. The underlying method is an analogue of the Reidemeister-Schreier method adapted for groupoids. We define a Schreier transversal for a connected free groupoid $\mathcal{F}(\mathcal{S})$ as a set of paths starting from a fixed object u_0 (the root of the transversal) and arriving at every object of $\mathcal{F}(\mathcal{S})$.

Proposition. *(Reidemeister-Schreier method for groupoids, Proposition A.13)*
Let $\mathcal{G} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a connected presented groupoid. A Schreier transversal T for $\mathcal{F}(\mathcal{S})$ rooted in the object u_0 induces an explicit presentation of the group $\mathcal{G}(u_0, u_0)$.

We consider the Springer groupoid $\mathcal{B}_{31}$ associated to the embedding of G_{31} inside G_{37} as the centralizer of a i-regular element. By choosing a particular Schreier transversal rooted in an object u, we are able to obtain a presentation of $\mathcal{B}_{31}(u,u) \simeq B(G_{31})$, where the generator are atomic loops. By considering a particular object of $\mathcal{B}_{31}$, we obtain the following theorem.

Theorem. *(Theorem 4.2 and Section 4.3.1)*
The complex braid group $B(G_{31})$ is given by the following presentation

$$B(G_{31}) \simeq \left\langle s,t,u,v,w \;\middle|\; \begin{array}{l} ts=st,\ vt=tv,\ wv=vw, \\ suw=uws=wsu, \\ svs=vsv,\ vuv=uvu,\ utu=tut,\ twt=wtw. \end{array} \right\rangle$$

where s,t,u,v,w are braided reflections.

This is the presentation of $B(G_{31})$ that was conjectured in [BMR98, Table 3] and [BM04, Conjecture 2.4]. Another approach to prove this presentation was proposed in [Bes15, Section 4], using the data of [BM04, Figure 3]. Unfortunately, this method cannot readily be carried out with these data as the formula for the discriminant of G_{31} given by [BM04, Figure 3] does not appear to satisfy the conditions of [Bes15, Corollary 4.5] (the discriminant considered in [Bes15, Corollary 4.5.(iii)] is zero everywhere).

By considering other objects of $\mathcal{B}_{31}$, we obtain other new presentations of $B(G_{31})$. They are all positive homogeneous with braided reflections (atomic loops) as generators. They give explicit examples of presentations whose existence was proven in [Bes01, Theorem 0.1] in the case of G_{31}. For each presentation, we give in particular an explicit isomorphism to the presentation given above in terms of images of the generators.

This work is part of my PhD thesis, with some results originating from my Master's thesis. Both are done under the supervision of Pr. Ivan Marin. I thank him for his precious advice during the preparation and redaction of this article.

1. Preliminaries on Garside categories

Garside categories were originally introduced near the end of the 2000s as a natural generalization of Garside monoids (see for instance [Kra08] or [Bes07]). A comprehensive survey of the general theory of Garside categories is made in [DDGKM]. Throughout this paper, all categories are assumed to be small categories.

1.1. **Definitions.** We start with a category $\mathcal{C}$. The set of objects of $\mathcal{C}$ will be denoted by $\mathrm{Ob}(\mathcal{C})$. For $u,v \in \mathrm{Ob}(\mathcal{C})$, the set of morphisms from u to v in $\mathcal{C}$ will be denoted by $\mathcal{C}(u,v)$. We follow the usual convention for composition of arrows in Garside categories: the composition of the diagram

$$x \xrightarrow{f} y \xrightarrow{g} z$$

will be denoted by fg.

For $u \in \mathrm{Ob}(\mathcal{C})$, we consider $\mathcal{C}(u,-)$ (resp. $\mathcal{C}(-,u)$) the set of morphisms in $\mathcal{C}$ with source u (resp. with target u). We define a relation $\preceq$ on $\mathcal{C}(u,-)$ by

$$\forall f,g \in \mathcal{C}(u,-),\ f \preceq g \Leftrightarrow \exists h \mid fh = g.$$

In particular, the source of h is the target of f, and the target of h is the target of g. We say that g is a right-multiple of f and that f left-divides g. Likewise, we define a relation $\succeq$ on $\mathcal{C}(-,u)$.

We want the relations $\preceq$ and $\succeq$ to be preorders. A classical condition to study this problem is that of (left- and right-)Noetherianity (cf [DDGKM, Definition II.2.26]).
However, for the purpose of this article, we can use the stronger condition of homogeneity.

Definition 1.1. *([Bes07, Definition 2.2])*
Let $\mathcal{C}$ be a category. A **length functor** *is a functor $\ell : \mathcal{C} \to (\mathbb{Z}_{\geqslant 0}, +)$, such that $\mathcal{C}$ is generated by morphisms of positive length. A category $\mathcal{C}$ endowed with a length functor ℓ is called a* **homogeneous** *category.*

A homogeneous category admits no nontrivial invertible morphism. Indeed if f is invertible, then we have $\ell(f)+\ell(f^{-1}) = \ell(1) = 0$, so $\ell(f) = \ell(f^{-1}) = 0$. Since $\mathcal{C}$ is generated by elements of positive lengths this implies that f is trivial.

A convenient way to prove that a category (or a monoid) is homogeneous is to define it by a homogeneous presentation (cf Lemma A.7).

Lemma 1.2. *Let $\mathcal{C} = (\mathcal{C}, \ell)$ be a homogeneous category, and let $u \in \mathrm{Ob}(\mathcal{C})$. The relations $\preceq$ and $\succeq$ are preorders on $\mathcal{C}(u,-)$ and $\mathcal{C}(-,u)$, respectively.*

Proof. We prove the result for $\preceq$, the result for $\succeq$ is obtained by working in $\mathcal{C}^{op}$. The relation $\preceq$ is obviously reflexive and transitive, it only remains to show that it is antisymmetric. Let $f, g \in \mathcal{C}(u,-)$ be such that $f \preceq g$ and $g \preceq f$. We have $\ell(f) \leqslant \ell(g)$ and $\ell(g) \leqslant \ell(f)$, so $\ell(g) = \ell(f)$. Let h be such that $fh = g$. We have $\ell(h) = \ell(g) - \ell(f) = 0$, thus h is trivial and $f = g$. □

Let $\mathcal{C}$ be category. A nontrivial element in $\mathcal{C}$ which admits no left-divisor (other than itself and the identity) is called an ***atom***. In a homogeneous category $(\mathcal{C}, \ell)$, a morphism of length f can always be written as a composition of at most $\ell(f)$ atoms.

Definition 1.3. *([DDGKM, Definition 2.9])*
Let u be an object of a homogeneous category $\mathcal{C}$.

- *The* **left-gcd** *of $f, g \in \mathcal{C}(u,-)$ is the meet of f and g in $(\mathcal{C}(u,-), \preceq)$ (should it exist), we denote it by $f \wedge g$.*
- *The* **right-lcm** *of $f, g \in \mathcal{C}(-,u)$ is the join of f and g in $(\mathcal{C}(u,-), \succeq)$ (should it exist), we denote it by $f \vee g$.*
- *Likewise, we define $f \wedge_R g$ and $f \vee_L g$ the* **right-gcd** *and the* **left-lcm** *of f and g, respectively.*

Of course, gcds and lcms need not exist in $\mathcal{C}$. We need two more general definitions before we move on to the definition of a Garside category.

Definition 1.4. *([DDGKM, Definition II.2.52])*
Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **cancellative** *if every equality of the form $fgh = fg'h$ in $\mathcal{C}$ implies $g = g'$. This is equivalent to the statement that every morphism in $\mathcal{C}$ is both a monomorphism and an epimorphism.*

Definition 1.5. *([DDGKM, Definition V.2.19])*
Let $\mathcal{C}$ be a homogeneous cancellative category. A **Garside map** *in $\mathcal{C}$ is a map $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ satisfying the following assumptions*

(1) For $u \in \mathrm{Ob}(\mathcal{C})$, the source of $\Delta(u)$ is u. The target of $\Delta(u)$ is denoted by $\phi(u)$.
(2) The map Δ is target injective: if $u \neq v$, then $\phi(u) \neq \phi(v)$.
(3) The families

$$\mathrm{Div}(\Delta) := \bigsqcup_{u \in \mathrm{Ob}(\mathcal{C})} ([1_u, \Delta(u)], \preceq) \quad \text{and} \quad \mathrm{Div}_R(\Delta) := \bigsqcup_{u \in \mathrm{Ob}(\mathcal{C})} ([1_{\phi(u)}, \Delta(u)], \succeq)$$

are equal. We say that Δ is a **balanced map**.
(4) The family $\mathcal{S} := \mathrm{Div}(\Delta)$ is finite and generates $\mathcal{C}$. We call its elements the **simple morphisms**.
(5) For every $g \in \mathcal{C}(u,-)$, the elements g and $\Delta(u)$ admit a left-gcd.

A homogeneous cancellative category $\mathcal{C}$ endowed with a Garside map Δ will be called a **homogeneous Garside category**.

Remark 1.6. This definition is actually stronger than [DDGKM, Definition V.2.19], we notably assume that $\mathcal{S} = \mathrm{Div}(\Delta)$ is finite. This implies in particular that $\mathcal{C}$ must have a finite number of objects.

Let from now on $(\mathcal{C}, \Delta)$ be a homogeneous Garside category. To avoid heavy expressions, we will often replace the assertion "$f = \Delta(u)$ where u is the source of f" with "$f \in \Delta$" or even "$f = \Delta$". This causes no confusion because $f = \Delta(u)$ implies that u is the source of f, thus $\Delta(u)$ is the only morphism of the form $\Delta(x)$ to which f may be equal.

For $s \in \mathcal{S}$, we denote by $\overline{s}$ (resp. s^*) the morphism in $\mathcal{S}$ such that $s\overline{s} \in \Delta$ (resp. $s^*s \in \Delta$). Both $\overline{s}$ and s^* are unique by cancellativity.

Proposition 1.7. *([DDGKM, Proposition V.1.28 and Proposition V.2.32])*
Let $(\mathcal{C}, \Delta)$ be a homogeneous Garside category. We define an automorphism ϕ of $\mathcal{C}$ by setting
- for $u \in \mathrm{Ob}(\mathcal{C})$, $\phi(u)$ is the target of $\Delta(u)$.
- for $f \in \mathcal{C}(u, v)$, $\phi(f)$ is the unique morphism in $\mathcal{C}$ such that $f\Delta(v) = \Delta(u)\phi(f)$.
Furthermore, ϕ has finite order and Δ is a natural transformation from the identity functor $1_{\mathcal{C}}$ of $\mathcal{C}$ to ϕ.

Let s be a simple morphism. We have $\phi(s) = \overline{\overline{s}}$ because $s\Delta = (s\overline{s})\overline{\overline{s}} = \Delta\overline{\overline{s}}$. In particular we see that ϕ maps $\mathcal{S}$ into itself. Since $\mathcal{S}$ is finite and generates $\mathcal{C}$, ϕ has finite order. We also get that ϕ is uniquely determined by the property that, for all $s \in \mathcal{S}$, we have $\phi(s) = \overline{\overline{s}}$.

Proposition 1.8. *([DDGKM, Proposition V.2.35])*
Let $(\mathcal{C}, \Delta)$ be a homogeneous Garside category, and let $u \in \mathrm{Ob}(\mathcal{C})$. The posets $(\mathcal{C}(u, -), \preceq)$ and $(\mathcal{C}(-, u), \succeq)$ are lattices. The posets $(\mathcal{S}(u, -), \preceq)$ and $(\mathcal{S}(-, u), \preceq)$ are sublattices of these lattices.

Remark 1.9. If we consider the particular case of monoids, that is categories with only one object, we recover the classical definition of a homogeneous Garside monoid. The relations $\preceq$ and $\succeq$ are the classical left- and right-divisibility relations. The Garside map Δ corresponds to an element of the monoid, which is the Garside element.

Let $(\mathcal{C}, \Delta)$ be a homogeneous Garside category with set of simples $\mathcal{S}$, and let ϕ be the automorphism of $\mathcal{C}$ introduced in Proposition 1.7. In the sequel, we will often be interested in subcategories of fixed points under some power of ϕ. Let $q \geqslant 0$ be an integer, and let $\mathcal{C}^{\phi^q}$ be the subcategory of $\mathcal{C}$ consisting of ϕ^q-invariant morphisms. We also consider the set $\mathcal{S}^{\phi^q}$ of simple morphisms which are ϕ^q-invariant.

Lemma 1.10. *Let $a, b \in \mathcal{C}$. If two of a, b and ab lie in $\mathcal{C}^{\phi^q}$, then so does the third.*

Proof. If both a and b are ϕ^q-invariant, then $\phi^q(ab) = \phi^q(a)\phi^q(b) = ab$. If both a and ab are ϕ^q-invariant, then $\phi^q(a)\phi^q(b) = \phi^q(ab) = ab = \phi^q(a)b$, and $\phi^q(b) = b$ by cancellativity. The third case is dual to the second. □

Lemma 1.11. *Let $s \in \mathcal{S}$ and suppose that the source of s is ϕ^q-invariant. We denote by $\psi(s)$ the right-lcm in $\mathcal{S}$ of all the $\phi^{iq}(s)$ for $i \in \mathbb{Z}_{\geqslant 0}$. We have $\psi(s) \in \mathcal{S}^{\phi^q}$, and for every $y \in \mathcal{S}^{\phi^q}$, we have $s \preceq y$ in $\mathcal{S}$ if and only if $\psi(s) \preceq y$ in $\mathcal{S}^{\phi^q}$.*

Proof. Let u be the source of s, and let n denote the order of ϕ^q. We have $\psi(s) = s \vee \phi^q(s) \vee \cdots \vee \phi^{(n-1)q}(s)$. As ϕ^q induces an automorphism of the lattice $\mathcal{S}(u, -)$, we have

$$\phi^q(\psi(s)) = \phi^q(s) \vee \phi^{2q}(s) \vee \cdots \vee \phi^{nq}(s) = \psi(s)$$

and $\psi(s) \in \mathcal{S}^{\phi^q}$. Let now $y \in \mathcal{S}^{\phi^q}$, if $\psi(s) \preceq y$, then $s \preceq \psi(s) \preceq y$. Conversely, if $x \preceq y$, then for all $i \in [\![1, n-1]\!]$, we have $\phi^{iq}(s) \preceq \phi^{iq}(y) = y$, thus $\psi(s) \preceq y$ by definition of the right-lcm. □

By [DDGKM, Proposition VII.4.2], the category $\mathcal{C}^{\phi^q}$, endowed with the restriction of the map Δ, is again a Garside category. Its simple morphisms are the elements of $\mathcal{S}^{\phi^q}$. In particular we see that $(\mathcal{S}^{\phi^q}(u, -), \preceq)$ is always a lattice.

1.2. **Normal forms, groupoid and conjugacy.** From now on, we fix a homogeneous Garside category $(\mathcal{C}, \Delta)$, and $\mathcal{S}$ its set of simple morphisms. Recall from [DDGKM, Definition II.1.28] that a $\mathcal{S}$-path is a sequence of composable elements of $\mathcal{S}$ in $\mathcal{C}$. By definition of a homogeneous Garside category, every morphism in $\mathcal{C}$ can be expressed by a $\mathcal{S}$-path.

Definition 1.12. *([DDGKM, Corollary V.1.54])*
A $\mathcal{S}$-path st of length 2 in $\mathcal{C}$ is called **greedy** *if s is the left-gcd of st with Δ, or equivalently if $\overline{s}$ and t are left-coprime. In general, a $\mathcal{S}$-path $s_1 \cdots s_r$ is called* **greedy** *if each subpath $s_i s_{i+1}$ is greedy for $i \in [\![1, r-1]\!]$.*

Remark 1.13. Let $s_1 \cdots s_r$ be a $\mathcal{S}$-path in $\mathcal{C}$. This path is greedy if and only if the path $\phi(s_1) \cdots \phi(s_r)$ is greedy.

Proposition 1.14. *([DDGKM, Proposition V.3.4])*
Every morphism f in $\mathcal{C}$ is expressed by a unique greedy $\mathcal{S}$-path. This path is called the **greedy normal form** *of f.*

Lemma 1.15. *Let s and t be two composable simple morphisms in $\mathcal{C}$. We set $d := \overline{s} \wedge t$, $s' := sd \in \mathcal{S}$ and $t' := d^{-1}t \in \mathcal{S}$. We have $st = s't'$ in $\mathcal{C}$, and the path $s't'$ is greedy.*

Proof. First, we have $st = sdt' = s't'$. Then, let p be such that $dp = \overline{s}$. We have $sdp = \Delta$, so $p = \overline{s'}$ and

$$d(\overline{s'} \wedge t') = d\overline{s'} \wedge dt' = s \wedge t = d.$$

Thus $\overline{s'} \wedge t'$ is trivial by cancellativity, and $s't'$ is a greedy path. □

This lemma gives an algorithmic way to compute the greedy normal form of a morphism in $\mathcal{C}$, provided that we know how to compute the left-gcd of two simple morphisms.

For every category $\mathcal{C}$, one can consider the ***enveloping groupoid*** $\mathcal{G}(\mathcal{C})$ of $\mathcal{C}$, defined by formally inverting all morphisms in $\mathcal{C}$ (see Lemma A.8). In the case where $(\mathcal{C}, \Delta)$ is a homogeneous Garside category, the natural functor $\mathcal{C} \to \mathcal{G}(\mathcal{C})$ is an embedding, and morphisms in $\mathcal{G}(\mathcal{C})$ can be conveniently described.

Notation 1.16. For a positive integer m, $\Delta^m(u)$ denotes the path $\Delta(u) \cdots \Delta(\phi^{m-1}(u))$. If m is a negative integer, $\Delta^m(u)$ denotes the inverse in $\mathcal{G}(\mathcal{C})$ of $\Delta^{-m}(u)$. Occasionally, we will write Δ^m for $\Delta^m(u)$ when there is no need to specify the source u explicitly (cf. [DDGKM, Convention 3.7]).

Proposition-Definition 1.17. *([DDGKM, Definition V.3.17 and Proposition V.3.18])*
Let $\mathcal{G}(\mathcal{C})$ be the enveloping groupoid of $\mathcal{C}$, and let f be a morphism in $\mathcal{G}(\mathcal{C})$. There is unique way to express f as a path of the form $f = \Delta^p s_1 \cdots s_r$ such that $p \in \mathbb{Z}$, $s_1 \neq \Delta$ and $s_1 \cdots s_r$ is a greedy path. The path $\Delta^p s_1 \cdots s_r$ in $\mathcal{G}(\mathcal{C})$ is called the **left-weighted factorization** *of f.*

In particular we see that, for every morphism f in $\mathcal{G}(\mathcal{C})$. There is a positive integer m such that $\Delta^m f$ lies in $\mathcal{C}$.

Definition 1.18. *([DDGKM, Definition V.3.23])*
Let f be a morphism in $\mathcal{G}(\mathcal{C})$, with left-weighted factorization $\Delta^p s_1 \cdots s_r$. The **infimum** *and* **supremum** *of f are defined by $\inf(f) := p$ and $\sup(f) := p + r$, respectively.*

In particular we see that an morphism f in $\mathcal{G}(\mathcal{C})$ lies in $\mathcal{C}$ is and only if $\inf(f) \geqslant 0$.

Definition 1.19. *([DDGKM, Definition VIII.1.1])*
Let x and x' be two endomorphisms in $\mathcal{G}(\mathcal{C})$. A morphism $f \in \mathcal{G}(\mathcal{C})$ **conjugates** *x to x' if $xf = fx'$ in $\mathcal{G}(\mathcal{C})$. The endomorphism x' will be denoted by x^f. As usual, conjugacy in $\mathcal{G}(\mathcal{C})$ induces an equivalence relation, and we can consider the* **conjugacy class** *in $\mathcal{G}(\mathcal{C})$ of the endomorphism x.*

Definition 1.20. *([DDGKM, Definition VIII.2.3 and Definition VIII.2.8])*
Let x be an endomorphism in $\mathcal{G}(\mathcal{C})$, with left-weighted factorization $\Delta^p s_1 \cdots s_r$. The **initial factor** *(resp.* **final factor***) of x is defined as $\phi^{-p}(s_1)$ (resp. s_r).*
The **cycling** *of x is defined as $\operatorname{cyc}(x) := x^{\phi^{-p}(s_1)} = \Delta^p s_2 \cdots s_r \phi^{-p}(s_1)$.*
The **decycling** *of x is defined as $\operatorname{dec}(x) = x^{s_r^{-1}} = s_r \Delta^p s_1 \ldots s_{r-1} = \Delta^p \phi^p(s_r) s_1 \ldots s_{r-1}$.*

Note that the expressions given for $\mathrm{cyc}(x)$ and $\mathrm{dec}(x)$ are not left-weighted factorizations a priori.

Definition 1.21. *([BGG07, Definition 3.1])*
Let x be an endomorphism in $\mathcal{G}(\mathcal{C})$, with left-weighted factorization $\Delta^p s_1 \dots s_r$. The morphism x is called **rigid** *if the path $s_r\phi^{-p}(s_1)$ is greedy, or if $r = 0$.*

Lemma 1.22. *Let $x \in \mathcal{G}(\mathcal{C})$ be a rigid endomorphism with left-weighted factorization $\Delta^p s_1 \cdots s_r$. The left-weighted factorizations of $\mathrm{cyc}(x)$ and $\mathrm{dec}(x)$ are given by*

$$\mathrm{cyc}(x) = \Delta^p s_2 \cdots s_r \phi^{-p}(s_1) \text{ and } \mathrm{dec}(x) = \Delta^p \phi^p(s_r) s_1 \dots s_{r-1}.$$

Furthermore, the morphisms $\mathrm{cyc}(x)$ and $\mathrm{dec}(x)$ are also rigid.

Proof. By definition, the path $s_2 \cdots s_r\phi^{-p}(s_1)$ is greedy, and $s_2 \neq \Delta$ (otherwise the path $s_1 s_2$ would not be greedy). Thus the path given for $\mathrm{cyc}(x)$ is its left-weighted factorization. The same reasoning holds for $\mathrm{dec}(x)$. Lastly, the fact that $s_1 \cdots s_r$ is a greedy path gives directly that $\mathrm{cyc}(x)$ and $\mathrm{dec}(x)$ are both rigid. □

Proposition-Definition 1.23. *([DDGKM, Definition VIII.2.12]*
Let x be an endomorphism in $\mathcal{G}(\mathcal{C})$. The conjugacy class of x in $\mathcal{G}(\mathcal{C})$ admits a well-defined subset $\mathrm{SSS}(x)$ on which each one of $\inf$ and $\sup$ takes a constant value. Furthermore, for every conjugate x' of x in $\mathcal{G}(\mathcal{C})$, we have

$$\inf(x') \leqslant \inf(\mathrm{SSS}(x)) \text{ and } \sup(x') \geqslant \sup(\mathrm{SSS}(x)).$$

The set $\mathrm{SSS}(x)$ is called the **super-summit set** *of x.*

Note that $\mathrm{SSS}(x)$ must be finite, because there is only a finite number of morphisms in $\mathcal{G}(\mathcal{C})$ with given inf and sup. Also, if $x \in \mathcal{C}$, then $\inf(x) \geqslant 0$, and $\mathrm{SSS}(x)$ is included in $\mathcal{C}$.

Proposition 1.24. *([DDGKM, Proposition VII.2.16])*
Let x be an endomorphism in $\mathcal{G}(\mathcal{C})$. One can go from x to an element of $\mathrm{SSS}(x)$ by a finite sequence of cycling, followed by a finite sequence of decycling.

This proposition, combined with Lemma 1.22, shows that a rigid element always lies inside its own super-summit set.

Proposition 1.25. *([DDGKM, Lemma VIII.2.19 and Proposition VIII.2.20])*
Let x be an endomorphism in $\mathcal{G}(\mathcal{C})$, and let x', x'' be in $\mathrm{SSS}(x)$. Let also f be a morphism in $\mathcal{C}$ with $x'^f = x''$. If $f = s_1 \dots s_r$ is the greedy normal form of f, then for all $i \in [\![1, r]\!]$ the morphism $x'^{s_1 \cdots s_i}$ lies in $\mathrm{SSS}(x)$.

Notice that, as ϕ preserves left-weighted factorizations, it stabilizes super-summit sets. In particular, last proposition also applies to the case where the conjugating element lies in $\mathcal{G}(\mathcal{C})$.

1.3. **Periodic elements and divided categories.** We fix $(\mathcal{C}, \Delta)$ a homogeneous Garside category, and $\mathcal{G}(\mathcal{C})$ its enveloping groupoid. The study of super-summit sets provides a solution to the conjugacy problem in $\mathcal{G}(\mathcal{C})$ for any two pair of endomorphisms. If we restrict our attention to the so-called periodic elements of $\mathcal{G}(\mathcal{C})$, then we obtain a more convenient solution to the conjugacy problem through the construction of a particular Garside category.

Definition 1.26. *([DDGKM, Definition V.3.2])*
*Let u be an object of $\mathcal{G}(\mathcal{C})$, and let $p, q \geqslant 1$ be integers. An endomorphism ρ of $\mathcal{G}(\mathcal{C})(u, u)$ is called (p, q)-***periodic** *if $\rho^p = \Delta^q(u)$.*

In particular, we need Δ^q to be an endomorphism of u, that is $\phi^q(u) = u$. Note that a (p, q)-periodic element in $\mathcal{G}$ is (np, nq)-periodic for every positive integer n.

Proposition 1.27. *([DDGKM, Corollary VIII.3.31 and Proposition VIII.3.34]) Let $p, q \geqslant 1$ be positive integers. Let d be the gcd of p and q, with $dp' = p$ and $dq' = q$. Any (p, q)-periodic element in $\mathcal{G}(\mathcal{C})$ is conjugate to a (p', q')-periodic element lying in $\mathcal{C}$.*

This proposition allows us to restrict our attention to (p, q)-periodic elements with p and q coprime. The study of periodic elements in a general Garside context was introduced in [Bes07], in which the following notion of divided category is introduced. We follow the exposition given in [DDGKM, Section XIV.1.1].

For $m \in \mathbb{Z}_{\geqslant 1}$, define

$$D_m(\Delta) := \{(u_0, \dots, u_{m-1}) \in \mathcal{S}^m \mid u_0 \cdots u_{m-1} = \Delta\}.$$

In particular, we require the u_i to be composable: the target of u_i must by the source of u_{i+1} for $i \in [\![0, m-2]\!]$. For every positive integer m, we define an action of the automorphism ϕ on m-tuples by setting

$$\phi.(u_0, \dots, u_{m-1}) := (u_1, \dots, u_{m-1}, \phi(u_0)) \in D_m(\Delta).$$

This not formally an action of the group $\langle \phi \rangle$, as $\phi^k = 1_{\mathcal{C}}$ doesn't necessarily mean that $\phi^k.u = u$ for all p-tuple u. This is rather an action of the free group $\mathbb{Z}$, which we denote by ϕ for convenience. Note that this action preserves $D_m(\Delta)$. Let now m, n be two positive coprime integers. We define

$$D_m^n(\Delta) := \{u \in D_m(\Delta) \mid \phi^n.u = u\}.$$

In order to study (p, q)-periodic elements for p and q coprime, we are going to define a categorical presentation using the sets $D_{kp}^{kq}(\Delta)$ with $k \in \{1, 2, 3\}$ (see Appendix A.1 for reminders on categorical presentations).

The subcategory $\mathcal{C}^{\phi^q}$ introduced in the end of Section 1.1 is useful for giving a more efficient description of the sets $D_{kp}^{kq}(\Delta)$. Those sets are a priori described by kp parameters lying in $\mathcal{C}$, but since p and q are coprime, we show that they depend only on k parameters lying in $\mathcal{C}^{\phi^q}$.

Lemma 1.28. *Let p, q be coprime positive integers with $p > 1$. There is a well-defined integer η such that, for all $f := (f_0, \dots, f_{p-1}) \in \mathcal{C}^p$, we have*

$$\phi^q.f = f \Leftrightarrow f_0 \in \mathcal{C}^{\phi^q} \text{ and } \forall i \in [\![1, p-1]\!], f_i = \phi^{i\eta}(f_0).$$

In particular, f depends only on f_0.

Proof. We begin with some arithmetic results. Let $q = kp + r$ be the Euclidean division of q by p (we have $r > 0$ since $p \neq 1$). For $n \in \mathbb{Z}_{\geqslant 1}$, let $nr = a_n p + b_n$ be the Euclidean division of nr by p. We have

$$\forall n \geqslant 1,\ b_{n+1} = \begin{cases} b_n + r & \text{if } b_n < p - r \\ b_n + r - p & \text{if } b_n \geqslant p - r \end{cases} \text{ and } a_{n+1} = \begin{cases} a_n & \text{if } b_n < p - r \\ a_n + 1 & \text{if } b_n \geqslant p - r \end{cases}$$

and $b_n \in [\![0, p-1]\!]$ is equal to nr modulo p. Consider now the sequence k_{nr}, defined by

$$k_{nr} = \begin{cases} k & \text{if } b_n < p - r \\ k + 1 & \text{if } b_n \geqslant p - r \end{cases} = k + a_{n+1} - a_n,\ \forall n \geqslant 1.$$

We set

$$h_{nr} = \sum_{i=0}^{n-1} k_{ir} = nk + a_n - a_0 = nk + a_n.$$

We have $h_{nr} p = -b_n$ modulo q. Indeed, as $pk + r = q$, we have

$$ph_{nr} = p(nk + a_n) = -nr + pa_n = -b_n \text{ modulo } q$$

As p and q are coprime, there is a smallest $m \in [\![1, p-1]\!]$ such that $b_m = 1$ (that is, $mr = 1$ modulo $[p]$). We have, for all $n \geqslant 1$

$$b_n h_{mr} = -ph_{nr}h_{mr} = h_{nr}b_m = h_{nr} \text{ modulo } q$$

We define $\eta = -h_{mr}$, and we have $\forall n \geqslant 0$, $b_n\eta = -h_{nr}$.

Let now $f := (f_0, \dots, f_{p-1})$ be a p-tuple of morphisms in $\mathcal{C}$. For the sake of readability, the index i of f_i is seen as an element of $\mathbb{Z}/p\mathbb{Z}$, so that $f_i = f_{i+p}$, $f_r = f_q$ and $f_{nr} = f_{b_n}$ by definition. We have

$$\phi^q.f = (\phi^k(f_r), \dots, \phi^k(f_{p-1}), \phi^{k+1}(f_0), \dots, \phi^{k+1}(f_{r-1})).$$

As p and q are coprime, for every $i \in [\![0, p-1]\!]$, there is a unique $n_i \in [\![0, p-1]\!]$ such that $b_{n_i} = i$. We obtain

$$\phi^q.f = f \Leftrightarrow \forall i \in [\![0, p-1]\!],\ f_i = f_{n_i r} = \phi^{k_{n_i r}}(f_{(n_i+1)r}) = \phi^{k_{n_i r}}(f_{i+r}).$$

By an immediate induction, we get $\phi^q.f = f \Leftrightarrow \forall n > 0,\ f_0 = \phi^{h_{nr}}(f_{nr})$.

If $\phi^q.f = f$, then we have in particular

$$f_0 = \phi^{h_{pr}}(f_0) = \phi^{pk+a_p}(f_0) = \phi^{pk+r}(f_0) = \phi^q(f_0).$$

Also, since $-h_{nr} \equiv b_n\eta$ modulo q, we deduce that

$$\forall i \in [\![0, p-1]\!], f_i = f_{n_i r} = \phi^{-h_{n_i r}}(f_0) = \phi^{b_{n_i}\eta}(f_0) = \phi^{i\eta}(f_0).$$

Conversely, if f is such that $\phi^q(f_0)$ and $f_i = \phi^{i\eta}(f_0)$ for all $i \in [\![1, p-1]\!]$, then for $n > 0$, we have $f_0 = \phi^{-b_n\eta}(f_{b_n}) = \phi^{h_{nr}}(f_{nr})$, which shows that $\phi^q.f = f$. □

Remark 1.29. In the above proof, we saw that $\eta = -mk - a_m$, where $q = pk + r$ is the Euclidean division of q by p, and $m \in [\![0, p-1]\!]$ is such that $mr = a_m p + 1$ is the Euclidean division of mr by p. In particular we see that η depends only on p and q.

The above result is also vacuously true for $p = 1$. From now on, we fix the integer η associated to p and q by Lemma 1.28 (we fix $\eta = 1$ when $p = 1$).

Proposition 1.30. *Let p, q be coprime positive integers.*

(a) The map $(u_0, \dots, u_{p-1}) \mapsto u_0$ induces a bijection between $D_p^q(\Delta)$ and the set

$$\left\{u_0 \in \mathcal{S}^{\phi^q} \mid u_0\phi^\eta(u_0)\dots\phi^{(p-1)\eta}(u_0) = \Delta\right\}.$$

(b) The map $(a_0, b_0, \dots, a_{p-1}, b_{p-1}) \mapsto (a_0, b_0)$ induces a bijection between $D_{2p}^{2q}(\Delta)$ and the set $\{(a, b) \in (\mathcal{S}^{\phi^q})^2 \mid ab \in D_p^q(\Delta)\}$.

(c) The map $(x_0, y_0, z_0, \dots, x_{p-1}, y_{p-1}, z_{p-1}) \mapsto (x_0, y_0, z_0)$ induces a bijection between $D_{3p}^{3q}(\Delta)$ and the set $\{(x, y, z) \in (\mathcal{S}^{\phi^q})^3 \mid xyz \in D_p^q(\Delta)\}$.

Proof. We first consider the case where $p = 1$. We have

$$\begin{aligned} D_1^q(\Delta) &= \{u_0 \in \mathcal{S} \mid u_0 = \Delta \text{ and } \phi^q(u_0) = u_0\}. \\ D_2^{2q}(\Delta) &= \{(a, b) \in \mathcal{S}^2 \mid ab = \Delta \text{ and } (a, b) = (\phi^q(a), \phi^q(b))\} \\ &= \{(a, b) \in (\mathcal{S}^{\phi^q})^2 \mid ab = \Delta\}. \end{aligned}$$

By Lemma 1.10, this set is equal to $\{(a, b) \in (\mathcal{S}^{\phi^q})^2 \mid ab \in D_1^q(\Delta)\}$. The same reasoning applies to $D_3^{3q}(\Delta)$.

We now assume that $p > 1$. The first claim is a direct consequence of Lemma 1.28 and the fact that $u_0 \cdots u_{p-1} = \Delta$. For the second claim, let $(a_0, \dots, b_{p-1})$ be a $2p$-tuple. This $2p$-tuple is ϕ^{2q}-invariant if and only if the two p-tuples $(a_0, \dots, a_{p-1})$ and $(b_0, \dots, b_{p-1})$ are both ϕ^q-invariant. By Lemma 1.28, this is equivalent to $\phi^q(a_0) = a_0, \phi^q(b_0) = b_0$ and

$$(a_0, b_0, \dots, a_{p-1}, b_{p-1}) = (a_0, b_0, \phi^\eta(a_0), \dots, \phi^{(p-1)\eta}(b_0))$$

which shows that the map $(a_0, b_0, \dots, a_{p-1}, b_{p-1}) \mapsto (a_0, b_0)$ is injective and maps $D_{2p}^{2q}(\Delta)$ into the considered set.
Conversely, for (a, b) in the considered set, the $2p$-tuple $(a_0, b_0, \phi^\eta(a_0), \dots, \phi^{(p-1)\eta}(b_0))$ lies inside $D_{2p}^{2q}(\Delta)$ and gives the preimage of (a, b). Again, the same reasoning applies to the third claim. □

From now on, we will use the alternative descriptions of $D_p^q(\Delta)$, $D_{2p}^{2q}(\Delta)$ and $D_{3p}^{3q}(\Delta)$ given by Proposition 1.30.

$$D_p^q(\Delta) \simeq \left\{ u \in \mathcal{S}^{\phi^q} \mid u\phi^\eta(u)\cdots\phi^{(p-1)\eta}(u) = \Delta \right\}.$$
$$D_{2p}^{2q}(\Delta) \simeq \left\{ (a, b) \in (\mathcal{S}^{\phi^q})^2 \mid ab \in D_p^q(\Delta) \right\}.$$
$$D_{3p}^{3q}(\Delta) \simeq \left\{ (x, y, z) \in (\mathcal{S}^{\phi^q})^3 \mid xyz \in D_p^q(\Delta) \right\}.$$

Under these descriptions, the action of ϕ is given by

$$\forall u \in D_p^q(\Delta),\ \phi.u = \phi^\eta(u).$$
$$\forall a, b \in D_{2p}^{2q}(\Delta),\ \phi.(a, b) = (b, \phi^\eta(a)).$$
$$\forall x, y, z \in D_{3p}^{3q}(\Delta),\ \phi.(x, y, z) = (y, z, \phi^\eta(x)).$$

This description of the action of ϕ also holds for $p = 1$ with $\eta = 1$.

We are now going to construct an oriented graph $\mathcal{S}_p^q$, which will serve as generators for our categorical presentation (see Appendix A.1 for definitions regarding oriented graphs).

Definition 1.31. *Let p, q be coprime positive integers. We denote by $\mathcal{S}_p^q$ the following oriented graph*
- *The objects are the elements of $D_p^q(\Delta)$.*
- *The arrows are the elements of $D_{2p}^{2q}(\Delta)$.*
- *For $(a, b) \in D_{2p}^{2q}(\Delta)$, the source (resp. the target) of (a, b) is given by ab (resp. $b\phi^\eta(a)$).*

We call $\mathcal{S}_p^q$ the **graph of simples** *(for the couple (p, q)).*

Note that, for $(a, b) \in \mathcal{S}_p^q$, the source and target of (a, b) are objects of $\mathcal{S}_p^q$. Indeed we have $(a, b) \in D_{2p}^{2q}(\Delta)$ and $\phi.(a, b) = (b, \phi^\eta(a)) \in D_{2p}^{2q}(\Delta)$, so both ab and $b\phi^\eta(a)$ lie in $D_p^q(\Delta)$ by Proposition 1.30.

Lemma 1.32. *The action of ϕ^2 on $D_{2p}^{2q}(\Delta)$ induces an automorphism ϕ_p of the graph $\mathcal{S}_p^q$, with $(\phi_p)^q = 1_{\mathcal{S}_p^q}$. For an object u of $\mathcal{S}_p^q$, we have $\phi_p(u) = \phi.u$.*

Proof. By definition, we have $\phi^2.(a, b) = (\phi^\eta(a), \phi^\eta(b))$ for $(a, b) \in D_{2p}^{2q}(\Delta)$. The source and target of $\phi^2.(a, b)$ are $\phi^\eta(ab) = \phi.(ab)$ and $\phi^\eta.(b\phi^\eta(a))$, respectively. As both ϕ and ϕ^2 are automorphisms of $\mathcal{C}$, they induce bijections on the sets $D_p^q(\Delta)$ and $D_{2p}^{2q}(\Delta)$, respectively. The fact that $(\phi_p)^q = 1_{\mathcal{S}_p^q}$ comes from the definition of the sets $D_p^q(\Delta)$ and $D_{2p}^{2q}(\Delta)$. □

We now endow the graph $\mathcal{S}_p^q$ with a set of relations R_p^q. The set R_p^q is in bijection with $D_{3p}^{3q}(\Delta)$. A $3p$-tuple (x, y, z) in $D_{3p}^{3q}(\Delta)$ induces the relation $(x, yz)(y, z\phi^\eta(x)) = (xy, z)$ between elements of $D_{2p}^{2q}(\Delta)$.

Definition 1.33. *The presented category $\mathcal{C}_p^q := \langle \mathcal{S}_p^q \mid R_p^q \rangle^+$ is called the* **category of (p, q)-periodic elements** *of $\mathcal{C}$. The enveloping groupoid of $\mathcal{C}_p^q$ will be denoted by $\mathcal{G}_p^q$.*

Lemma 1.34. *The function ℓ defined on arrows of $\mathcal{S}_p^q$ by $\ell(a, b) := \ell(a)$ extends to a length functor on $\mathcal{C}_p^q$, making $\mathcal{C}_p^q$ into a homogeneous category.*

Proof. First, the function ℓ extends to $\mathcal{C}_p^q$ as it is compatible with R_p^q. Indeed, for $(x,y,z)\in D_{3p}^{3q}(\Delta)$, we see that $\ell(xy,z)=\ell(xy)=\ell(x)+\ell(y)=\ell(x,yz)+\ell(y,z\phi^\eta(x))$. We then have to show that $\mathcal{C}_p^q$ is generated by elements of positive length. As $\mathcal{C}_p^q$ is generated (by definition) by $\mathcal{S}_p^q$, we have to show that the elements of length 0 in $\mathcal{S}_p^q$ are trivial in $\mathcal{C}_p^q$. By definition of ℓ, the only elements of $\mathcal{S}_p^q$ of length 0 are elements of the form $(1,u)$ for $u\in D_p^q(\Delta)$. We claim that the element $(1,u)\in\mathcal{S}_p^q$ is the identity morphism of the object u in $\mathcal{C}_p^q$. The source and target of $(1,u)$ are both u. Furthermore, for $(a,b)\in\mathcal{S}_p^q$ with source u, the element $(1,a,b)\in D_{3p}^{3q}(\Delta)$ expresses the relation $(1,u)(a,b)=(a,b)$. Similarly, for $(d,e)\in\mathcal{S}_p^q$ with target u, the element $(d,1,e)$ in $D_{3p}^{3q}(\Delta)$ expresses the relation $(d,e)(1,u)=(d,e)$. Thus $(1,u)$ represents the morphism 1_u in $\mathcal{C}_p^q$. ☐

Theorem 1.35. *([DDGKM, Proposition XIV.1.5] and [DDGKM, Proposition VII.4.2]) The category $\mathcal{C}_p^q$ is cancellative. Furthermore, defining*

$$\forall u\in \mathrm{Ob}(\mathcal{C}_p^q),\ \Delta_p(u):=(u,1)\in\mathcal{C}_p^q(u,\phi_p(u))$$

induces a Garside map Δ_p on $\mathcal{C}_p^q$. The simple morphisms associated to Δ_p are exactly the image in $\mathcal{C}_p^q$ of the graph $\mathcal{S}_p^q$.

Lemma 1.36. *The automorphism ϕ_p of $\mathcal{S}_p^q$ extends to an automorphism of $\mathcal{C}_p^q$. Moreover, ϕ_p is the automorphism induced by the Garside map Δ_p.*

Proof. As we already know that ϕ_p is an automorphism of $\mathcal{S}_p^q$, we only have to show that it is compatible with the set of relations. Let $(x,y,z)\in D_{3p}^{3q}(\Delta)$, the element $(\phi^\eta(x),\phi^\eta(y),\phi^\eta(z))$ of $D_{3p}^{3q}(\Delta)$ induces the relation

$$\begin{aligned}&(\phi^\eta(x),\phi^\eta(y)\phi^\eta(z))(\phi^\eta(y),\phi^\eta(x)\phi^{2\eta}(z))=(\phi^\eta(x)\phi^\eta(y),\phi^\eta(z))\\ \Leftrightarrow&(\phi^\eta(x),\phi^\eta(yz))(\phi^\eta(y),\phi^\eta(x\phi^\eta(z)))=(\phi^\eta(xy),\phi^\eta(z))\\ \Leftrightarrow&\phi_p(x,yz)\phi_p(y,z\phi^\eta(x))=\phi_p(xy,z)\end{aligned}$$

which proves that ϕ_p is compatible with R_p^q. Now, in order to prove that ϕ_p is indeed the automorphism induced by the Garside map Δ_p, we only need to show that $\phi_p(s)=\overline{\overline{s}}$ for any simple morphism s. Let $s=(a,b)\in\mathcal{S}_p^q$, the element $(a,b,1)$ in $D_{3p}^{3q}(\Delta)$ expresses the relation $(a,b)(b,\phi^\eta(a))=(ab,1)=\Delta_p(ab)$, which proves that $\overline{(a,b)}=(b,\phi^\eta(a))$. From this we deduce $\overline{\overline{s}}=(\phi^\eta(a),\phi^\eta(b))=\phi_p(a,b)$ as claimed. ☐

Definition 1.37. *([Bes15, Definition B.23]) The map defined on $\mathcal{S}_p^q$ by $(a,b)\mapsto a$ induces a functor $\pi_p:\mathcal{C}_p^q\to\mathcal{C}$, which is called the* **collapse functor**.

Lemma 1.38. *Let u be an object of $\mathcal{C}_p^q$, and let $x:=\pi_p(u)$ be the source of u in $\mathcal{C}$. The collapse functor restricts to an isomorphism of posets*

$$(\mathcal{S}_p^q(u,-),\preceq)\simeq(\{s\in\mathcal{S}^{\phi^q}\mid s\preceq u\},\preceq)\subset\mathcal{S}^{\phi^q}(x,-)\subset\mathcal{S}(x,-).$$

In particular, if $s:=(a,b)$ and $s':=(a',b')$ are two simples in $\mathcal{C}_p^q$ with source u, then $s\preceq s'$ in $\mathcal{C}_p^q$ if and only if $a\preceq a'$ in $\mathcal{C}$.

Proof. Let π denote the restriction of the collapse functor π_p to the set $\mathcal{S}_p^q(u,-)$. For $s:=(a,b)\in\mathcal{S}_p^q(u,-)$, we have $ab=u$ by definition, so $a\preceq u$ and $\phi^q(a)=a$. Conversely, if $a\preceq u$ is such that $\phi^q(a)=a$, then there is a unique $b\in\mathcal{S}$ such that $ab=u$. We have $\phi^q(b)=b$ by Lemma 1.10, and the simple morphism (a,b) is then the unique preimage by π of a in $\mathcal{S}_p^q(u,-)$.

Let now $s:=(a,b)$ and $s':=(a',b')$ be two elements of $\mathcal{S}_p^q(u,-)$. If $s\preceq s'$, then there is some $(x,y,z)\in D_{3p}^{3q}(\Delta)$ such that $(x,yz)=(a,b)$ and $(xy,z)=(a',b')$. We have in particular $ay=xy=a'$ and $a\preceq a'$. Conversely, if $a\preceq a'$, there is some $y\in\mathcal{S}$ such that $ay=a'$. Again we have $\phi^q(y)=y$ by Lemma 1.10, and the relation induced by (a,y,b) then gives $s\preceq s'$. ☐

Corollary 1.39. *Let $s := (a,b)$ and $t := (\alpha,\beta)$ be two composable simple morphisms in $\mathcal{C}_p^q$. The greedy normal form of the path st in $\mathcal{C}_p^q$ is given by*

$$st = (ad, d^{-1}b)(d^{-1}\alpha, \beta\phi^\eta(d)))$$

where $d = \alpha \wedge b$. In particular the path st is greedy if and only if b and α are coprime.

Proof. The composability of s and t is equivalent to $\alpha\beta = b\phi^\eta(a)$. By Lemma 1.38, we have $t \wedge \overline{s} = (\alpha,\beta) \wedge (b, \phi^\eta(a)) = (d,e)$, where e is defined by $de = \alpha\beta$. We have

$$\begin{aligned} st &= (a,b)(\alpha,\beta) = (a,b)(d,e)(d^{-1}\alpha,\beta) \\ &= (ad, d^{-1}b)(d^{-1}\alpha, \beta\phi^\eta(d)). \end{aligned}$$

This last term is the greedy normal form of st because of Lemma 1.15. $\square$

Theorem 1.40. *([DDGKM, Proposition XIV.1.8])*

(1) Let u be an object of $\mathcal{G}_p^q$. The element $\pi_p(\Delta_p^q(u))$ is a (p,q)-periodic element in $\mathcal{G}(\mathcal{C})$.
(2) Every (p,q)-regular element of $\mathcal{G}(\mathcal{C})$ is conjugate to an element of the form $\pi_p(\Delta_p^q(u))$.
(3) The collapse functor induces a functor $\mathcal{G}_p^q \to \mathcal{G}(\mathcal{C})$, which in turn induces a group isomorphism between $\mathcal{G}_p^q(u,u)$ and the centralizer of $\pi_p(\Delta_p^q(u))$ in $\mathcal{G}(\pi_p(u),\pi_p(u))$.

1.3.1. *The case $p = 1$.* If $p = 1$, then the graph $\mathcal{S}_p^q$ is simply the subgraph $\mathcal{S}^{\phi^q}$ of $\mathcal{S}$. Indeed we have

$$\begin{aligned} D_1^q(\Delta) &= \{u \in \mathcal{S}^{\phi^q} \mid u = \Delta\} = \{\Delta(u) \in \mathcal{S} \mid \phi^q(\Delta(u)) = \Delta(\phi^q(u)) = \Delta(u)\} \\ &\simeq \{u \in \mathrm{Ob}(\mathcal{C}) \mid \phi^q(u) = u\}. \\ D_2^q(\Delta) &= \{(a,b) \in (\mathcal{S}^{\phi^q})^2 \mid ab = \Delta\} \\ &= \{(a,\overline{a}) \in (\mathcal{S}^{\phi^q})^2\} \simeq \mathcal{S}^{\phi^q}. \end{aligned}$$

Let (x,y,z) be in $D_3^{3q}(\Delta)$. We have $z = \overline{xy}$, thus (x,y,z) expresses the relation $x.y = (xy)$ between elements of $\mathcal{S}^{\phi^q}$.

By [DDGKM, Proposition VI.1.11], the category $\mathcal{C}_1^q$ is simply the subcategory $\mathcal{C}^{\phi^q}$. The Garside map of Theorem 1.35 is then the restriction to $\mathcal{C}^{\phi^q}$ of the Garside map of $\mathcal{C}$. The same goes for the automorphism ϕ_1.

The collapse functor π_1 sends $a \in \mathcal{S}^{\phi^q}$ to $a \in \mathcal{S}$: it is the inclusion $\mathcal{C}^{\phi^q} \hookrightarrow \mathcal{C}$. Lemma 1.38 expresses that, for an object u of $\mathcal{C}^{\phi^q}$, the inclusion $\mathcal{S}^{\phi^q}(u,-) \subset \mathcal{S}(u,-)$ is a morphism of lattices, which is also a consequence of Lemma 1.10. Corollary 1.39 ensures that the greedy normal form of a path in $\mathcal{C}^{\phi^q}$ is the same as its greedy normal form in $\mathcal{C}$.

Lastly, Theorem 1.40 expresses that every $(1,q)$-periodic element of $\mathcal{C}$ is conjugate to some $\Delta^q(u)$, and that $\mathcal{G}(\mathcal{C})^{\phi^q}(u,u)$ is the centralizer of $\Delta^q(u)$ inside of $\mathcal{G}(\mathcal{C})(u,u)$.

2. Complex reflection groups, braid groups and braid categories

2.1. Reminders on complex reflection groups and their braid groups.

2.1.1. *Definitions, basic invariants.* We follow the exposition of [LT09]. Let V be a finite dimensional complex vector space. An nontrivial element $s \in \mathrm{GL}(V)$ is called a ***reflection*** if it pointwise fixes some hyperplane of V and has finite order. The hyperplane $\mathrm{Ker}(s-\mathrm{Id})$ is then called the ***reflecting hyperplane*** of s. A subgroup $W < \mathrm{GL}(V)$ is called a ***complex reflection group*** if it is finite and generated by reflections of V. The integer $n = \dim V$ is the ***rank*** of W.

A complex reflection group $W < \mathrm{GL}(V)$ is ***irreducible*** if there are no nontrivial W-stable subspaces in V. It is well-known that every complex reflection group decomposes as a product of irreducible complex reflection groups. This allows us to restrict our attention to irreducible groups. These groups were classified by Shephard and Todd in [ST54], with on the one hand an infinite series of monomial matrices (groups denoted by $G(de,e,n)$ for integers d,e,n), and on the other hand a list of 34 exceptional cases, labeled G_4 to G_{37}.

An irreducible complex reflection group of rank n is called ***well-generated*** if it can be generated by a set of n reflections. An irreducible group of rank n which is not well-generated is ***badly-generated***. Such a group can always be generated by $n+1$ reflections.

From now on we fix $W < \mathrm{GL}(V)$ an irreducible complex reflection group of rank n. The action of W on V extends to an action of W on the algebra S of polynomial functions on V. By the Chevalley-Shephard-Todd theorem ([LT09, Theorem 3.20]), the algebra S^W of W-invariant elements of S is a polynomial algebra in n variables. A n-tuple $f = (f_1, \dots, f_n)$ of homogeneous algebraically independent generators of S^W is called a ***system of basic invariants***. The degrees d_i of the f_i doesn't depend on the choice of f. They are the ***degrees*** of the reflection group W.

Likewise, one can define the ***codegrees*** $d_1^*, \dots, d_n^*$ of W by considering the module of invariant derivations on the algebra S ([Bes15, Definition 1.2]).

2.1.2. *Braid groups, braided reflections.* We consider the subset $X \subset V$ consisting of points not belonging to any of the reflecting hyperplanes associated to reflections of W. By Steinberg's Theorem ([LT09, Theorem 9.44]), the action of W on X is free and induces a covering map from X to X/W. We fix a basepoint $x_0 \in X$ and we set

$$P(W) := \pi_1(X, x_0) \text{ and } B(W) := \pi_1(X/W, W.x_0)$$

the ***pure braid group*** of W and the ***braid group*** of W, respectively. The covering map $X \twoheadrightarrow X/W$ induces a short exact sequence

$$1 \longrightarrow P(W) \longrightarrow B(W) \longrightarrow W \longrightarrow 1\ .$$

Note that, as X is path connected, a change of basepoint $x_0 \to x_1$ yields a (non canonical) isomorphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & P(W) \arrow[r] \arrow[d, "\simeq"] & B(W) \arrow[r] \arrow[d, "\simeq"] & W \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(X, x_1) \arrow[r] & \pi_1(X/W, W.x_1) \arrow[r] & W \arrow[r] & 1. \end{tikzcd}$$

Let $f := (f_1, \dots, f_n)$ be a system of basic invariants. The isomorphism between S^W and $\mathbb{C}[f_1, \dots, f_n]$ induces in turn an algebraic isomorphism $V/W \simeq \mathbb{C}^n$. Let $\mathcal{H}$ denote the image in V/W of the reunion of the reflecting hyperplanes of W, it is an algebraic hypersurface of $V/W \simeq \mathbb{C}^n$. Since the braid group $B(W)$ is defined as the fundamental group of the complement of $\mathcal{H}$ in V/W, it is generated by particular elements called ***braided reflections*** (around the irreducible divisors of $\mathcal{H}$).

We quickly recall from [BMR98, Appendix 1] the definition of braided reflections, as it will be useful in Section 3.4. Let H be a reflecting hyperplane of W, and let $D \subset \mathcal{H}$ be the image of H in V/W.

Definition 2.1. *A* **path from** x_0 **to** D **in** X/W *is a path* $\gamma : [0,1] \to V/W$ *such that* $\gamma(0) = x_0$*,* $\gamma(1) \in \widetilde{D}$ *and* $\gamma(t) \in X/W$ *for* $t < 1$*. Two paths* γ, γ' *from* x_0 *to* D *in* X/W *are* D-**homotopic** *if there exist a homotopy* $T : [0,1] \times [0,1] \to V/W$ *from* γ *to* γ' *such that*

- *For all* $t \in [0,1[$ *and* $u \in [0,1]$ $T(t,u) \in X/W$*.*
- *For all* $u \in [0,1]$*,* $T(0,u) = x_0$ *and* $T(1,u) \in \widetilde{D}$*.*

The D*-homotopy class of* γ *is denoted by* $[\gamma]$*.*

Let γ be a path from x_0 to D in X/W. Let also U be a connected open neighborhood of $\gamma(1)$ in $X/W \cup \widetilde{D}$ such that $U \cap X/W$ has a fundamental group free of rank 1. Let $u \in [0,1[$ be such that $\gamma(t) \in U$ for $t \geqslant u$. The orientation of $U \cap X$ allows us to choose a "positive" generator λ of $\pi_1(U \cap X/W, \gamma(u))$.

We set $\gamma_u(t) := \gamma(ut)$ for $t \in [0,1]$ and $\rho_{\gamma,\lambda} := \gamma_u * \lambda * \gamma_u^{-1}$. The homotopy class of $\rho_{\gamma,\lambda}$ depends only on $[\gamma]$ and is denoted by $\rho_{[\gamma]}$. It is by definition a braided reflection of $B(W)$ (around the divisor D). The image of $\rho_{[\gamma]}$ inside W is a reflection with hyperplane H.

Proposition 2.2. *([BMR98, Appendix 1])*
All meridians around D form a conjugacy class of $B(W)$. In particular the set of all braided reflections is stable under conjugacy in $B(W)$.

Remark 2.3. Let $x_0 \to x_1$ be a change of basepoint. The induced isomorphism between $B(W)$ and $\pi_1(X/W, W.x_1)$ maps braided reflections to braided reflections.

Lastly, we give the definition of a particular element in $B(W)$, which plays a particular role when considering centers and centralizers.

Definition 2.4. *([BMR98, Notation 2.3])*
The **full-twist** *is the homotopy class in $B(W)$ of the loop*

$$\begin{array}{cccc} \gamma_\pi : & [0,1] & \longrightarrow & X/W \\ & t & \longmapsto & \exp(2i\pi t)x_0. \end{array}$$

It lies inside $P(W) \cap Z(B(W))$.

2.1.3. *Regular elements, regular braids.* Let d be a positive integer. We denote by μ_d the group of d-th roots of unity in $\mathbb{C}$. We also consider μ_d^* the subgroup of μ_d consisting of primitive d-th roots of unity, and $\zeta_d := \exp(\frac{2i\pi}{d}) \in \mu_d^*$.

Definition 2.5. *([LT09, Definition 11.21])*
Let ζ be a root of unity in $\mathbb{C}$. An element $g \in W$ is called **ζ-regular** *if it admits a ζ-eigenvector lying in X. An integer d is* **regular** *for W if there is a ζ_d-regular element g in W.*

If $g \in W$ is a ζ-regular element, then for every integer k, g^k is ζ_d^k-regular. Thus, if g is ζ-regular for some $\zeta \in \mu_d^*$, then d is a regular integer for W. It is known that ζ-regular elements (should they exist) form a conjugacy class in W.

Theorem 2.6. *([LT09, Theorem 11.24])*
Let $g \in W$ be a ζ-regular element for some $\zeta \in \mu_d^*$. The centralizer $C_W(g)$ of g in W acts on the eigenspace $\operatorname{Ker}(g - \zeta \operatorname{Id})$ as a complex reflection group. Its degrees (resp. codegrees) coincide with those degrees (resp. codegrees) of W which are divisible by d.

Let $g \in W$ be a ζ_d-regular element, and let $W_g := C_W(g)$. It is possible to study the braid group $B(W_g)$ by embedding it inside $B(W)$. Let V_g denote the eigenspace $\operatorname{Ker}(g - \zeta_d Id)$ on which W_g acts as a reflection group. Let also X_g denote the space of regular vectors inside V_g. By [LT09, Theorem 11.33], we have $X_g = X \cap V_g$.

The scalar action of $\mathbb{C}^*$ on V induces in particular an action of μ_d on V/W. By [Bes15, Theorem 1.9], the embedding $V_g \to V$ induces two homeomorphisms $V_g/W_g \simeq (V/W)^{\mu_d}$ and $X_g/W_g \simeq (X/W)^{\mu_d}$. In particular $B(W_g)$ identifies with the fundamental group of $(X/W)^{\mu_d}$. It was shown in [Bes15, Theorem 12.4] and [Gar23, Theorem 1.2] that the fundamental group $(X/W)^{\mu_d}$ can itself be identified with the centralizer in $B(W)$ of any d-th root of the full-twist. Such d-th roots of the full-twist will be called ***d-regular braids***.

Example 2.7. A guiding example is the case of the groups G_{31} and G_{37}. The degrees (resp. codegrees) of G_{37} are

2	8	12	14	18	20	24	30
0	6	10	12	16	18	22	28

The integer 4 is regular for G_{37}. The centralizer of a $i = \zeta_4$-regular element in G_{37} is isomorphic to G_{31}. Its degrees (resp. codegrees) are $8, 12, 20, 24$ (resp. $0, 12, 16, 28$). Note that 4 is the gcd of the degrees of G_{37} which it divides.

Lemma 2.8. *Let $W < \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C})$ be a real reflection group which contains $-\operatorname{Id}$ and for which 4 is regular. If $g \in W$ is a i-regular element and r is a reflection of W, then r and r^g commute.*

Proof. First, the element $-\operatorname{Id}$ is a -1-regular element in W. Since $-\operatorname{Id}$ is central in W, and since all -1-regular elements in W are conjugate to $-\operatorname{Id}$, we get that $-\operatorname{Id}$ is in fact the only -1-regular element in W. Now, since g is i-regular, g^2 is $i^2 = -1$-regular, hence equal to $-\operatorname{Id}$.

We assume that $r \neq r^g$ (otherwise our claim is immediate). We fix $\langle -, - \rangle$ a W-invariant scalar product on $\mathbb{R}^n$. Let α be a root for r (that is, a generator of $\operatorname{Ker}(r-\operatorname{Id})^{\perp}$). We have

$$g^2 r g^{-2} = r \Rightarrow g r g^{-1} = g^{-1} r g.$$

Thus $g^{-1}(\alpha)$ and $g(\alpha)$ are two roots of the same reflection r^g. Let $\alpha \in \mathbb{R}$ such that $g^{-1}(\alpha) = \lambda g(\alpha)$. We have $\alpha = \lambda g^2(\alpha) = -\lambda\alpha$ and $\lambda = -1$,thus

$$\langle g(\alpha), \alpha \rangle = \left\langle \alpha, g^{-1}(\alpha) \right\rangle = \langle \alpha, -g(\alpha) \rangle = -\langle g(\alpha), \alpha \rangle .$$

The two roots α and $g(\alpha)$ are then orthogonal for $\langle -, - \rangle$ and r and r^g commute. $\square$

This lemma applies to the case of G_{31} seen inside G_{37} as the centralizer of a i-regular element. Indeed, G_{37} is the complexified version of the exceptional Coxeter group E_8, which contains $-\operatorname{Id}$.

2.2. **Dual braid monoid for well-generated groups.** Let $W < \operatorname{GL}(V)$ be a well-generated complex reflection group of rank n. In [Bes03] and [Bes15], Bessis defines a particular Garside monoid, the ***dual braid monoid***, which admits the braid group $B(W)$ as its group of fraction. This monoid is the first step from which it is possible to define a Garside category suitable for studying centralizers of regular braids in $B(W)$.

Here we consider the combinatorial definition of the dual braid monoid as an interval monoid. We will be discussing the topological definition of the dual braid monoid in the next section. We follow [Bes15, Section 8].

Let R be the set of reflections of W. We have $R = R^{-1}$, and R generates W by definition. For $w \in W$, one can define $\ell_R(w)$ to be the minimal length of a decomposition of w as a product of reflections. Because the set R is globally invariant under conjugation, the value of $\ell_R(w)$ depends only on the conjugacy class of w in W. The function ℓ_R induces relations $\preceq$ and $\succeq$ on W defined by

$$\forall w, v \in W,\ v \preceq w \Leftrightarrow \ell_R(v) + \ell_R(v^{-1}w) = \ell_R(w),$$
$$\forall w, v \in W,\ w \succeq v \Leftrightarrow \ell_R(wv^{-1}) + \ell_R(v) = \ell_R(w).$$

Let v, w be in W. Since ℓ_R takes constant values on conjugacy classes in W, we get that $v \preceq w \Leftrightarrow w \succeq v$. Thus we will always work with $\preceq$ and only consider $\succeq$ for readability purposes.

As W is well-generated, we know that the highest degree h of W is regular for W (see the proof of [Bes15, Theorem 2.4]). A ***Coxeter element*** of W is then an element $c \in W$ which is regular for the eigenvalue $\zeta_h := \exp(\frac{2i\pi}{h})$. Let c be a Coxeter element in W. We define

$$R_c := \{r \in R \mid r \preceq c\} \text{ and } I_c := [1, c]_{\preceq} = \{w \in W \mid w \preceq c\}$$

We also consider formal copies $\mathcal{R} \subset \mathcal{I}$ of $R_c \subset I_c$ (their elements will be denoted in bold font). The ***interval monoid*** associated to R and c is then defined by the presentation:

$$M(c) := \langle \mathcal{I} \mid \mathbf{st} = \mathbf{u} \Leftrightarrow (st = u \text{ and } \ell_R(s) + \ell_R(t) = \ell_R(u)) \rangle$$

We know that $(I_c, \preceq)$ is a lattice ([Bes15, Lemma 8.6]). We will denote $s \wedge t$ (resp. $s \vee t$) the gcd (resp. lcm) of s and t in I_c. We do not have to distinguish between left- and right-gcd and lcm since all simple elements are balanced (their left- and right-divisor are the same).

Let c and c' be two Coxeter elements of W. Since they are regular elements of W for the same eigenvalue ζ_h, there is some $w \in W$ such that $wcw^{-1} = c'$. Since ℓ_R is invariant under conjugacy, one readily sees that conjugation by w induces an isomorphism between the two intervals I_c and $I_{c'}$. In particular we have $M(c) \simeq M(c')$.

Theorem 2.9. *([Bes15, Theorem 8.2])*
The monoid $M(c)$ is a homogeneous Garside monoid, with Garside element $\Delta := \mathbf{c}$ and set of simples $\mathcal{I}$. The atoms of $M(c)$ are the elements of $\mathcal{R}$.

The ***Hurwitz relations*** with respect to W and c are the formal relations of the form

$$\mathbf{r}\mathbf{r}' = \mathbf{r}'\mathbf{r}''$$

where $r, r', r'' \in R_c$ are such that $r \neq r'$, $rr' \in I_c$ and $rr' = r'r''$ holds in W. We know from [Bes15, Lemma 8.8] that the monoid $M(c)$ admits a particular presentation with $\mathcal{R}$ as a set of generators, endowed with the Hurwitz relations.

Remark 2.10. We can be a little less specific and say that $M(c)$ (and $G(c)$) is presented by relations of the form

$$\mathbf{s}\mathbf{s}' = \mathbf{r}\mathbf{r}'$$

with $s, s', r, r' \in R$, and $ss' \in I_c$. That is $M(c)$ is presented by its atoms and the equality between decompositions of simple elements of length 2. It is this precise rephrasing which we will show holds in Springer categories (see Theorem 3.27).

Lemma 2.11. *For $\mathbf{a} \in \mathcal{I}$, we have $\mathbf{a}^2 \in \mathcal{I}$ if and only if $\mathbf{a} = 1$.*

Proof. If $\mathbf{a}^2 \in \mathcal{I}$, the relation $\mathbf{a}\mathbf{a} = \mathbf{a}^2$ must be a defining relation of M, so we must have $2\ell_R(a) = \ell_R(a^2)$. If $a \neq 1$, then let $s \in R_c$ be a divisor of a with $sa' = a$ and $a''s = a$. We have $a^2 = a''ssa = a''s^2a$, so $\ell_R(a^2) \leqslant \ell_R(a'') + \ell_R(a') + 1 = 2\ell_R(a) - 1$, which contradicts $2\ell_R(a) = \ell_R(a^2)$. ☐

Proposition 2.12. *Let $\mathbf{a}, \mathbf{a}'$ be in $\mathcal{I}$. If the product $\mathbf{a}\mathbf{a}'$ is also in $\mathcal{I}$, then $\mathbf{a}\mathbf{a}' = \mathbf{a} \vee \mathbf{a}'$ and $\mathbf{a} \wedge \mathbf{a}'$ is trivial.*

Proof. First, note that in I_c, we have $aa' = a'a^{a'} = {}^{a}a'a$. Now, because ℓ_R is invariant under conjugation, we have

$$\ell_R(aa') = \ell_R(a) + \ell_R(a') = \ell_R(a') + \ell_R\left(a^{a'}\right) = \ell_R\left({}^{a}a'\right) + \ell_R(a)$$

This means that $aa' \succeq a^{a'}$ and ${}^{a}a' \preceq aa'$: we have $a^{a'}, {}^{a}a' \in I_c$, and $\mathbf{a}\mathbf{a}'$ is a common left and right multiple of $\mathbf{a}$ and $\mathbf{a}'$. Now let $\mathbf{a}\mathbf{b} = \mathbf{a}'\mathbf{b}'$ be the right-lcm of $\mathbf{a}$ and $\mathbf{a}'$. By definition, there is some $\mathbf{x} \in \mathcal{I}$ such that $\mathbf{b}\mathbf{x} = \mathbf{a}'$ and $\mathbf{b}'\mathbf{x} = \mathbf{a}^{\mathbf{a}'}$. In particular, we have $\mathbf{a}' \succeq \mathbf{x}$, $\mathbf{a}^{\mathbf{a}'} \succeq \mathbf{x}$ and $\mathbf{x} \preceq \mathbf{a}^{\mathbf{a}'}$ since every element is balanced. So $\mathbf{a}\mathbf{a}'$ admits $\mathbf{x}^2$ as a factor and $\mathbf{x}^2$ is a simple. By Lemma 2.11, $\mathbf{x} = 1$, $\mathbf{a}' = \mathbf{b}$, $\mathbf{b}' = \mathbf{a}^{\mathbf{a}'}$ and $\mathbf{a}\mathbf{a}' = \mathbf{a} \vee \mathbf{a}'$.

If $\mathbf{x}$ is a common divisor of $\mathbf{a}$ and $\mathbf{a}'$, then we have $\mathbf{a} \succeq \mathbf{x}$ and $\mathbf{x} \preceq \mathbf{a}'$. We get that $\mathbf{x}^2$ is a simple element in $M(c)$, it must then be trivial. ☐

Remark 2.13. By definition of an interval monoid, there is a natural bijection between the simples of $M(c)$ and the interval $I_c = ([1, c], \preceq_R)$ in W. This bijection induces in turn a bijection between the sets $D_m(\Delta)$ and the sets $D_m(c)$, defined by

$$D_m(c) := \left\{ (u_0, \ldots, u_{p-1}) \in W^p \;\middle|\; \begin{cases} u_0 \cdots u_{p-1} = c \\ \ell_R(u_0) + \cdots + \ell_R(u_{p-1}) = \ell_R(c) \end{cases} \right\}.$$

The automorphism ϕ of $M(c)$ acts on $D_m(c)$ by $\phi.(u_0, \ldots, u_{p-1}) := (u_1, \ldots, u_{p-1}, u_0^c)$. The bijection between $D_m(\Delta)$ and $D_m(c)$ restricts to a bijection between $D_m^n(\Delta)$ and

$$D_m^n(c) := \{(u_0, \ldots, u_{p-1}) \in D_m(c) \mid \phi^q.(u_0, \ldots, u_{p-1}) = (u_0, \ldots, u_{p-1})\}$$

The results of Proposition 1.30 also allow for an alternative description of the sets $D_n^m(c)$.

2.3. **Lyashko-Looijenga map, cyclic labels.** Here we recall the preliminaries which are necessary to construct both the dual braid monoid from a topological point of view, and the Springer category associated to a regular number. Most of this section comes from [Bes15, Section 5, 6].

Let again $W < \mathrm{GL}(V)$ be an irreducible well-generated complex reflection group with highest degree h, together with a system of basic invariants f such that the discriminant has the following form:

$$\Delta_f = X_n^n + \alpha_2 X_n^{n-2} + \cdots + \alpha_n$$

where $\alpha_i \in \mathbb{C}[X_1, \cdots, X_{n-1}]$ (such a system always exists for well-generated groups by [Bes15, Theorem 2.4 and Section 2.5]).

The system of basic invariants f provides an isomorphism $V/W \simeq \mathbb{C}^n$, which sends the orbit $W.v$ of $v \in V$ to $(f_1(v), \cdots, f_n(v)) \in \mathbb{C}^n$. We denote $v_i := f_i(v)$.

Definition 2.14. *([Bes15, Definition 7.23]) Let $x = W.v \in V/W$. The multiset $\overline{\mathrm{LL}}(x)$ is defined as the solutions in T of the polynomial*

$$\Delta_f(v_1, \cdots, v_{n-1}, T + v_n) \in \mathbb{C}[T]$$

It is by definition an element of $E_n := \mathbb{C}^n/\mathfrak{S}_n$. The map $\overline{\mathrm{LL}} : V/W \to E_n$ is called the (extended) **Lyashko-Looijenga morphism***.*

Remark 2.15. The original approach in [Bes15] was to study a slightly different application, denoted by LL (cf [Bes15, Definition 5.1]). Bessis then suggested that the application $\overline{\mathrm{LL}}$ was in fact a far better choice.

For $x \in V/W$, we have $x \in X/W \Leftrightarrow 0 \notin \overline{\mathrm{LL}}(x)$. By definition this is equivalent to $\overline{\mathrm{LL}}(x) \in E_n^\circ := (\mathbb{C}^*)^n/\mathfrak{S}_n$. We endow V/W with the quotient of the scalar action of $\mathbb{C}$ on V. We also endow the space of configurations E_n with the scalar action of $\mathbb{C}$.

Lemma 2.16. *([Bes15, Lemma 11.1])*
Let $x \in V/W$ and let $\lambda \in \mathbb{C}^*$. We have $\overline{\mathrm{LL}}(\lambda x) = \lambda^h \overline{\mathrm{LL}}(x)$.

The ***fat basepoint*** for X/W is then defined as the following set

$$\mathcal{U} := \{x \in V/W \mid \overline{\mathrm{LL}}(x) \cap i\mathbb{R}_{\geqslant 0} = \varnothing\}.$$

That is, $x \in \mathcal{U}$ if no point of $\overline{\mathrm{LL}}(x)$ is directly above 0. This subset of X/W is open and contractible ([Bes15, Lemma 6.3]). It can be used as a fat basepoint for defining the braid group of W ([Bes15, Definition 6.4]). From now on we consider $B(W) := \pi_1(X/W, \mathcal{U})$.

Let $x \in X/W$, the points of $\overline{\mathrm{LL}}(x)$ are ordered clockwise starting from "right after noon" (that is, points lying on the half-line $i\mathbb{R}_{\geqslant 0}$ will be at the end). Points with the same argument are ordered by increasing module. If $x \in V/W$, the sequence $(x_1, \ldots, x_k)$ of points of $\overline{\mathrm{LL}}(x)$ ordered in this way is called the ***cyclic support*** of x ([Bes15, Definition 11.8]).

Remark 2.17. The set of points $x \in \mathcal{U}$ such that the points in the cyclic support of x all have distinct arguments is dense in X/W. The ordering we choose allows us to define univalent desingularization, which preserves the ordering of the points of the cyclic support.

Definition 2.18. *([Bes15, Definition 11.24])*
A **circular semitunnel** *is a couple $T = (x, L) \in \mathcal{U} \times [0, \frac{2\pi}{h}]$. The path γ_T associated with a semitunnel T is the path*

$$\begin{array}{rccc} \gamma_T : & [0,1] & \longrightarrow & X/W \\ & t & \longmapsto & e^{itL}x \end{array}$$

We say that T is a **circular tunnel** *if it satisfies the additional condition $\gamma_T(1) \in \mathcal{U}$.*

Let $T := (x, L)$ be a circular (semi)tunnel. Since $\overline{\mathrm{LL}}$ is homogeneous of degree h, the path $\overline{\mathrm{LL}} \circ \gamma_T$ corresponds to a continuous rotation of angle hL. Since the path γ_T starts and ends in $\mathcal{U}$, it induces a well defined element in $B(W) = \pi_1(X/W, \mathcal{U})$. From now on we amalgamate a circular tunnel T with the path γ_T it induces in X/W.

Theorem 2.19. *([Bes15, Proposition 8.5 and Lemma 11.10])*

(i) The homotopy class Δ of the circular tunnel $T = (x, \frac{2\pi}{h})$ does not depend on $x \in \mathcal{U}$.
(ii) The projection c of Δ in W is a Coxeter element of W.
(iii) Consider S the set of homotopy classes of circular tunnels in $B(W)$. It is endowed with the relation $\preceq$ defined by $s \preceq s'$ if $s^{-1}s'$ is homotopic to a circular tunnel. The projection map $B(W) \twoheadrightarrow W$ induces an isomorphism of posets $(S, \preceq) \simeq (I_c, \preceq)$. This isomorphism induces in turn an isomorphism $B(W) \simeq G(c)$.

From now on we fix the Coxeter element c so that we have the isomorphism $G(c) \simeq B(W)$ induced by the above theorem. By [Bes15, Lemma 6.13], the element Δ^h in $B(W)$ represents the full-twist.

Remark 2.20. In [Bes15], the notion of circular tunnel is preceded by the notion of tunnel (cf [Bes15, Definition 6.6]). However, [Bes15, Lemma 11.10] and [Bes15, Corollary 6.18] show that the two notion are actually synonymous, in the sense that an element of $B(W)$ is represented by a circular tunnel if and only if it is represented by a tunnel.

The notion of circular tunnel allows us to define the notion of cyclic label, which will prove crucial in the study of $(X/W)^{\mu_d}$. Let $x \in \mathcal{U}$, and let $(x_1, \ldots, x_k)$ be the cyclic support of x. We assume that all the x_i have distinct arguments. There is a minimal $\theta \in \mathbb{R}_{>0}$ such that $e^{i\theta}x \notin \mathcal{U}$. The ***head*** of x is defined as the element c_1 of $B(W)$ represented by the circular tunnel $T = (x, \theta + \varepsilon)$, which doesn't depend on $\varepsilon > 0$ small enough. We can then consider the head of $\gamma_T(1) = e^{i(\theta+\varepsilon)t}x$ and so on, until all the points of the cyclic support of x have been labeled.

Definition 2.21. *([Bes15, Definition 11.9])*
Let $x \in \mathcal{U}$ be such that all the points in $\overline{\mathrm{LL}}(x)$ have distinct arguments. The sequence $(s_1, \cdots, s_k)$ defined above is the **cyclic label** *of x, denoted by* $\mathrm{clbl}(x)$.
If different points in the cyclic support of x have the same argument, we define $\mathrm{clbl}(x)$ *as the cyclic label of some desingularization of x, as in Remark 2.17.*

Since the elements of $\mathrm{clbl}(x)$ are defined as circular tunnels, they are simple elements in $M(c)$. Thus we can see them as elements of $I_c \subset W$. Theorem 2.19 proves that, for any $x \in X/W$, the product of all the terms of $\mathrm{clbl}(x)$ is equal to Δ in $B(W)$, as it is represented by the circular tunnel $(x, \frac{2\pi}{h})$. Thus, if the cyclic support of x contains k points, then $\mathrm{clbl}(x)$ lies inside $D_k(c)$.

Theorem 2.22. *([Bes15, Proposition 11.13])*
A pair $(x, (a_1, \cdots, a_k)) \in E_n^\circ \times D_k(c)$ is **compatible** *if the cyclic support of x contains k points, and their respective multiplicities coincide with $(\ell_R(a_1), \cdots, \ell_R(a_k))$.*
The map $(\overline{LL}, \mathrm{clbl})$ induces a bijection between X/W and the set $E_n^\circ \boxdot D(c)$ of compatible pairs. This bijection induces a topology on the latter.

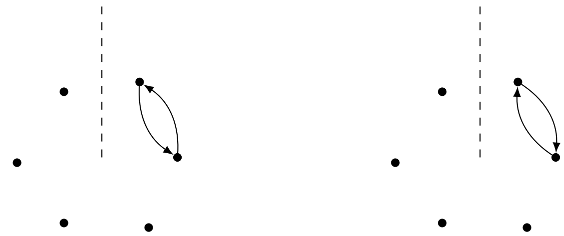
By [Bes15, Remark 7.21], a path γ in E_n admits a unique lift $\widetilde{\gamma}$ in X/W with fixed starting point provided that points are only merged and not unmerged in $\gamma(t)$ when t increases. We want to understand how the cyclic label of $\widetilde{\gamma}(t)$ changes depending on γ.

Lemma 2.23. *(Hurwitz Rule)*
Let $x \in X/W$, and let $T := (x, L)$ be a circular tunnel. Let also $\gamma : [0,1] \to X/W$ be a path starting at x and such that, for all $t \in [0,1]$, $(\gamma(t), L)$ is a circular tunnel. For all $t \in [0,1]$, the elements in $B(W)$ represented by $(\gamma(t), L)$ and T are equal.

Proof. Let $t \in [0,1]$. Defining $H(r,s) := e^{isL}\gamma(rt)$ yields a homotopy between the paths associated to T and to the circular tunnel $(\gamma(t), L)$. □

This easy result is actually quite useful. For instance it implies directly that altering the module of one (or several) points of the cyclic support of some $x \in X/W$ does not affect the cyclic label.

Let now x be in X/W and let $(x_1, \cdots, x_k)$ be its cyclic support. Suppose that we swap two consecutive points x_i and x_{i+1} of $\overline{\mathrm{LL}}(x)$, there are two natural ways to do so, one going "farther" than the other:



Let γ_i^+ and γ_i^- denote the first path and the second path, respectively.

Lemma 2.24. *Let $x \in X/W$, with cyclic support $(x_1, \ldots, x_k)$ and cyclic label $(s_1, \ldots, s_k)$. The cyclic labels of $\gamma_i^+(1)$ and $\gamma_i^-(1)$ are given by*

$$\mathrm{clbl}(\gamma_i^+(1)) = (s_1, \ldots, s_{i-1}, s_{i+1}, s_i^{s_{i+1}}, s_{i+2}, \ldots, s_k)$$
$$\mathrm{clbl}(\gamma_i^-(1)) = (s_1, \ldots, s_{i-1}, {}^{s_i}s_{i+1}, s_i, s_{i+2}, \ldots, s_k).$$

Proof. The concatenation of γ_i^- and γ_i^+ gives a homotopically trivial path from x to itself. In particular the assertion about $\mathrm{clbl}(\gamma_i^-(1))$ follows from that on $\mathrm{clbl}(\gamma_i^+(1))$. By construction of the cyclic label, we can assume that all the arguments of the points in $\overline{\mathrm{LL}}(x)$ are distincts. Let then θ be some angle strictly between the arguments of x_{i-1} and x_i. By [Bes15, Lemma 11.1], one can replace x by $e^{i\theta/h}x$ and consider $x' := \gamma_1^+(x)$.

Now if θ' denotes an angle strictly included between the arguments of x_2 and x_3, then the circular tunnels (x, θ') and (x', θ') represent the same element in $B(W)$ by the Hurwitz rule. That is the product of the first two terms of $\mathrm{clbl}(x)$ and $\mathrm{clbl}(x')$ are equal. The Hurwitz rule also shows that the terms of $\mathrm{clbl}(x)$ and $\mathrm{clbl}(x')$ are equal for $i > 2$.

Lastly, the assertion that the first two terms of $\mathrm{clbl}(x')$ are $(s_2, s_1^{s_2})$ follows from [Bes15, Lemma 11.11], using the notion of reduced label ([Bes15, Definition 7.14]) and a modified version of the Hurwitz rule ([Bes15, Lemma 6.15]). $\square$

By induction, we see that in general, moving a point of the cyclic support of some $x \in X/W$ may only affect the terms of the cyclic label corresponding to points of the support with lower module.

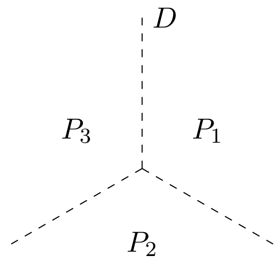
2.4. **Regular numbers and topological groupoids.** We can now move on to the definition of the Springer category associated to a regular number. We keep the definitions and notation of the last section. Let d be a regular number for W, let $g \in W$ be a ζ_d-regular element. Let also $W_g := C_W(g)$ be the centralizer in W of g. We set

$$p := \frac{d}{d \wedge h},\ q := \frac{h}{d \wedge h}$$

where $d \wedge h$ is the gcd of d and h. The categories we are going to consider are defined as fundamental groupoids with fat basepoints (cf. [Bes15, Appendix A]) having several contractible connected components. Define

$$D := \bigcup_{\zeta \in \mu_p} \zeta i \mathbb{R}_{\geqslant 0},$$
$$\mathcal{U}_p := \{x \in X/W \mid \overline{\mathrm{LL}}(x) \cap D = \varnothing\},$$
$$\mathcal{U}^{\mu_d} := \{x \in (X/W)^{\mu_d} \mid \overline{\mathrm{LL}}(x) \cap D = \varnothing\}.$$

The set D is composed of p half-lines starting at 0. It cuts the plane in p sectors $P_1, \ldots P_p$, labeled clockwise starting from the vertical half-line $i\mathbb{R}_+ \subset D$.
In the following example we have $p = 3$:



Lemma 2.25. *([Bes15, Lemma 11.22])*
Let $x \in \mathcal{U}_p$ with cyclic label $(c_1, \cdots, c_k)$. For $i \in [\![1, d']\!]$, define u_i as the product of the c_j corresponding to points inside the sector P_i. The **cyclic content** *of x is defined as $\mathrm{cc}_p(x) := (u_1, \ldots, u_p)$.*
The map cc_p induces a bijection between the connected components of $\mathcal{U}_p$ (resp. $\mathcal{U}^{\mu_p}$) and $D_p(c)$ (resp $D_p^q(c)$). Furthermore the connected components of $\mathcal{U}_p$ (resp. $\mathcal{U}^{\mu_d}$) are contractible.

Since the connected components of $\mathcal{U}_p$ and $\mathcal{U}^{\mu_d}$ are contractible, we can use them as fat basepoint for groupoids in the sense of [Bes15, Definition A.4].

Definition 2.26. *([Bes15, Definition 11.23])*
The **Springer category** *associated to W and d is the groupoid*

$$B_p^q(W) = \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d})$$

The functoriality of π_1 gives a natural functor $B_p^q(W) \to B(W)$.

Note that $B_p^q(W)$ is equivalent to the braid group of W_g (in particular, it is a connected groupoid). We now establish an isomorphism between the topological groupoid $B_p^q(W)$ and the category $M(c)_p^q$. Let $s := (a, b) \in D_{2p}^{2q}(c)$ be a morphism in $M(c)_p^q$. By Theorem 2.22, there is a unique element $x_s \in \mathcal{U}^{\mu_d}$ such that $\mathrm{clbl}(x) = s$ and $\overline{\mathrm{LL}}(x_s)$ consists of the points $e^{i\pi(\frac{1}{2} - \frac{2j+1}{2p})}$ such that the j-th term of s is nontrivial. The circular tunnel $(x_s, \frac{\pi}{d'h})$ then defines an element of $B_p^q(W)$ which we denote by b_s.

Theorem 2.27. *([Bes15, Theorem 11.28])*
The map $s \mapsto b_s$ extends to a groupoid isomorphism $\mathcal{G}(M(c)_p^q) \to B_p^q(W)$.

We have the following diagram of functors

$$\begin{tikzcd}
M(c)_p^q \arrow[r, "\pi_p"] \arrow[d, hook] & M(c) \arrow[d, hook] \\
\mathcal{G}(M(c)_p^q) \arrow[r] \arrow[d, "\simeq"] & G(c) \arrow[d, "\simeq"] \\
B_p^q(W) \arrow[r] & B(W)
\end{tikzcd}$$

As $\Delta^h \in M(c)$ represents the full-twist in $B(W)$, a d-th root of the full-twist is a (d, h)-periodic element of $M(c)$. Such an element is conjugate to a (p, q)-periodic element of $M(c)$ by Proposition 1.27. Let u be an object of $B_p^q(W)$, that is an object of $M(c)_p^q$ or an element of $D_p^q(c)$. By Theorem 1.40, the morphism $B_p^q(W) \to B(W)$ sends the automorphism group $B_p^q(W)(u, u)$ to the centralizer in $B(W)$ of some (p, q)-periodic element of $M(c)$, in particular a d-th root of.

Lemma 2.28. *Let u be an object of $B_p^q(W)$. The group isomorphisms*

$$B(W_g) \simeq B_p^q(W)(u, u) \simeq \mathcal{G}(M(c)_p^q)(u, u)$$

maps the full-twist in $B(W_g)$ to $\Delta^{ph}(u)$.

Proof. The morphism $\Delta_p(u)$ in $M(c)_p^q$ corresponds to a rotation of angle $\frac{2\pi}{ph}$, while the full-twist corresponds to a rotation of angle 2π. □

3. Properties of Springer categories

Theorem 2.27 gives a way to study the braid group of the centralizer of a regular element in a well-generated complex reflection group using Garside theory. This section is dedicated to the study of the categories of periodic elements associated to dual braid monoids, and to the results we deduce for the associated braid groups.

Let $W < \mathrm{GL}(V)$ be an irreducible well-generated complex reflection group of rank n, with highest degree h. Let d be a regular number for W. We set again

$$p := \frac{d}{d \wedge h},\ q := \frac{h}{d \wedge h}$$

We choose some Coxeter element c in W and we denote by $\mathcal{C}$ the category $M(c)_p^q$ of (p,q)-periodic elements of the dual braid monoid $M(c)$. The enveloping groupoid $\mathcal{B}$ of $\mathcal{C}$ is equivalent to the braid group of the centralizer of some ζ_d-regular element in W by Theorem 2.27.

Note again that the full-twist of $B(W)$ is equal to Δ^h, where h is the highest degree of W. The category of $(1,h)$-periodic elements is then simply the monoid $M(c)$ itself. Thus the content of this sections also applies to the dual braid monoid of a well-generated irreducible complex reflection group.

We denote by I_c^q the set of elements of I_c which are invariant under ϕ^q (that is, invariant under conjugation by c^q).

3.1. Elementary properties.

Recall from Definitions 1.31 and 1.33 that $\mathcal{C}$ depends on the sets $D_p^q(\Delta)$, $D_{2p}^{2q}(\Delta)$ and $D_{3p}^{3q}(\Delta)$. Thanks to Remark 2.13 and Proposition 1.30, we can replace these by

$$\begin{aligned}
\mathrm{Ob}(\mathcal{C}) &:= D_p^q(c) \simeq \left\{ u \in I_c^q \;\middle|\; \begin{cases} p\ell_R(u) = \ell_R(c) = n \\ u(u^{c^\eta}) \cdots (u^{c^{(p-1)\eta}}) = c \end{cases} \right\} \\
\mathcal{S}_p^q &:= D_{2p}^{2q}(c) = \left\{ (a,b) \in (I_c^q)^2 \mid ab \in \mathrm{Ob}(\mathcal{C}) \right\} \\
R_p^q &:= D_{3p}^{3q}(c) = \left\{ (x,y,z) \in (I_c^q)^3 \mid xyz \in \mathrm{Ob}(\mathcal{C}) \right\}
\end{aligned}$$

where η is the integer associated to (p,q) introduced in Lemma 1.28 and Remark 1.29.

Under these definitions, a simple morphism (a,b) in the graph of simples $\mathcal{S}_p^q$ goes from ab to ba^{c^η} in $\mathcal{C}$, and an element (x,y,z) of $D_{3p}^{3q}(c)$ induces the relation $(x,yz)(y,zx^{c^\eta}) = (xy,z)$ in R_p^q. The following lemma is an obvious consequence of the definition of $\mathrm{Ob}(\mathcal{C})$.

Lemma 3.1. *Let $u \in \mathrm{Ob}(\mathcal{C})$, the morphism $\Delta_p(u) = (u,1)$ has length $\ell_R(u) = \frac{n}{p}$. This length doesn't depend on u and every simple morphism in $\mathcal{C}$ has length at most $\frac{n}{p}$.*

Let $u \in D_p^q(c)$ be an object of $\mathcal{C}$. The collapse functor $\pi_p : \mathcal{C} \to M(c)$ sends $\Delta_p^q(u)$ to some (p,q)-periodic element in $M(c)$. We know that $\Delta_p(u) = (u,1)$ and $\Delta_p(\phi_p^n(u)) = (u^{c^{\eta n}},1)$. We then have

$$\begin{aligned}
\pi_p(\Delta_p^p(u)) &= \pi_p\Big((u,1)(u^{c^\eta},1)\cdots(u^{c^{(p-1)\eta}},1)\Big) \\
&= uu^{c^\eta}\cdots u^{c^{(p-1)\eta}} = c.
\end{aligned}$$

If $q := pk + r$ is the Euclidean division of q by p, then we get

$$\pi_p(\Delta^q(u)) = c^k u^{c^{pk\eta}} \cdots u^{c^{(pk+r-1)\eta}}.$$

We know from Theorem 1.40 that this is a (p,q)-regular element in $M(c)$. This gives an explicit formula for roots of the full-twist in $B(W)$, provided that one knows how to compute elements of the sets $D_m^n(c)$.

Lemma 3.2. *The atoms of $\mathcal{C}$ are exactly the simples $s = (a, b) \in \mathcal{S}_p^q(c)$ such that a admits no proper divisors in I_c^q. In particular if ϕ^q is trivial, a simple morphism of $\mathcal{C}$ is an atom if and only if it has length 1.*

Proof. The first statement follows directly from Lemma 1.38. If ϕ^q is trivial (that is, if Δ^q is central in $M(c)$), then Lemma 1.38 gives an isomorphism between $\mathcal{S}_p^q(u, -)$ and $[1, u]_\preceq$ where u is the source of s. Thus s is an atom if and only if a lies in R_c. Since by definition $\ell(s) = \ell(a)$, we get the desired result. □

Proposition 3.3. *If $p > 1$, then there are no pairs of parallel simples in $\mathcal{C}$ that is, a simple morphism is uniquely determined by its source and target. In particular, for $u \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{S}_p^q(u, u) = \{1_u\}$ and $\mathcal{S}_p^q(u, \phi_p(u)) = \{\Delta_p(u)\}$.*

Proof. Let $s := (a, b)$ be a simple morphism in $\mathcal{C}$, and let u, v denote the source and target of s, respectively. We have $u = ab$ and $v = ba^{c^\eta}$. We show that b is the left-gcd of u and v in I_c. The result is obvious when $a = 1$, so we assume $a \neq 1$. We have $ab = ba^b$, so b is an obvious left-divisor of ab and ba^{c^η}. If b is not the gcd of ab and ba^{c^η}, then there is a nontrivial common divisor d of a and a^{c^η}. Since $p > 1$, we have $\Delta = aba^{c^\eta}b^{c^\eta} \cdots (ab)^{c^{(p-1)\eta}}$ and aba^{c^η} is simple. Since all elements of I_c are balanced, we obtain that d^2 divides aba^{c^η}, thus $d^2 \in I_c$, which contradicts Lemma 2.11.

Now, let $(a', b') \in \mathcal{S}_p^q$ have the same source and target as (a, b). We have $u = ab = a'b'$ and $v = ba^{c^\eta} = b'a'^{c^\eta}$. We have $b \preceq b'$ and $b' \preceq b$, so $b = b'$ and $(a, b) = (a', b')$. □

In the case $p = 1$. There is only one object in $\mathcal{C}$, which corresponds to $\Delta \in D_1(\Delta)$. The proposition is then false in this case.

Lemma 3.4. *(Lifting words expressing simples)*
Let $s := (a, b)$ be a simple morphism in $\mathcal{C}$, and let $a_1 \cdots a_r$ be a word in I_c^q expressing a in $M(c)$. There is a unique path $s_1 \cdots s_r$ in $\mathcal{S}_p^q$ expressing s in $\mathcal{C}$ and such that $\pi_p(s_i) = a_i$ for all $i \in [\![1, r]\!]$

Proof. Let $u := ab$ be the source of s. We proceed by $\succeq$-induction on s. If s is an atom, then a is an atom of I_c^q by Lemma 3.2. The only word in I_c^q expressing a is then a itself and the result is trivial. Now for the general case, we have $a_1 x = a$ with $x = a_2 \cdots a_r$. By Lemma 1.38, $s_1 := (a_1, xb)$ is the only atom in $\mathcal{C}$ with source u and such that $\pi_p(s_1) = a_1$. By induction hypothesis, there is a unique path $s_2 \cdots s_r$ expressing $(x, ba_1^{c^\eta})$ in $\mathcal{C}$ and such that $\pi_p(s_i) = a_i$ for $i \in [\![2, r]\!]$. The path $s_1 s_2 \cdots s_r$ is then the unique path expressing s and such that $\pi_p(s_i) = a_i$ for $i \in [\![1, n]\!]$. □

Let $s := (a, b) \in \mathcal{S}_p^q$ be a simple morphism in $\mathcal{C}$. Its source is $ab = b\phi^\eta(a^{bc^{-\eta}})$ and its target is $b\phi^\eta(a) = a^{c^\eta b^{-1}}b$. By Lemma 1.38 and Lemma 1.10, we deduce the existence of the following simple morphisms in $\mathcal{C}$

$$s^\flat := (a^{bc^{-\eta}}, b) \text{ and } s^\# := (a^{c^\eta b^{-1}}, b).$$

Lemma 3.5. *Let $s := (a, b)$ be a simple morphism in $\mathcal{C}$.*
(a) The target of $s^\flat$ is the source of s, and the target of s is the source of $s^\#$.
(b) We have $(s^\flat)^\# = (s^\#)^\flat = s$.
(c) The paths $s^\flat s$ and $ss^\#$ are both in greedy normal form in $\mathcal{C}$.
(d) We have $\phi_p(s^\#) = (\phi_p(s))^\#$ and $\phi_p(s^\flat) = (\phi_p(s))^\flat$.

Proof. (a) The target of $s^\flat$ is $ba^b = ab$ and the source of $s^\#$ is $a^{c^\eta b^{-1}}b = ba^{c^\eta}$.
(b) We have $s^{\flat\#} = (a^{bc^{-\eta}}, b)^\# = (a^{bc^{-\eta}c^\eta b}, b) = (a, b) = s$. The same reasoning proves that $(s^\#)^\flat = s$.
(c) Since $s = (a, b)$ is a simple morphism, $ab = u$ is a simple element in $M(c)$. In particular a and b are coprime by Proposition 2.12. We get that the path $s^\flat s$ is greedy by Corollary

1.39. The path $ss^{\#} = (s^{\#})^{\flat} s^{\#}$ is also greedy by the same argument.
(d) We have

$$(\phi_p(s))^{\flat} = (a^{c^{\eta}}, b^{c^{\eta}})^{\flat} = (a^{c^{\eta} b^{c^{\eta}} c^{-\eta}}, b^{c^{\eta}}) = (a^{b}, b^{c^{\eta}}) = \phi_p(s^{\flat})$$

and

$$(\phi_p(s))^{\#} = (\phi_p(s^{\#\flat}))^{\#} = ((\phi_p(s^{\#}))^{\flat})^{\#} = \phi_p(s^{\#})$$

□

The transformation $s \mapsto s^{\#}$ is a bijection of the finite set $D_{2p}^{2q}(c)$: it has finite order. As the source of $s^{\#}$ is the target of s, there is then a smallest integer n such that $s^{(n\#)} = s$.

Definition 3.6. *Let $s : u \to v$ be a simple morphism in $\mathcal{C}$. The* **simple loop** *(of the object u) associated to s is the morphism*

$$\lambda(s) := ss^{\#}s^{\#\#} \cdots s^{\flat\flat}s^{\flat} \in \mathcal{C}(u,u)$$

If s is an atom of $\mathcal{C}$, then we say that $\lambda(s)$ is an **atomic loop**.

Lemma 3.7. *Let s be a simple morphism in $\mathcal{C}$. The simple loop $\lambda(s)$ is rigid, in particular it lies in its own super-summit set.*

Proof. By Lemma 3.5 (c), the greedy normal form of $\lambda(s)$ is given by

$$\lambda(s) := ss^{\#}s^{\#\#} \cdots s^{\flat\flat}s^{\flat}.$$

Since $s^{\flat}s$ is greedy, we get that $\lambda(s)$ is rigid. By Lemma 1.22, we have $\mathrm{cyc}(\lambda(s)) = \lambda(s^{\#})$ and $\mathrm{dec}(\lambda(s)) = \lambda(s^{\flat})$. In particular we see that cycling and decycling a simple loop doesn't change its inf or its sup. By Proposition 1.24, we deduce that $\lambda(s)$ lie in its own super-summit set. □

Remark 3.8. Let u be an object of $M(c)_p$. We have

$$\Delta_p(u)^{\#} = (u,1)^{\#} = (u^{c^{\eta}}, 1) = \phi_p(\Delta_p(u)) = \Delta_p(\phi_p(u))$$

So $\lambda(\Delta_p(u)) = \Delta_p^k(u)$ where k is the smallest integer such that $\phi_p^k(u) = u$. By definition of $\mathcal{C}$, we have that k divides q: there is some k' such that $kk' = q$. The element $(\Delta_p)^q(u)$ is then equal to $\lambda(\Delta_p(u))^{k'}$. By Theorem 1.40, the group $\mathcal{B}(u,u)$ then identifies with the centralizer in $G(c)$ of $\pi_p(\lambda(\Delta_p(u))^{k'})$.

3.2. **The case** $p = 1$**.** Exceptionally in this Section, we denote by $I_c(W)$ the interval associated to c in W for the reflection length. Let q be a positive integer. A $(1,q)$-periodic element in the dual braid monoid $M(c)$ is simply given by Δ^q. The associated category of $(1,q)$-periodic elements is then the submonoid $M(c)^{\phi^q}$ of $M(c)$. Its set of simple elements corresponds to the set $I_c(W)^{c^q}$ of simple elements which commute with c^q in W. We have by Theorem 2.27 that $M(c)^{\phi^q}$ is a Garside monoid for the braid group of the centralizer $W_{c^q} := C_W(c^q)$.

Lemma 3.9. *Let W be a well-generated irreducible group, and let q be a positive integer. The group $W_{c^q} = C_W(c^q)$ (acting on the eigenspace $V_{c^q} = \mathrm{Ker}(c^q - \zeta_h^q)$) is well-generated.*

Proof. By [Bes15, Theorem 2.4], we known that an irreducible complex reflection group is well-generated if and only if the sum of its i-th degree (in increasing order) with its i-th codegree (in decreasing order) is constant and equal to the highest degree. Let n be the rank of W, we denote the degrees (resp. codegrees) of W by $d_1, \ldots, d_n$ in increasing order (resp. $d_1^*, \ldots, d_n^*$ in decreasing order). Let d be the order of ζ_h^q. The integer d is regular for W and we know by Theorem 2.6 that the degrees (resp. codegrees) of W_{c^q} are precisely the d_i (resp. the d_i^*) that are divisible by d. Since d divides $h = d_i + d_i^*$ for all $i \in [\![1,n]\!]$, we have that d divides d_i if and only if it divides d_i^*. Thus, if the degrees of W_{c^q} are $d_{i_1}, \ldots, d_{i_k}$, then the codegrees of W_{c^q} are $d_{i_1}^*, \ldots, d_{i_k}^*$. In particular for $j \in [\![1,k]\!]$, we have $d_{i_j} + d_{i_j}^* = h = d_{i_k}$ and W_{c^q} is well-generated. □

We have $c \in W_{c^q}$ and, as $\operatorname{Ker}(c-\zeta_h) \subset \operatorname{Ker}(c^q-\zeta_h^q) = V_{c^q}$, it is a Coxeter element of W_{c^q}. Because of this lemma, we can consider the interval $I_c(W_{c^q})$ associated to c in W_{c^q} for the reflection length in W_{c^q}. This section is devoted to the proof of the following theorem.

Theorem 3.10. *Let W be a well-generated irreducible group with a Coxeter element c, and let q be a positive integer. The posets $I_c(W)^{c^q} = I_c(W_{c^q})$ are equal. That is, we have $I_c(W)^{c^q} = I_c(W_{c^q})$ as subsets of W_{c^q} and, for $s, t \in I_c(W_{c^q})$, we have $s \preceq t$ in W if and only if $s \preceq t$ in W_{c^q}, for the respective reflection length of W and W_{c^q}.*

Corollary 3.11. *Let W be a well-generated irreducible group with a Coxeter element c, and let q be a positive integer. The monoid $M(c)^{\phi^q}$ is isomorphic to the dual braid monoid $M'(c)$ associated to W_{c^q}.*

The corollary comes from the fact that the defining presentation of an interval monoid only depends on the poset structure of the associated interval $(I, \preceq)$: the defining relations are all relations of the form $s(s^{-1}t) = t$ for $s, t \in I$ with $s \preceq t$.

The proof of Theorem 3.10 is ultimately going to rely on a case by case analysis, but we can do some easy reductions.

Lemma 3.12. *We keep the notation of Theorem 3.10. If Theorem 3.10 holds when q divides the highest degree of W, then it holds for all values of q.*

Proof. Let h be the highest degree of W. Since h is the order of c, we have that the order d of c^q divides h. we set $q' := h/d$. The two elements c^q and $c^{q'}$ both have order d in the cyclic group $\langle c \rangle$ of order h. There are two integers a and b with $c^{aq} = c^{q'}$ and $c^{q'b} = c^q$. We then have $W_{c^q} = W_{c^{q'}}$, $I_c(W)^{c^q} = I_c(W)^{c^{q'}}$ and $M(c)^{\phi^q} = M(c)^{\phi^{q'}}$. Thus Theorem 3.10 holds for q if and only if it holds for q', which divides h. $\square$

From now on we assume that the integer q divides h, and we set $d = h/q$. The element c^q is $\zeta_d = \zeta_h^q$-regular. If d divides all the degrees of W, then c^q is central in W and we have $W_{c^q} = W$, Theorem 3.10 is obvious in this case. If h is the only degree of W divisible by d, then W_{c^q} is a complex reflection group of rank 1: it is cyclic and equal to $\langle c \rangle$. We have $I_c(W_{c^q}) = \{\mathrm{Id}, c\}$. On the other hand, any $s \in I_c^q$ lies in $W_{c^q} = \langle c \rangle$. We then have $I_c^q = \{\mathrm{Id}, c\}$ by [Bes15, Lemma 12.2]. The poset structure is induced by $\mathrm{Id} \preceq c$ in both cases and Theorem 3.10 holds.

Note that these two extreme cases are sufficient to prove Theorem 3.10 when W has rank 2. We now distinguish whether or not W belongs in the infinite series.

The case where W is a well-generated irreducible exceptional group is handled by computer.

If W is a well-generated irreducible group of rank $n \geqslant 3$ belonging to the infinite series then we have either

- $W \simeq \mathfrak{S}_{n+1}$ acting the the hyperplane H in $\mathbb{C}^{n+1}$ given by the equation $x_1 + \cdots + x_{n+1} = 0$. This is not the group $G(1,1,n+1)$, as the latter acts on a space of dimension $n+1$ and is not irreducible. However, the sets of reflections of these two groups are equal (they are the transpositions in $\mathfrak{S}_{n+1}$), and their sets of Coxeter elements are the same (they are the $n+1$-cycles in $\mathfrak{S}_{n+1}$). Thus the interval monoids given by W and $G(1,1,n+1)$ are the same and we can restrict our attention to the latter group.
- $W \simeq G(m,1,n)$ for $m \geqslant 2$.
- $W \simeq G(e,e,n)$ for $e \geqslant 2$ (this group is always irreducible when $n \geqslant 3$).

3.2.1. $W = G(1,1,n)$ *for* $n \geqslant 4$. Let ℓ_W denote the reflection length in W. Our approach is largely modeled on that of [BW02, Section 3 and Section 4], which covers the case where n is even and $q = \frac{n}{2}$.

Lemma 3.13. *Let $w = c_1 \cdots c_k$ be a product of disjoint cycles in W. We have*

$$\ell_W(w) = \sum_{i=1}^{k} \ell_W(c_i) = \sum_{i=1}^{k} (\ell(c_i) - 1)$$

where $\ell(c)$ denotes the size of the support of c.

Proof. The group W is a complexified real reflection group by [DDGKM, Lemma IX.2.19]. The reflection length $\ell_W(w)$ is then given by the codimension of the fixed space of w acting on $\mathbb{C}^n$. Since the c_i all have disjoint support, the first equality is immediate. For the second equality, let $c = (i_1 \ \cdots \ i_k)$ be a k cycle in W. The fixed space of c acting on $\mathbb{C}^n$ is given by $\{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_{i_1} = x_{i_2} = \cdots = x_{i_k}\}$ and has codimension $k - 1$. $\square$

The highest degree of W is n. Let q be a divisor of n, with $dq = n$. We label the canonical basis of $\mathbb{C}^n$ in the following way

$$\{e(0,1), e(0,2), \dots, e(0,q), e(1,1), \dots, e(d-1,q)\}$$

A guiding idea is to think of $e(j,i)$ as $\zeta_d^j e_i$ in the vector space $\mathbb{C}^q$ (with canonical basis $e_1, \dots, e_q$). We consider the following element of W

$$c(1,1,n) := (e(0,1) \ e(0,2) \ \cdots \ e(0,q) \ e(1,1) \ \cdots \ e(d-1,q))$$

It is a Coxeter element of W as the vector $\sum_{i=1}^{q} \sum_{j=0}^{d-1} \zeta_n^{n-i-qj} e(j,i)$ is a ζ_n-eigenvector for $c(1,1,n)$ which is regular. We have

$$c(1,1,n)^q = (e(0,1) \ e(1,1) \ \cdots \ e(d-1,1))(e(0,2) \ \cdots \ e(d-1,2)) \cdots (e(0,q) \ \cdots \ e(d-1,q))$$

Lemma 3.14. *The eigenspace $V := \operatorname{Ker}(c(1,1,n)^q - \zeta_d \operatorname{Id})$ has dimension q and admits a basis $v := \{v_1, \cdots, v_q\}$, where*

$$v_i := \sum_{j=0}^{d-1} \zeta_d^{d-j} e(j,i).$$

The isomorphism of vector spaces $V \simeq \mathbb{C}^q$ given by the basis v induces an isomorphism between $W_{c(1,1,n)^q}$ and $G(d,1,q)$.

Proof. The degrees of $W_{c(1,1,n)^q}$ acting on V are the degrees of $G(1,1,n)$ which are divided by d, that is $d, 2d, \dots, dq$. In particular, $W_{c(1,1,n)^q}$ has rank q and V has dimension q. As the v_i are clearly linearly independent, v is a basis for V.

The groups $W_{c(1,1,n)^q}$ and $G(d,1,q)$ share the same degrees (cf. [BMR98, Table 2]), and the same cardinality (since the order of a complex reflection group is the product of its degrees). We then only have to show that the image of $W_{c(1,1,n)^q}$ in $\mathrm{GL}_q(\mathbb{C})$ contains a generating set of $G(d,1,q)$, like the one given in [LT09, Section 2.7]. This is a direct check:

- The cycle $(e(0,1) \ e(1,1) \cdots e(d-1,1))$ acts trivially on v_j for $j \neq 1$ and sends v_1 to $\zeta_d v_1$.
- For $i \in [\![1, q-1]\!]$, the permutation $(e(0,i) \ e(0,i+1)) \cdots (e(d-1,i) \ e(d-1,i+1))$ swaps v_i and v_{i+1}.

$\square$

From now on, we set $W' = W_{c(1,1,n)^q}$ and $\ell_{W'}$ the reflection length for elements of W' regarding the reflections of W' acting on V. The action of $c(1,1,n)$ on V is given (in the basis v) by the matrix

$$c(d,1,q) := \begin{pmatrix} 0 & & & \zeta_d \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

which is indeed a Coxeter element for the group $G(d,1,q)$ (see Section 3.2.2). The group $G(d,1,q)$ is endowed with a character χ, sending a monomial matrix to the product of its nonzero entries. The isomorphism $W' \simeq G(d,1,q)$ of Lemma 3.14 allows us to define χ on W'.

Let $c := (e(j_1,i_1)\ \cdots\ e(j_k,i_k)) \in W$ be a cycle. We set

$$c^{(1)} := c^{c(1,1,n)^q} = (e(j_1-1,i_1)\ \cdots\ e(j_k-1,i_k)).$$

Proposition 3.15. *An element $\sigma \in W$ lies in W' if and only if can be written as decomposition as a product of disjoint cycles of the form*

$$\sigma = c_1c_1^{(1)}\cdots c_1^{(d-1)}c_2c_2^{(1)}\cdots c_2^{(d-1)}\cdots c_ac_a^{(1)}\cdots c_a^{(d-1)}\gamma_1\cdots\gamma_b$$

where $\gamma_i^{(1)} = \gamma_i$ for $i \in [\![1,b]\!]$.

Proof. Our proof is an adaptation of the proof of [BW02, Proposition 3.1], which deals with the case $d=2$. First, it is clear that elements of the given form lie in W'. Conversely, let $\sigma = c_1\cdots c_r$ be a product of disjoint cycles in $\mathfrak{S}_n$. We have that $c(1,1,n)^q$ centralizes σ if and only if $c_1\cdots c_r = c_1^{(1)}\cdots c_r^{(1)}$. By uniqueness (up to reordering) of cycle decompositions in $\mathfrak{S}_n$, for each i either $c_i = c_j^{(1)}$ for some $j \neq i$ or else $c_i = c_i^{(1)}$. An immediate induction then gives that σ has the required decomposition. □

We note in particular that the cycles $c_i\cdots c_i^{(d-1)}$ in the decomposition of Proposition 3.15 are disjoint.

Definition 3.16. *Let $c \in W$ be a cycle such that $c, c^{(1)},\ldots,c^{(d-1)}$ are disjoint. The product $cc^{(1)}\cdots c^{(d-1)}$ will be denoted by $\widetilde{c}$ and called a* **saturated cycle**. *If $\gamma \in W$ is a cycle lying in W', we say that γ is a* **balanced cycle**.

Proposition 3.15 states that elements of W' are the products of disjoint saturated cycles and disjoint balanced cycles.

Lemma 3.17. *Let $\widetilde{c}$ be a saturated cycle in W'. We have $\chi(\widetilde{c}) = 1$ and $\ell_{W'}(c) = \ell(c)-1$. Let γ be a balanced cycle in W'. We have $\chi(\gamma) = \zeta_d$ and $\ell_{W'}(\gamma) = \ell(\gamma)/d$.*

Proof. Let $c = (e(j_1,i_1)\ \cdots\ e(j_k,i_k))$ be a cycle such that $c,\ldots,c^{(d-1)}$ are all disjoint. By assumption, $i_1,\ldots,i_k$ are all distinct. One readily sees that $\widetilde{c}$ acts on V by

$$\begin{cases} \widetilde{c}.v_{i_m} = \zeta_d^{j_{m+1}-j_m}v_{i_{m+1}} & \forall m \in [\![1,k-1]\!], \\ \widetilde{c}.v_{i_k} = \zeta_d^{j_1-j_k}v_{i_1}. \end{cases}$$

In particular, we have $\chi(\widetilde{c}) = \zeta_d^{j_2-j_1}\cdots\zeta_d^{j_k-j_{k-1}}\zeta_d^{j_1-j_k} = 1$. The fixed space of $\widetilde{c}$ acting on V is generated by all the v_i with $i \notin \{i_1,\ldots,i_k\}$ and $v_{i_1}+\cdots+v_{i_k}$. Thus $\ell_{W'}(\widetilde{c}) = k-1 = \ell(c)-1$ by Lemma 3.14 and [Shi07, Theorem 2.1].

Let γ be a balanced cycle. It can be written as

$$\gamma = (e(0,i_1)\ e(j_2,i_2)\ \cdots\ e(j_k,i_k)\ e(1,i_1)\ \cdots\ e(j_k+d-1,i_k)).$$

The action of γ on V is then given by

$$\begin{cases} \gamma.v_{i_1} = \zeta_d^{j_2}v_{i_2}, \\ \gamma.v_{i_m} = \zeta_d^{j_{m+1}-j_m}v_{i_{m+1}} & \forall m \in [\![2,k-1]\!], \\ \gamma.v_{i_k} = \zeta_d^{1-j_k}v_1. \end{cases}$$

In particular we have $\chi(\gamma) = \zeta_d^{j_2}\cdots\zeta_d^{j_k-j_{k-1}}\zeta_d^{1-k} = \zeta_d$. The fixed space of γ acting on V is generated by all the v_i with $i \notin \{i_1,\ldots,i_k\}$. Thus $\ell_{W'}(\gamma) = k = \ell(\gamma)/d$ again by Lemma 3.14 and [Shi07, Theorem 2.1]. □

By combining Lemma 3.17 and Lemma 3.13, we obtain the following proposition.

Proposition 3.18. *Let $\sigma = \widetilde{c_1}\cdots\widetilde{c_a}\gamma_1\cdots\gamma_b \in W'$. The reflection length of σ in W and W' are given by*

$$\ell_W(\sigma) = \sum_{i=1}^{a} d(\ell(c_i)-1) + \sum_{j=1}^{b}(\ell(\gamma_i)-1) \text{ and } \ell_{W'}(\sigma) = \sum_{i=1}^{a}(\ell(c_i)-1) + \sum_{j=1}^{b}\frac{\ell(\gamma_i)}{d}.$$

In particular, we have $\ell_W(\sigma) + b = d\ell_{W'}(\sigma)$.

This relation between reflection lengths in W and in W' will be the key element in our proof of Theorem 3.10 for the case $W = G(1,1,n)$.

Lemma 3.19. *Let D be a nontrivial diagonal matrix in $G(d,1,q)$. We have $D \preceq c(d,1,q)$ in $G(d,1,q)$ if and only if D is a diagonal reflection s with $\chi(s) = \zeta_d$.*

Proof. By [Shi07, Theorem 2.1], the reflection length $\ell_{G(d,1,q)}(D)$ of D in $G(d,1,q)$ is the number of nontrivial diagonal entries of D. The underlying permutation of $D^{-1}c(d,1,q)$ is an n-cycle. Thus, again by [Shi07, Theorem 2.1], we have

$$\ell_{G(d,1,q)}(D^{-1}c(d,1,q)) = \begin{cases} n-1 & \text{if } \chi(D) = \chi(c(d,1,q)) = \zeta_d \\ n & \text{otherwise} \end{cases}$$

If D is non trivial (i.e if its reflection length is nonzero), we get $D \preceq c(d,1,q)$ if and only if $\chi(D) = \zeta_d$ and D has exactly one nontrivial diagonal entry. □

Proposition 3.20. *Let $\sigma = \widetilde{c_1}\cdots\widetilde{c_a}\gamma_1\cdots\gamma_b \in W'$. If $\sigma \in I_{c(1,1,n)}(W')$ or if $\sigma \in I_{c(1,1,n)}(W)$, then $b \in \{0,1\}$ and $\chi(\sigma) = \zeta_d^b$.*

Proof. Let $\gamma = (e(0,i_1)\ e(j_2,i_2)\ \cdots\ e(j_k,i_k)\ e(1,i_1)\ \cdots\ e(j_k+d-1,i_k))$ be a balanced cycle. The diagonal reflection of W' sending v_i to $\zeta_d v_i$ is given by $s_i := (e(0,i)\cdots e(d-1,i))$. We have that $s_{i_1}^{-1}\gamma = (e(0,i_1)\cdots e(j_k,i_k))\cdots(e(d-1,i_1)\cdots e(j_k+d-1,i_k))$ is a saturated cycle. Proposition 3.18 then gives that $\ell_{W'}(s_{i_1}) + \ell_{W'}(s_{i_1}^{-1}\gamma) = 1+k-1 = k = \ell_{W'}(\gamma)$ and $s_{i_1} \preceq \gamma$ in W'.

Assume that $\sigma \in I_{c(1,1,n)}(W')$ and $b > 1$. Since γ_1 and γ_2 have disjoint support, we deduce that there are two diagonal reflections s, s' of W' such that $s \preceq \gamma_1$, $s' \preceq \gamma_2$, $ss' \neq \mathrm{Id}$ and $\chi(ss') = \zeta_d^2$. We then have $ss' \preceq c(1,1,n)$ in W' which contradicts Lemma 3.19.

Assume now that $\sigma \in I_{c(1,1,n)}(W)$. By [DDGKM, Proposition IX.2.7], the set theoretic partition of the set μ_n induced by σ is noncrossing (in the sense of [BC06, Section 1.2]). If $b > 1$, then the orbits of γ_1 and γ_2 induce two parts of μ_n whose convex hull contain 0. Thus the partition of μ_n induced by σ is crossing and $\sigma \notin I_{c(1,1,n)}(W)$.

The assumption on $\chi(\sigma)$ is a direct consequence of Lemma 3.17. □

Proposition 3.21. *The two posets $I_{c(1,1,n)}(W)^{c(1,1,n)^q}$ and $I_{c(1,1,n)}(W')$ are equal.*

Proof. First, we show that the two sets $I_{c(1,1,n)}(W)^{c(1,1,n)^q}$ and $I_{c(1,1,n)}(W')$ are equal. Let $\sigma \in I_{c(1,1,n)}(W')$. By Proposition 3.20, we have $\chi(\sigma) \in \{1, \zeta_d\}$ and $\chi(\sigma^{-1}c(1,1,n)) = \chi(\sigma)^{-1}\zeta_d$. Proposition 3.18 then gives

$$\begin{aligned} \ell_W(\sigma) + \ell_W(\sigma^{-1}c(1,1,n)) &= d\ell_{W_{c(1,1,n)^q}}(\sigma) + d\ell_{W_{c(1,1,n)^q}}(\sigma^{-1}c(1,1,n)) - 1 \\ &= d(\ell_{W_{c(1,1,n)^q}}(\sigma) + \ell_{W_{c(1,1,n)^q}}(\sigma^{-1}c(1,1,n))) - 1 \\ &= d\ell_{W_{c(1,1,n)^q}}(c(1,1,n)^q) - 1 \\ &= \ell_W(c(1,1,n)) \end{aligned}$$

and $\sigma \in I_{c(1,1,n)}(W)$. The same reasoning gives $I_{c(1,1,n)}(W)^{c(1,1,n)^q} \subset I_{c(1,1,n)}(W')$.

Let now $\sigma, \tau \in I_{c(1,1,n)}(W')$, we have to show that $\sigma \preceq \tau$ in W' if and only if $\sigma \preceq \tau$ in W. We consider four cases

- $\chi(\sigma)=1$ and $\chi(\tau)=1$. We have $\ell_W(\sigma)=d\ell_{W'}(\sigma)$ and $\ell_W(\tau)=d\ell_{W'}(\tau)$. If $\sigma\preceq\tau$ in W (resp. in W'), then $\sigma^{-1}\tau\in I_{c(1,1,n)}(W')$ is such that $\chi(\sigma^{-1}\tau)=1$. We then have $\ell_W(\sigma^{-1}\tau)=d\ell_{W'}(\sigma^{-1}\tau)$ and $\sigma\preceq\tau$ in W' (resp. in W).
- $\chi(\sigma)=\zeta_d$ and $\chi(\tau)=1$. We cannot have $\sigma\preceq\tau$ in either W or W' since this would imply that $\sigma^{-1}\tau$ is an element of $I_{c(1,1,n)}(W')$ with $\chi(\sigma^{-1}\tau)=\zeta_d^{-1}$ (we can assume $\zeta_d\neq -1$ since the case $d=2$ is known by [BW02, Lemma 4.8]).
- $\chi(\sigma)=1$ and $\chi(\tau)=\zeta_d$. We have $\ell_W(\sigma)=d\ell_{W'}(\sigma)$ and $\ell_W(\tau)+1=d\ell_{W'}(\tau)$. If $\sigma\preceq\tau$ in W (resp. in W'), then $\sigma^{-1}\tau\in I_{c(1,1,n)}(W')$ is such that $\chi(\sigma^{-1}\tau)=\zeta_d$. We then have $\ell_W(\sigma^{-1}\tau)+1=d\ell_{W'}(\sigma^{-1}\tau)$ and $\sigma\preceq\tau$ in W' (resp. in W).
- $\chi(\sigma)=\zeta_d$ and $\chi(\tau)=\zeta_d$. We have $\ell_W(\sigma)+1=d\ell_{W'}(\sigma)$ and $\ell_W(\tau)+1=d\ell_{W'}(\tau)$. If $\sigma\preceq\tau$ in W (resp. in W'), then $\sigma^{-1}\tau\in I_{c(1,1,n)}(W')$ is such that $\chi(\sigma^{-1}\tau)=1$. We then have $\ell_W(\sigma^{-1}\tau)=d\ell_{W'}(\sigma^{-1}\tau)$ and $\sigma\preceq\tau$ in W' (resp. in W).

□

This closes the proof of Theorem 3.10 in this case. We have the following corollary, which substantiates [BC06, Remark at the end of Section 7], for which we could not find a proof in the literature.

Corollary 3.22. *The map from $G(d,1,q)$ to the set of partitions of the regular n-gon that sends every element of $G(d,1,q)$ to the partition given by the orbits of its image in $\mathfrak{S}_n$ induces a poset isomorphism from $I_{c(d,1,q)}(G(d,1,q))$ to $\mathrm{NCP}(d,1,q):=\mathrm{NCP}(1,1,n)^{\mu_d}$, where $\mathrm{NCP}(1,1,n)$ denotes the set of noncrossing partitions of the regular n-gon (in the sense of [BC06, Section 1.2]).*

Proof. The embedding $G(d,1,q)\to G(1,1,n)$ as the centralizer of $c(1,1,n)^q$ identifies the posets $I_{c(d,1,q)}(G(d,1,q))$ and $I_{c(1,1,n)}(W')=I_{c(1,1,n)}(W)^{c(1,1,q)}$. On the other hand, the map from $G(1,1,n)$ to the set of partitions of the regular n-gon that sends every element of $G(1,1,n)$ to the partition given by its orbits induces a poset isomorphism from $I_{c(1,1,n)}(G(1,1,n))$ to $\mathrm{NCP}(1,1,n)$ ([DDGKM, Proposition IX.2.7]).
Under this isomorphism, the action of $c(1,1,n)$ by conjugation corresponds to a rotation of angle $\frac{2\pi}{n}$. Thus the isomorphism $I_{c(1,1,n)}(W)\simeq\mathrm{NCP}(1,1,n)$ induces an isomorphism between $I_{c(1,1,n)}(W)^{c(1,1,n)^q}$ and $\mathrm{NCP}(1,1,n)^{\mu_d}$ as claimed. □

3.2.2. *$W=G(m,1,n)$ for $m\geqslant 2$.* The highest degree of W is mn. We endow W with the Coxeter element

$$c(m,1,n)=\begin{pmatrix}0&&&&\zeta_m\\1&0&&&\\&1&\ddots&&\\&&\ddots&0&\\&&&1&0\end{pmatrix}$$

A ζ_{mn}-eigenvector for $c(m,1,n)$ which is regular is given by $\zeta_{mn}^{n-1}e_1+\zeta_{mn}^{n-2}e_2+\cdots+e_n$.

We have seen in Section 3.2.1 that $G(m,1,n)$ can be realized as the centralizer in $W':=G(1,1,mn)$ of $c(1,1,mn)^m$, and this identification sends $c(m,1,n)$ to $c(1,1,mn)$. Let q be a divisor of mn. We have

$$W_{c(m,1,n)^q}\simeq (W_{c(1,1,mn)^m})_{c(1,1,mn)^q}=W'_{c(1,1,mn)^{q\vee m}}$$

This identification induces the following isomorphisms of posets

$$I_{c(m,1,n)}(W)^{c(m,1,n)^q}\simeq (I_{c(1,1,mn)}(W'_{c(1,1,mn)^m}))^{c(1,1,mn)^q}$$

$$I_{c(m,1,n)}(W_{c(m,1,n)^q})\simeq I_{c(1,1,mn)}(W'_{c(1,1,mn)^{q\vee m}})$$

By Section 3.2.1, we have equalities of posets

$$\begin{aligned}(I_{c(1,1,mn)}(W'_{c(1,1,mn)^m}))^{c(1,1,mn)^q} &= (I_{c(1,1,mn)}(W')^{c(1,1,mn)^m})^{c(1,1,mn)^q} \\ &= I_{c(1,1,mn)}(W')^{c(1,1,mn)^{m\vee q}} \\ &= I_{c(1,1,mn)}(W'_{c(1,1,mn)^{q\vee m}})\end{aligned}$$

which gives the desired results.

3.2.3. $W = G(e,e,n)$ *for* $e \geqslant 2$, $n \geqslant 3$. The highest degree of W is $e(n-1)$. We start by considering the group $W' := G(e,1,n-1)$. The character $\chi : W' \to \mathbb{C}^*$ giving the product of the nonzero entries allows us to define an embedding

$$\begin{array}{rccc} i: & W' & \longrightarrow & W \\ & M & \longmapsto & \begin{pmatrix} \chi(M)^{-1} & 0 \\ 0 & M \end{pmatrix} \end{array}$$

The element $c(e,e,n) = i(c(e,1,n-1))$ is a Coxeter element for W. For instance $\zeta_{e(n-1)}^{n-2}e_2 + \cdots + \zeta_{e(n-1)}e_{n-1} + e_n$ is a $\zeta_{e(n-1)}$-eigenvector for $c(e,e,n)$ which is regular. Let q be a divisor of $e(n-1)$. We have

$$c(e,e,n)^q = \begin{pmatrix} \zeta_e^{-q} & 0 \\ 0 & c(e,1,n-1)^q \end{pmatrix}$$

We can compute directly the centralizer of this element in W.

Proposition 3.23. *Let q be a positive integer dividing $e(n-1)$.*
(a) If $\frac{e(n-1)}{e\wedge n}$ divides q, then $W = W_{c(e,e,n)^q}$. In particular Theorem 3.10 holds.
(b) If $\frac{e(n-1)}{e\wedge n}$ does not divide q, then i induces an isomorphism between $W'_{c(e,1,n-1)^q}$ and $W_{c(e,e,n)^q}$.

Proof. (a) Assume that $\frac{e(n-1)}{e\wedge n}j = q$ for some integer j. Since $dq = e(n-1)$, we obtain that $d = \frac{e\wedge n}{j}$ divides $e \wedge n$, which is the gcd of the degrees of W (cf. [BMR98, Table 2]). The degrees of $W_{c(e,e,n)}$ are then the same as that of W.
(b) We denote d the integer such that $dq = e(n-1)$. Let $w \in W$. We write w as a block matrix.

$$w = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

with $X \in \mathcal{M}_1(\mathbb{C}), Y \in \mathcal{M}_{1,n-1}(\mathbb{C}), Z \in \mathcal{M}_{n-1,1}(\mathbb{C})$ and $T \in \mathcal{M}_{n-1,n-1}(\mathbb{C})$. We find

$$w \in W_{c(e,e,n)^q} \Leftrightarrow \begin{cases} c(e,1,n-1)^q Z = \zeta_e^{-q} Z \\ Y c(e,1,n-1)^q = \zeta_e^{-q} Y \\ T c(e,1,n-1)^q = c(e,1,n-1)^q T \end{cases}$$

as w is a monomial matrix, Z and Y have at most one nonzero coefficient. Suppose that $Z \neq 0$, then $c(e,1,n-1)^q$ has a diagonal coefficient equal to ζ_d^{-q}. The underlying permutation of $c(e,1,n-1)$ is a $(n-1)$-cycle. Thus $c(e,1,n-1)^q$ has a nonzero diagonal coefficient if and only if $(n-1)$ divides q.

Let j be an integer such that $j(n-1) = q$. Since $dq = e(n-1)$, we have $dj = e$. We also have $c(e,1,n-1)^q = (c(e,1,n-1)^{n-1})^j = \zeta_e^j \operatorname{Id}$. Thus $c(e,1,n-1)^q Z = \zeta_e^{-q} Z$ if and only if $\zeta_e^j = \zeta_e^{-q}$, that is, $j = -q \bmod e$. Since $j(n-1) = q$, we obtain that there is some integer k with $jn = kjd$. Thus $kd = n$ and d divides n. We have already shown that d divides e, we obtain that d divides $e \wedge n$ and that $\frac{e(n-1)}{e\wedge n}$ divides q.

The same reasoning shows that $Y \neq 0$ implies that $\frac{e(n-1)}{e\wedge n}$ divides q. We obtain $Y = 0$, $Z = 0$, and $w \in W_{c(e,e,n)^q}$ if and only if $w = i(T)$ with $T \in W'_{c(e,1,n-1)^q}$. $\square$

From now on, we suppose that $\frac{e(n-1)}{e\wedge n}$ does not divide q. The isomorphism between $W_{c(e,e,n)^q}$ and $W'_{c(e,1,n-1)^q}$ induced by i induces in turn an isomorphism of posets

$$I_{c(e,1,n-1)}(W'_{c(e,1,n-1)^q}) \simeq I_{c(e,e,n)}(W_{c(e,e,n)^q}).$$

On the other hand, we know from Section 3.2.2 that the two posets

$$I_{c(e,1,n-1)}(W)^{c(e,1,n-1)^q} \text{ and } I_{c(e,1,n-1)}(W'_{c(e,1,n-1)^q})$$

are equal. To show that Theorem 3.10 holds in this case, it only remains to show that the morphism i induces an isomorphism of posets between $I_{c(e,1,n-1)}(W')^{c(e,1,n-1)^q}$ and $I_{c(e,e,n)}(W)^{c(e,e,n)^q}$. By [BC06, Remark after Lemma 1.22], $c(e,e,n)$ induces a free action (of a cyclic group of order $\frac{e(n-1)}{e\wedge n}$) on the set $I_{c(e,e,n)}(W) \setminus i(I_{c(e,1,n-1)}(W'))$. Since $\frac{e(n-1)}{e\wedge n}$ does not divide q by assumption, we get

$$I_{c(e,e,n)}(W)^{c(e,e,n)^q} = i(I_{c(e,1,n-1)}(W'))^{c(e,e,n)^q} = i\left(I_{c(e,1,n-1)}(W')^{c(e,1,n-1)^q}\right)$$

which finishes the proof.

3.3. **Presentation by Hurwitz relations.** As a category of periodic elements, $\mathcal{C}$ has been defined by the categorical presentation $\langle \mathcal{S}_p^q \mid R_p^q\rangle^+$. In practice, we want a presentation with less generators, for instance with only the atoms of $\mathcal{C}$ as generators. Such a result is known in the case of the dual braid monoid $M(c)$, which is presented by its atoms, endowed with the Hurwitz relations (see Remark 2.10).

Let $\mathcal{A}$ be the set of atoms of $\mathcal{C}$. Recall from Lemma 3.2 that $\mathcal{A}$ is made of the simples (a,b) such that a admits no proper ϕ^q-invariant left-divisor in $M(c)$.

Lemma 3.24. *Let $s := (a,b)$ and $t := (d,e)$ be two composable atoms in $\mathcal{C}$. The product st is either its own greedy normal form, or it is equal to the simple $r = (ad, d^{-1}b)$.*

Proof. If st is not a greedy path, then the left-gcd of t and $\overline{s}$ is nontrivial. Since t is an atom, we then get that $t \preceq \overline{s}$, i.e $d \preceq b$ in $M(c)$. In which case $st = (ad, d^{-1}b)$ is a simple morphism. ☐

Lemma 3.25. *Consider a square of atoms in $\mathcal{C}$*

$$\begin{tikzcd} u \arrow[r, "s"] \arrow[d, "\sigma"'] & v \arrow[d, "t"] \\ v' \arrow[r, "\tau"'] & w \end{tikzcd}$$

The square is commutative if and only if one of the following conditions is met:
(1) $s = \sigma$ and $t = \tau$.
(2) $\pi_p(\sigma\tau) = \pi_p(st) \in I_c$. In this case, $st = \sigma\tau$ is a simple morphism.

Proof. Let us denote $s = (a,b), t = (d,e), \sigma = (\alpha,\beta)$ and $\tau = (\delta,\varepsilon)$.
Suppose that we have $st = \sigma\tau$ and $s \neq \sigma$. Since both s and σ are atoms, we have $s \not\preceq \sigma$ and $\sigma \not\preceq s$: the paths $st, \sigma\tau$ are not in greedy normal form. By Lemma 3.24, $\sigma\tau = st$ is a simple morphism, and we have $\pi_p(\sigma\tau) = \pi_p(st) \in I_c$.

Conversely, assume that $\pi_p(\sigma\tau) = \alpha\delta = ad = \pi_p(st)$ lies in I_c. If $p = 1$ then π_p is injective by Section 1.3.1 and $\sigma\tau = st$, thus we can suppose that $p > 1$. Let $\alpha x = ay$ be the right-lcm of α and a in I_c. Let also $\psi(x)$ and $\psi(y)$ be the elements introduced in Lemma 1.11. We have $\alpha x \preceq \alpha d$ and $x \preceq d$. By Lemma 1.11, we have $\psi(x) \preceq d$, thus $\psi(x) = d$ by Lemma 3.2. The same argument gives $\psi(y) = \delta$. Since s and σ share the same source, we have $ab = \alpha\beta$. We deduce $x \preceq b, y \preceq \beta$, and $d \preceq b, \delta \preceq \beta$ again by Lemma 1.11. This proves that st and $\sigma\tau$ are simple morphisms with the same source and target: they are equal by Proposition 3.3. ☐

Definition 3.26. *We call* **Hurwitz relations** *on $\mathcal{C}$ the relations of the form $st = \tau\sigma$, where s, t, τ, σ are atoms in $\mathcal{C}$, and $\pi_p(\sigma\tau) = \pi_p(st) \in I_c$. We call $\mathcal{H}$ the set of Hurwitz relations in $\mathcal{C}$.*

Theorem 3.27. *The category $\mathcal{C}$ is presented by the atoms and the Hurwitz relations, that is $\mathcal{C} = \langle \mathcal{A} \mid \mathcal{H} \rangle^+$.*

Proof. If $p = 1$, this is already known from Corollary 3.11, since dual braid monoids are presented by their atoms, endowed with Hurwitz relations (cf. [Bes15, Lemma 8.8]). Suppose now that $p > 1$. We know that $\mathcal{C}$ is generated by its atoms, and that the defining relations of $\mathcal{C}$ imply the Hurwitz relations. It remains to show that the defining relations of $\mathcal{C}$ are implied by the Hurwitz relations.

Let $st = u$ be a defining relation of $\mathcal{C}$. We consider three paths of atoms in $\mathcal{C}$

$$s_1 \cdots s_r,\ t_1 \cdots t_k,\ u_1 \cdots u_m$$

expressing s, t and u, respectively. We set $a_i = \pi_p(s_i)$ for $i \in [\![1, r]\!]$, $b_i := \pi_p(t_i)$ for $i \in [\![1, k]\!]$ and $\alpha_i := \pi_p(u_i)$ for $i \in [\![1, m]\!]$. In $M(c)^{\phi^q}$, the two words

$$a_1 \ldots a_r b_1 \cdots b_k \text{ and } \alpha_1 \cdots \alpha_m$$

express the same element $\pi_p(u)$. Since $M(c)^{\phi^q}$ is presented by the Hurwitz relations (this is the case $p = 1$), there is a sequence of words $\mu_1, \ldots, \mu_n$ in the atoms of $M(c)^{\phi^q}$ such that

- $\mu_1 = a_1 \cdots a_r b_1 \cdots b_r$
- $\mu_n = \alpha_1 \cdots \alpha_m$
- For $i \in [\![1, n-1]\!]$, μ_i and μ_{i+1} are related by a Hurwitz relation in $M(c)^{\phi^q}$.

In particular, each of the μ_i expresses the element $\pi_p(u)$ in $M(c)^{\phi^q}$. By Lemma 3.4, each μ_i admits a unique lift p_i in $\mathcal{C}$, and each of the p_i expresses u in $\mathcal{C}$. By Lemma 3.25, the paths p_i and p_{i+1} are related by a Hurwitz relation in $\mathcal{C}$ for $i \in [\![1, n-1]\!]$. In particular the equality $st = u$ is implied by the Hurwitz relations in $\mathcal{C}$. □

This presentation can be seen as an analogue of Remark 2.10 for categories of periodic elements of dual braid monoids. This new presentation will be useful for computational purposes in Section 4.

3.4. **Braided reflections and atomic loops.** In this section we give a description of braided reflections in $B(W_g)$ in terms of atomic loops in the category $\mathcal{C}$. Namely we prove that a braided reflection is conjugate to an atomic loop. We keep the same notation as in the last sections.

Since the homeomorphism between X_g/W_g and $(X/W)^{\mu_d}$ doesn't depend on a choice, we can identify the two spaces. The choice of a basepoint $x \in \mathcal{U}^{\mu_d}$ induces an isomorphism between $B(W_g) \simeq \pi_1(X_g/W_g, x)$ and $\mathcal{B}(u, u)$, where u is the connected component of x in $\mathcal{U}^{\mu_d}$. By Remark 2.3, a change of basepoint preserves the set of braided reflections.

We explicit the definition of braided reflections in the context of the topological tools of Section 2.3 and 2.4.

Proposition 3.28. *([Bes15, Definition 11.12 and Proposition 11.13])*
The bijection $(\overline{\mathrm{LL}}, \mathrm{clbl})$ of Theorem 2.22 restricts to a bijection between $(X/W)^{\mu_d}$ and the set $(E_n^\circ)^{\mu_p} \boxdot \mathcal{S}_p^q$ of compatible pairs where the first term is μ_p-invariant, and the second term belongs in $\mathcal{S}_p^q$.

Following this proposition, we can now see the cyclic label of some $x \in (X/W)^{\mu_d}$ as an element of $\mathcal{S}_p^q$, with the convention introduced in Section 3.1.

Let $a \in (V/W)^{\mu_d}$ be on the smooth part of some irreducible divisor of the discriminant hypersurface in $(V/W)^{\mu_d}$. For $\zeta \in \mathbb{C}^*$, the point ζa lies on the smooth part of the same irreducible divisor as a. In particular, we can replace a by some ζa for $\zeta \in \mathbb{S}^1$ so that no point of $\overline{\mathrm{LL}}(a)$ lies on D. From now on we assume that this condition is met.

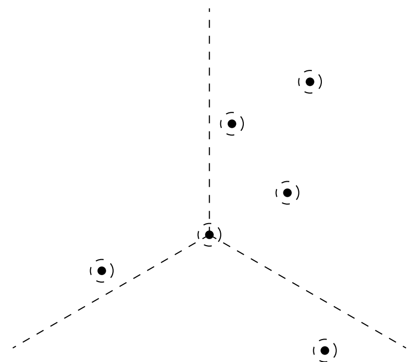
Since $\overline{\mathrm{LL}}$ is a (ramified) covering ([Bes15, Theorem 5.3]), it is an open map. The choice of a neighborhood of a then induces a neighborhood of $\overline{\mathrm{LL}}(a)$ in E_n that is, a neighborhood in $\mathbb{C}$ of each point of the cyclic support of $\overline{\mathrm{LL}}(a)$.

- If $x \in \mathbb{C}^*$ is in the support of $\overline{\mathrm{LL}}(a)$, the associated neighborhood is called an ***outer neighborhood***.
- If a lies on the discriminant hypersurface, that is if $0 \in \overline{\mathrm{LL}}(a)$, we call ***central neighborhood*** the neighborhood of $0 \in \overline{\mathrm{LL}}(a)$.

Definition 3.29. *Let $a \in V/W$ be a point such that $\overline{\mathrm{LL}}(a) \cap D = \varnothing$. We we say that a neighborhood of a in V/W is* **confining** *if the induced neighborhood of $\overline{\mathrm{LL}}(a)$ satisfies the following conditions*

(1) The outer neighborhoods do not overlap in module the central neighborhood.
(2) The outer neighborhoods never contain points in D.

It is easy to see that any confining neighborhood are stable under intersection and that they make up a basis of neighborhoods of a in V/W. The goal of a confining neighborhood is to "isolate" the central neighborhood, so that the exterior points of $\overline{\mathrm{LL}}(a)$ cannot interfere with the points in the central neighborhood, as seen in the example below

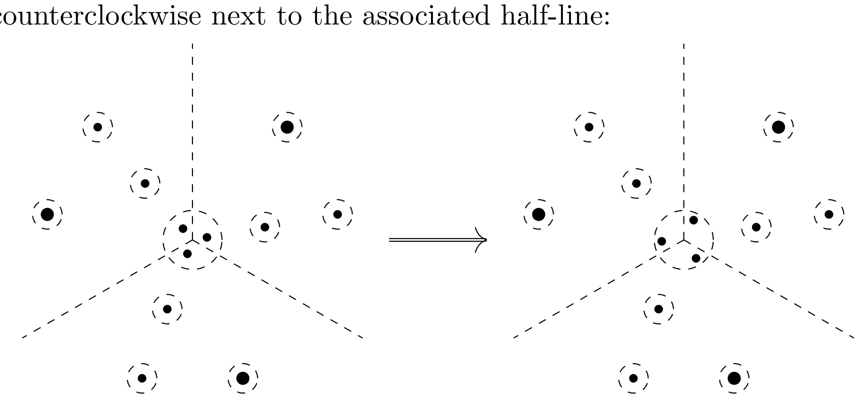


Let now $a \in (V/W)^{\mu_d}$ be such that no point of $\overline{\mathrm{LL}}(a)$ lies on D, and let U be a confining neighborhood of a in V/W. The set $U \cap (V/W)^{\mu_d}$ is a neighborhood of a in $(V/W)^{\mu_d}$. In particular we have that $U \cap \mathcal{U}^{\mu_d}$ is nonempty since $\mathcal{U}^{\mu_d}$ is dense in $(V/W)^{\mu_d}$.

Lemma 3.30. *Let $a \in (V/W)^{\mu_d}$ be such that no point of $\overline{\mathrm{LL}}(a)$ lies on D, and let U be a confining neighborhood of a in V/W. Let also $x \in U \cap \mathcal{U}^{\mu_d}$ and let u be the connected component of x in $\mathcal{U}^{\mu_d}$. There is some $s = s(x) \in \mathcal{S}_p^q(u, -)$ such that the image of $\pi_1(U \cap (X/W)^{\mu_d}, x)$ in $\mathcal{B}(u,u)$ contains all the simple loops $\lambda(t)$ with $t \preceq s$ in $\mathcal{C}$.*

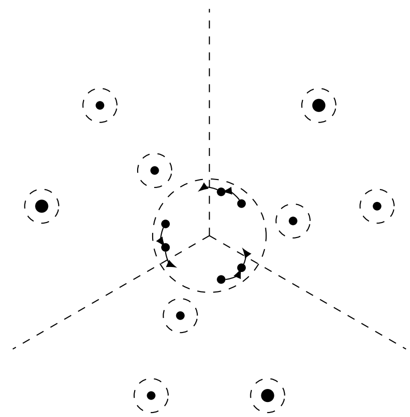
Proof. We denote by $\beta = \beta(x)$ the product (in clockwise order) of the terms of $\mathrm{clbl}(x)$ corresponding to the outer points in the sector P_1. Let γ be a path in $U \cap (X/W)^{\mu_d}$ starting from x. For all $t \in [0,1]$ such that $\gamma(t) \in \mathcal{U}^{\mu_d}$, we have $\beta(\gamma(t)) = \beta(x)$ by Lemma 2.24 and the following discussion.

We consider the path γ which consists in sliding the central points in each sector together and then counterclockwise next to the associated half-line:



This standardization motion is homotopically trivial and only impacts the cyclic label. Let $\alpha = \alpha(x)$ be the first term of the cyclic label of $\gamma(1)$. The cyclic label of $\gamma(1)$ is given by $(\alpha, s_1, \ldots, s_k)$ with $s_1 \cdots s_k = b$. We associate to x the well defined element $s = s(x) := (\alpha, \beta) \in \mathcal{S}_p^q$ (α is also defined by the equality $u = \alpha\beta$).

Let $t = (d, e)$ be a left-divisor of s in $\mathcal{C}$. There is some $m \in \mathcal{S}^{\phi^q}$ such that $dm = \alpha$ and $m\beta = e$. We can desingularize $\gamma(1)$ so that we obtain a point $x' \in U \cap \mathcal{U}^{\mu_d}$ with cyclic label $(d, m, s_1, \ldots, s_k)$. The path in $U \cap (X/W)^{\mu_d}$ starting from x' and consisting in rotating the central points to the left, so that the first point of the cyclic support goes into the sector P_p, represents the simple morphism t in $\mathcal{B}$.



Let x_1 denote the endpoint of this motion. The cyclic label of x_1 is $(m, s_1, \ldots, s_k, d^{c^\eta})$. By applying a standardization motion γ_1, we get a point $\gamma_1(1)$ with cyclic label

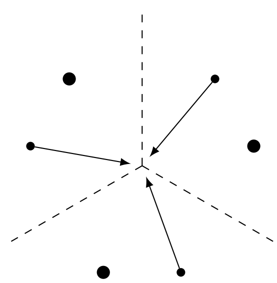
$$(d^{c^\eta m s_1 \cdots s_k}, m, s_1, \ldots, s_k) = (d^{c^\eta e}, m, s_1, \ldots, s_k)$$

We see that the circular tunnel associated to $\gamma(1)$ represents $t^\#$ in $\mathcal{C}$. By an immediate induction, we obtain that $\lambda(t)$ lies in the image of $\pi_1(U \cap (X/W)^{\mu_d}, x)$ in $\mathcal{B}(u, u)$. $\square$

Theorem 3.31. *Let u be an object of $\mathcal{C}$, and let $s \in \mathcal{C}(u, -)$ be an atom. The atomic loop $\lambda(s)$ is a braided reflection in the group $\mathcal{B}(u, u) \simeq B(W_g)$. Conversely, any braided reflection in $B(W_g) \simeq \mathcal{B}(u, u)$ is conjugate in $\mathcal{B}$ to some atomic loop.*

Proof. First, if $p = 1$, then $\mathcal{B} = M(c)^{\phi^q}$ is isomorphic to the dual braid monoid associated to W_g by Corollary 3.11. The result is already known in this case: the braided reflections in $B(W_g)$ are exactly the elements that are conjugated to atoms of the dual braid monoid.

From now on we assume that $p > 1$. Let $s = (\alpha, \beta)$ be an atom of $\mathcal{C}$. We consider the element x_s of s used to define the circular tunnel b_s of Theorem 2.27. Let γ be the path consisting of sliding the first point of $\overline{\mathrm{LL}}(x_s)$ in each sector towards the center



The endpoint of γ lies on the discriminant hypersurface. Let U be a confining neighborhood of $\gamma(1)$ and let $r > 0$ be such that $\gamma(r) \in U$. The cyclic label of $\gamma(r)$ is (α, β). By Lemma 3.30, the image of $\pi_1(U \cap (X/W)^{\mu_d}, \gamma(r))$ in $\mathcal{B}(u, u)$ contains $\lambda(s)$.

Next, let $h \in \pi_1(U \cap (X/W)^{\mu_d}, \gamma(r))$ be represented by a loop θ. We can choose θ so that, at any given $t \in [0, 1]$, at most one point of $\overline{\mathrm{LL}}(\theta(t))$ lies in the half-line $i\mathbb{R}^+$ (as points which do not satisfy this form a a subspace of real codimension 2). This expresses

θ as a concatenation of paths in $U\cap(X/W)^{\mu_d}$ homotopic to circular tunnels (and their inverses).

Let $t=(d,e)$ be a simple morphism represented by a path inside $U\cap(X/W)^{\mu_d}$. By Lemma 2.24 and the following discussion, both the source and target of t are elements of $M(c)$ divisible by β. Since $p>1$, the proof of Proposition 3.3 gives that e which is the gcd in $M(c)$ of the source and target of t. We then have that β divides e and t left-divides s. The same reasoning (by induction) gives that the only morphisms represented by a concatenation of paths in $U\cap(X/W)^{\mu_d}$ (starting from $\gamma(r)$) homotopic to circular tunnels are of the form $ss^{\#}s^{\#\#}\cdots$ or $(ss^{\#}\cdots)^{-1}$. We obtain that $\pi_1(U\cap(X/W)^{\mu_d},\gamma(r))$ is cyclic and generated by a path representing $\lambda(s)$ in $\mathcal{B}$, which proves that $\lambda(s)$ is a braided reflection in $\mathcal{B}(u,u)$.

Conversely, let γ be a path from some $x\in u$ to the smooth part of some irreducible divisor of the discriminant in $X_g/W_g\simeq(X/W)^{\mu_d}$. Let also ρ_γ denote the braided reflection associated to γ. Up to conjugacy in $\mathcal{B}(u,u)$, we can assume that the endpoint a of γ is such that no point of $\overline{\mathrm{LL}}(a)$ lies on D and that, for all $t\in[0,1[$, there is some $r\in[t,1[$ such that $\overline{\mathrm{LL}}(\gamma(r))\in\mathcal{U}^{\mu_d}$.

Let U be a small enough confining neighborhood of a so that the fundamental group of $U\cap(X/W)^{\mu_d}$ is infinite cyclic. Let also $r\in[0,1]$ be such that $y:=\gamma(r)$ lies in $U\cap\mathcal{U}^{\mu_d}$. We denote by v the connected component of y in $\mathcal{U}^{\mu_d}$.

Consider the element $s=s(y)$ introduced in Lemma 3.30. We claim that s is an atom of $\mathcal{C}$. Otherwise, s admits two distinct left-divisors σ and σ' (indeed, $s(x)$ is the right-lcm of the atoms which divide it). Lemma 3.30 implies that the image of $\pi_1(U\cap(X/W)^{\mu_d},y)$ in $\mathcal{B}(v,v)$ contains both $\lambda(\sigma)$ and $\lambda(\sigma')$. Since $\pi_1(U\cap(X/W)^{\mu_d},x)$ is cyclic, there are two integers k and k' such that $\lambda(\sigma)^k=\lambda(\sigma')^{k'}$. Since both $\lambda(\sigma)$ and $\lambda(\sigma')$ are rigid, this is impossible.

Again, as every element of $\pi_1(U\cap(X/W)^{\mu_d},y)$ is represented by a path homotopic to a concatenation of circular tunnels, we obtain that the image of $\pi_1(U\cap(X/W)^{\mu_d},y)$ in $\mathcal{B}(v,v)$ is generated by $\lambda(s)$. The path $\gamma_r:t\mapsto\gamma(rt)$ then gives a conjugator in $\mathcal{B}$ between the braided reflection ρ_γ in $\mathcal{B}(u,u)$ and $\lambda(s)\in\mathcal{B}(v,v)$. □

3.5. **Conjugacy of atomic loops and centers of finite index subgroups.** In this section we study the super-summit set of (powers of) atomic loops in $\mathcal{C}$. We deduce results on the center of finite index subgroups of $B(W_g)$. More precisely we obtain new Garside theoretic proofs of [Bes15, Corollary 12.7] and of [DMM11, Theorem 1.4] in the case of the centralizer of a regular element in a well-generated irreducible complex reflection group.

Note again that our approach covers in particular the dual braid monoid associated to a well-generated irreducible complex reflection group.

Definition 3.32. *Let $\mathcal{C}$ be a small category. The* **center** *of $\mathcal{C}$ is the set $Z(\mathcal{C})$ of natural transformation from the functor $1_{\mathcal{C}}$ to itself.*

The data of an element z of $Z(\mathcal{C})$ is equivalent to the data, for every object u of $\mathcal{C}$, of a morphism z_u such that

$$\forall f\in\mathcal{C}(u,v), z_u f=f z_v$$

If $\mathcal{C}=M$ is a monoid, we recover the classical definition of the center. Note that $Z(\mathcal{C})$ is always a monoid. Moreover, the center $Z(\mathcal{G})$ of a groupoid $\mathcal{G}$ is a group.

Lemma 3.33. *Let $\mathcal{G}$ be a connected groupoid, and let u be an object of $\mathcal{G}$. The map*

$$\begin{array}{rccc} r: & Z(\mathcal{G}) & \longrightarrow & Z(\mathcal{G}(u,u)) \\ & z & \longmapsto & z_u \end{array}$$

is an isomorphism of groups.

Proof. The map r is clearly a morphism of groups. We choose, for every $v\in\mathrm{Ob}(\mathcal{G})$, a morphism $m_v:u\to v$. Let $z_0\in Z(\mathcal{G}(u,u))$ be a central element. We define an element z

of $Z(\mathcal{G})$ by defining, for every $v \in \mathrm{Ob}(\mathcal{G})$:

$$z_v := m_v^{-1} z_0 m_v$$

We have in particular $z_u = m_u^{-1} z_0 m_u = z_0$ since z_0 is central. Let $f : v \to w$ be a morphism in $\mathcal{G}$, we have

$$z_v f = m_v^{-1} z_0 m_v f = m_v^{-1} z_0 (m_v f m_w^{-1}) m_w = m_v^{-1} (m_v f m_w^{-1}) z_0 m_w = f z_w$$

and z is indeed in $Z(\mathcal{G})$. The map $z_0 \mapsto z$ is the inverse of r. □

It is known from [BMR98, Theorem 2.24] and [Bes15, Theorem 12.3 and Corollary 12.7] that the center of an irreducible complex braid group is cyclic. The center of the Springer groupoid $\mathcal{B}$ is also cyclic since $\mathcal{B}$ is connected. Under a combinatorial assumption on the integer d, we get that the center of $\mathcal{C}$ (and $\mathcal{B}$) is actually generated by some power of the Garside map Δ.

Proposition 3.34. *If d is the gcd of the degrees of W which it divides, then both $Z(\mathcal{C})$ and $Z(\mathcal{B})$ are cyclic and generated by Δ_p^q.*

Proof. First, as Δ_p is a natural transformation from $1_{\mathcal{C}}$ to ϕ_p, and as $\phi_p^q = 1_{\mathcal{C}}$, we get that Δ_p^q lies in the center of $\mathcal{C}$.

We claim that $Z(\mathcal{B})$ is generated by $Z(\mathcal{C})$ and Δ_p^{-q}. Let $z \in Z(\mathcal{B})$, and let $u \in \mathrm{Ob}(\mathcal{B})$. We set $n_u := \inf(z_u)$. Since $\mathcal{B}$ admits a finite number of objects ($D_p^q(c)$ is finite by definition), there is some kq for $k \in \mathbb{Z}_{\geqslant 0}$ such that $kq + n_u > 0$ for all object u. We then have that $\Delta_p^{kq} z \in Z(\mathcal{C})$ and $z = \Delta_p^{-kq}(\Delta_p^{kq} z)$ as claimed.

Now, let ρ be a generator of the center of $B(W_g) \simeq \mathcal{B}(u,u)$. As $\Delta_p^q(u) \in Z(\mathcal{B}(u,u))$, there is some integer k such that $\rho^k = \Delta_p^q(u)$. That is ρ is a (pk,q)-regular element in $\mathcal{B}$. By applying π_p, we get that $\pi_p(\rho)$ is a (pk,q)-regular element of $B(W)$. By assumption, we have

$$\pi_p(\mathcal{B}(u,u)) = C_{B(W)}(\pi_p(\Delta_p^q)) \subset C_{B(W)}(\pi_p(\rho)).$$

We also have $C_{B(W)}(\pi_p(\Delta^q)) \supset C_{B(W)}(\pi_p(\rho))$ since $\pi_p(\Delta_p^q)$ is a power of $\pi_p(\rho)$. The element $\pi_p(\rho)$ is then a dk-regular braid in $B(W)$ with the same centralizer as a d-regular braid. Since d is maximal regarding to divisibility, we get that $k = \pm 1$. Thus $\rho = \Delta_p^{\pm q}(u)$ which shows the proposition. □

Remark 3.35. The assumption that d is the gcd of the degrees of W which it divides is important, otherwise $Z(\mathcal{C})$ is generated by some root of Δ_p^q. For instance in the group $W = G_{37}$, the integers $d_1 = 5$ and $d_2 = 10$ are regulars. We have $(p_1, q_1) = (1,6)$ and $(p_2, q_2) = (1,3)$. The associated categories of periodic elements are then monoids, given by $C_{M(c)}(\Delta^3)$ and $C_{M(c)}(\Delta^6)$, respectively. Because d_1 and d_2 both divide 2 degrees of W, those centralizers are equal, and their centers are both equal to $\langle \Delta^3 \rangle^+$.

Lemma 3.36. *Let $\lambda(s)$ be an atomic loop of some object u of $\mathcal{C}$. Let $t : u \to v$ be an atom in $\mathcal{C}$. If the path $s^\flat t$ is not greedy, then we have $\lambda(s)t = t\lambda(s')$ for some atomic loop $\lambda(s')$ of v.*

Proof. Let $\lambda(s) = s_n \cdots s_1$ where $s_1 = s^\flat$ and $s_{i+1} := s_i^\flat$ for $i \in [\![1, n-1]\!]$ (in particular we have $s_n = s$). We denote $t := (\alpha, \beta)$ and $s_i := (a_i, b)$ for $i \in [\![1, n]\!]$. By Corollary 1.39, if $s^\flat t$ is not greedy, then we have $\alpha \preceq b$. Let then x be such that $\alpha x = b$. We have

$$s^\flat t = (a_1, b)(\alpha, \beta) = (a_1 \alpha, x) = (\alpha, a_1^\alpha x)(a_1^\alpha, x\alpha^{c^\eta}).$$

We set $t_1 := (\alpha, a_1^\alpha x)$ and $\sigma_1 = (a_1^\alpha, x\alpha^{c^\eta})$. Again, as $\alpha \preceq b$, the path $s_2 t$ is not greedy, and we have

$$s_2 t_1 = (a_2, b)(\alpha, a_1^\alpha x) = (a_2 \alpha, x) = (\alpha, a_2^\alpha x)(a_2^\alpha, x\alpha^{c^\eta}),$$

and we set again $t_2 := (\alpha, a_2^\alpha x)$, $\sigma_2 := (a_2^\alpha, x\alpha^{c^\eta})$. By an immediate induction, we get $\lambda(s)t = t_n \sigma_n \cdots \sigma_1$, with $t_n := (\alpha, a_n^\alpha x)$ and $\sigma_i := (a_i^\alpha, x\alpha^{c^\eta})$ for $i \in [\![1, n]\!]$. The two simple morphisms t and t_n share the same source, and we have $\pi_p(t) = \pi_p(t_n)$. By Lemma 1.38, we have $t = t_n$. For $i \in [\![1, n-1]\!]$, the target of σ_{i+1} is the source of σ_i, and the second terms of σ_i and σ_{i+1} are both equal to $x\alpha^{c^\eta}$. Thus σ_i is equal to $\sigma_{i+1}^{\#}$, again by Lemma 1.38.

We claim that $\sigma_n \cdots \sigma_1 = \lambda(\sigma_n)$. Since $\sigma_i = \sigma_{i+1}^{\#}$ for $i \in [\![1, n-1]\!]$, we only have to show that σ_1 is the first σ_i with target v. The target of σ_i is $x(\alpha a_i^{\alpha})^{c^\eta}) = x(a_i\alpha)^{c^\eta}$, and the source of σ_n is the target of σ_1, that is $v = x(a_1\alpha)^{c^\eta}$. By cancellativity of $M(c)$, we get that the target of σ_i is v only if $i = 1$, for $i \in [\![1, n]\!]$. $\square$

Theorem 3.37. *Let $\lambda(s) \in \mathcal{C}(u, u)$ be an atomic loop of some object u, and let $f \in \mathcal{G}$. If there is some endomorphism $z \in \mathcal{C}$ such that $\lambda(s)^n f = fz$ for some $n \geqslant 1$, then $z = \lambda(s')^n$ for some atomic loop $\lambda(s')$ such that $\lambda(s)f = f\lambda(s')$.*

Proof. We do the proof in several steps.

1. f is an atom. By assumption, we have $f \preceq \lambda(s)^n f$. If $\lambda(s)^n f$ is in greedy normal form, we get $f \preceq s$. Thus f is trivial or $f = s$. We have $\lambda(s)^f = \lambda(s)$ in the first case, and $\lambda(s)^f = \lambda(s^{\#})$ in the second case. If $\lambda(s)^n f$ is not in greedy normal form, then $s^\flat f$ is not in greedy normal form. By Lemma 3.36, we get that $\lambda(s)^f$ is an atomic loop, hence the desired result.
2. f is a simple morphism. Since $\mathcal{C}$ is homogeneous, we can proceed by $\succeq$-induction on f. The case where f is an atom has already been dealt with. If f is not an atom, then $\lambda(s)^n f$ cannot be in greedy normal form as $f \preceq s$ is impossible. Thus $s^\flat f$ is not greedy, and there is some decomposition $f = tf'$ with t an atom such that $s^\flat t$ is not greedy. By the first point we have that $\lambda(s)^t$ is of the form $\lambda(s')$. Since $z = \lambda(s)^f = \lambda(s)^{tf'} = \lambda(s')^{f'}$, the induction hypothesis gives the desired result.
3. f is an arbitrary morphism in $\mathcal{B}$. We have that $\Delta_p^q(u)$ is central in $\mathcal{B}(u, u)$, thus $(\lambda(s)^n)^{\Delta_p^{kq}(u)f} = (\lambda(s)^n)^f = z$ for all $k \in \mathbb{Z}$. By choosing some k big enough so that $kq + \inf(f) \geqslant 0$, we can replace f with $\Delta_p^{kq}(u)f$ and assume that $f \in \mathcal{C}$. We claim that both $\lambda(s)^n$ and z lie in $\mathrm{SSS}(\lambda(s)^n)$. First, as atomic loops are rigid, we have
$$\mathrm{cyc}(\lambda(s)) = (s^{\#} \cdots s^{\flat\flat} s^\flat)s = \lambda(s^{\#}) = \mathrm{dec}(\lambda(s)).$$
Powers of atomic loops are also rigid, and we have $\mathrm{cyc}(\lambda(s)^n) = \mathrm{dec}(\lambda(s)^n) = \lambda(s^{\#})^n$. Because of Proposition 1.24, we get that $\lambda(s)^n$ lies in $\mathrm{SSS}(\lambda(s)^n)$. Then, we have $\inf(z) \geqslant 0$ as z lies in $\mathcal{C}$, so $\inf(z) = 0$ as $\inf(z) \leqslant \inf(\lambda(s)^n)$. Lastly, we have $\sup(z) \geqslant \sup(\lambda(s)^n) = nm$ (where m is the length of $\lambda(s)$). Since every simple morphism has length at least 1, and since $\inf(z) = 0$, we get $mn = \ell(z) \geqslant \sup(z) \geqslant nm$ and $\sup(z) = nm$, so $z \in \mathrm{SSS}(\lambda(s)^n)$. Let $s_1 \cdots s_r$ be the greedy normal form of $f \in \mathcal{C}$. By Proposition 1.25, each $(\lambda(s)^n)^{s_1 \cdots s_i}$ lies in $\mathrm{SSS}(\lambda(s)^n)$, in particular is positive. By an immediate induction using the second case, we get that z is of the form $\lambda(t)^n$, with $\lambda(s)^f = \lambda(t)$.

$\square$

This theorem is an analogue of [DMM11, Proposition 2.2] in the context of Springer categories. It gives in particular a complete description of super-summit sets of atomic loops and their powers.

Corollary 3.38. *Let $u \in \mathrm{Ob}(\mathcal{B})$, and let $\sigma \in B(W_g) \simeq \mathcal{B}(u, u)$ be a braided reflection. The super-summit set of σ in $\mathcal{B}$ consists of all the atomic loops to which σ is conjugate in $\mathcal{B}$. Furthermore, for $n \geqslant 1$, we have*
$$\mathrm{SSS}(\sigma^n) = \{\lambda(s)^n \mid \lambda(s) \in \mathrm{SSS}(\sigma)\}$$

Proof. We already showed that σ is conjugate to some atomic loop $\lambda(s) \in \mathcal{B}$, and that $\lambda(s) \in \mathrm{SSS}(\lambda(s))$ for all atomic loops $\lambda(s)$. If $g \in \mathrm{SSS}(\lambda(s))$, then g is a positive conjugate of $\lambda(s)$ and Theorem 3.37 gives that g is an atomic loop.

We also have $\lambda(s)^n \in \mathrm{SSS}(\sigma^n)$. If $g \in \mathrm{SSS}(\sigma^n)$, then we also have that g is a positive conjugate of $\lambda(s)^n$. We get that g is of the form $\lambda(s')^n$ for some conjugate $\lambda(s')$ of $\lambda(s)$ by Theorem 3.37. $\square$

Like in [DMM11], our theorem on conjugacy of atomic loops has the following corollary on the center of finite index subgroups.

Corollary 3.39. *Let U be a finite index subgroup of $B(W_g) \simeq \mathcal{B}(u,u)$. We have $Z(U) \subset Z(B(W_g))$.*

Proof. Let $\sigma \in B(W_g)$ be a braided reflection, and let $x \in Z(U)$. Since U is of finite index, there is some $n \geqslant 1$ with $x\sigma^n = \sigma^n x$. We claim that $\sigma x = x\sigma$.

By Theorem 3.31, there is some morphism $f : u \to v$ in $\mathcal{B}$ such that $\sigma^f = \lambda(s)$ is an atomic loop. We define $x' := x^f$ and $U' := U^f \subset \mathcal{B}(v,v)$. By Theorem 3.37, since $\lambda(s)^{nx} = \lambda(s)^n$, we have $\lambda(s)x = x\lambda(s')$ for some atomic loop $\lambda(s')$ such that $\lambda(s')^n = \lambda(s)^n$. Since atomic loops are rigid, the equality $\lambda(s)^n = \lambda(s')^n$ implies $\lambda(s) = \lambda(s')$ and x' commutes with $\lambda(s)$. We obtain that x commutes with every braided reflection in $\mathcal{B}(u,u)$, and thus lies in $Z(\mathcal{B}(u,u))$. $\square$

3.5.1. *The case $p = 1$.* In this case, for any simple s, we have $s^\# = s^\flat = s$ and $\lambda(s) = s$. Lemma 3.36 is then a mere consequence of the fact that every simple element is balanced. Theorem 3.37 is just [DMM11, Proposition 2.2] in the case of the dual braid monoid.

4. Presentations of the braid group $B(G_{31})$

The goal of this section is to apply the previous results to the complex braid group $B(G_{31})$. In particular we prove several positive homogeneous presentations of this group, where the generators are braided reflections.

We first consider the complex reflection group G_{37}, which is isomorphic to the real reflection group of type E_8. Let $M(c)$ be the dual braid monoid associated to some Coxeter element c of G_{37}. We denote again its set of simple elements by I_c. We have seen in Example 2.7 that the integer 4 is regular for G_{37}, and that the associated centralizer is isomorphic to G_{31}. Since the Coxeter number of G_{37} is $h = 30$, we have $(p,q) = (2,15)$ in this case. By Theorem 2.27, the groupoid $\mathcal{G}(M(c)_2^{15})$ is equivalent to the complex braid group $B(G_{31})$. We denote $\mathcal{C}_{31} := M(c)_2^{15}$ and $\mathcal{B}_{31} := \mathcal{G}(M(c)_2^{15})$.

The integer η associated to p,q by Lemma 1.28 is $\eta = -7$. As $c^{15} = -\mathrm{Id}$ is central in G_{37}, we have

$$\begin{aligned}
\mathrm{Ob}(\mathcal{C}_{31}) &:= D_2^{15}(c) = \{u \in I_c \mid uu^{c^8} = c \text{ and } \ell_R(u) = 4\},\\
\mathcal{S}_2^{15} &:= D_4^{30}(c) = \{(a,b) \in (I_c)^2 \mid ab \in \mathrm{Ob}(\mathcal{C})\},\\
R_2^{15} &:= D_6^{45}(c) = \{(x,y,z) \in (I_c)^3 \mid xyz \in \mathrm{Ob}(\mathcal{C})\}.
\end{aligned}$$

By Lemma 3.2, the atoms of $\mathcal{C}_{31}$ are exactly the elements of length 1. Since 4 is the gcd of degrees of G_{37} which it divides, we can apply Proposition 3.34 and Corollary 3.39. We get

Theorem 4.1. *The centers of $\mathcal{C}_{31}$ and $\mathcal{B}_{31}$ are cyclic and generated by Δ_2^{15}. If U is a finite index subgroup of $B(G_{31})$, then $Z(U) \subset Z(B(G_{31}))$. In particular, the center of the pure braid group $P(G_{31})$ is cyclic and generated by the full-twist.*

Proof. The only nontrivial part is that the full-twist is a generator of $Z(P(G_{31}))$. Let $B(G_{31}) \simeq \mathcal{B}_{31}(u,u)$, the center of $P(G_{31})$ is cyclic and generated by the smallest power of $\Delta_2^{15}(u)$ which lies in $P(G_{31})$. Since the collapse functor $\mathcal{C}_{31} \to M(c)$ sends $\Delta_2^{15}(u_0)$ to some element g with $g^2 = \Delta_2^{15}$, the smallest power of Δ_2^{15} lying in $P(G_{31})$ is the full-twist Δ^{60}. $\square$

The remainder of this section is devoted to the study of presentations of $B(G_{31})$. Let u be an object of $\mathcal{C}_{31}$, we have an isomorphism $\mathcal{B}_{31}(u,u) \simeq B(G_{31})$, which sends atomic loops to braided reflections.

On the one hand, we build a conjectural presentation of $\mathcal{B}_{31}(u,u)$ with atomic loops as generators. On the other hand, the Reidemeister-Schreier method for groupoids (cf. Annex A.2) gives a presentation of $\mathcal{B}_{31}(u,u)$ which we know holds. We then prove that these two presentations are equivalent.

4.1. **The method.** Let u be an object of $\mathcal{B}_{31}$. We start by considering the submonoid L_u^+ of $\mathcal{C}_{31}(u,u)$ generated by atomic loops of u. Note that we have $L_u^+ \neq \mathcal{C}_{31}(u,u)$ in general.

Our first goal is to construct a (conjectural) group presentation using L_u^+. We consider the following algorithm.

Algorithm 4.1 Compute shortest right-multiple of atomic loops in L_u^+

Input: Two atomics loops $\lambda(s)$ and $\lambda(t)$ of u.
Output: If $\lambda(s), \lambda(t)$ admit a common right-multiple in L_u^+, then the output is a pair of words $\theta(\lambda(s),\lambda(t)), \theta(\lambda(t),\lambda(s))$ in L_u^+ such that $\lambda(s)\theta(\lambda(s),\lambda(t)) = \lambda(t)\theta(\lambda(t),\lambda(s))$. No output otherwise.
put $i := 1$
compute the set S_i of words of length i in L_u^+
while $\lambda(s)m_1 \neq \lambda(s)m_2$ in $\mathcal{C}_{31}$ for all $(m_1,m_2) \in S_i \times S_i$ **do**
  put $i := i+1$
  compute the set S_i of words of length i in L_u^+
end while
put $S := \{(m_1,m_2) \in S_i \times S_i \mid \lambda(s)m_1 = \lambda(t)m_2 \in \mathcal{C}_{31}\}$
return $(m_1,m_2) \in S$ such that m_1 is the lowest possible in the lexicographic order, and m_2 is the lowest possible in the lexicographic order among the words m such that $(m_1,m) \in S$.

Algorithm 4.1, running on two atomic loops of u, terminates if and only if they admit a common right-multiple in L_u^+. The fact that it terminates on every pair of atomic loops of u (which we checked by computer using the data of Section 4.2) proves that all pair of atomic loops of u admit a common right-multiple. We now consider the following presentation:

- The set of generators is a set $X_u := \{\lambda'(s)\}$ in bijection with atomic loops of u in $L_u^+ \subset \mathcal{C}_{31}(u,u)$.
- The set R_u is given by relations of the form
$$\lambda'(s)\theta(\lambda'(s),\lambda'(t)) = \lambda'(t)\theta(\lambda'(t),\lambda'(s))$$
where $\theta(\lambda'(s),\lambda'(t))$ is the same word as $\theta(\lambda(s),\lambda(t))$ but with letters in X_u.

We define $H_u := \langle X_u \mid R_u\rangle$ and $H_u^+ := \langle X_u \mid R_u\rangle^+$. By [DDGKM, Lemma II.4.3], the presentation of H_u^+ is ***right-complemented*** in the sense of [DDGKM, Definition II.4.1]. That is for $a \neq b \in X_u$, there is exactly one relation of the form $a\ldots = b\ldots$, namely $a\theta(a,b) = b\theta(b,a)$.

The goal of this section is to show the following theorem

Theorem 4.2. *Let $u \in \mathrm{Ob}(\mathcal{C}_{31})$. The natural map from X_u to the set of atomic loops of u induces a group isomorphism $H_u \simeq \mathcal{B}_{31}(u,u)$. In particular, $\langle X_u \mid R_u\rangle$ gives a positive homogeneous presentations of $B(G_{31})$ with braided reflections as generators.*

We first notice that, by definition of R_u and X_u, the natural map from X_u to the set of atomic loops of u induces a group morphism $f_u : H_u \to \mathcal{B}_{31}(u,u)$. We only have to show that the said morphism is an isomorphism. We show this by case by case analysis on the objects of $\mathcal{C}_{31}$.

Lemma 4.3. *Let $u \in \mathrm{Ob}(\mathcal{C}_{31})$. If f_u is an isomorphism, then $f_{\phi_2(u)}$ is also an isomorphism. In particular we only need to show that Theorem 4.2 holds for a system of representative of ϕ_2-orbits in $\mathcal{B}_{31}$.*

Proof. The automorphism ϕ_2 gives an isomorphism between $\mathcal{C}_{31}(u,u)$ and $\mathcal{C}_{31}(\phi(u),\phi(u))$. Because of Lemma 3.5, ϕ_2 induces a bijection between the sets of atomic loops of u and of $\phi(u)$. We obtain that ϕ_2 induces bijections between X_u and $X_{\phi(u)}$, and between R_u and $R_{\phi(u)}$, respectively. Thus $H_u \simeq H_{\phi(u)}$ which shows the claim. □

Remark 4.4. The atoms of the monoid H_u^+ are in bijection with atomic loops of u. In particular H_u^+ is not isomorphic to the monoid $\mathcal{C}_{31}(u,u)$ in general. As a matter of fact, we will see that the monoid H_u^+ is never cancellative, and thus cannot be isomorphic to either L_u^+ or $\mathcal{C}_{31}(u,u)$. The isomorphism we consider only occurs at the level of groups.

Note that the monoid H_u^+ is homogeneous by definition. In particular we have a solution to the word problem in H_u^+, given by Algorithm 4.2.

Algorithm 4.2 Check equality between two words in H_u^+

Input: Two words m_1, m_2 in the atoms of H_u^+
Output: **true** if m_1 and m_2 represent the same element of H_u^+, **false** otherwise.
put $S := \varnothing$.
put $S_1 := \{m_1\}$.
while $S \neq S_1$ **do**
  put $S := S_1$.
  put S_1 the set of words obtained from elements of S_1 by applying one relation of R_u.
end while
if $m_2 \in S$ **then**
  return true
else
  return false
end if

Since H_u^+ is homogeneous, two words representing the same element have the same length, thus there is a finite number of words that represent the same element of H_u^+ and Algorithm 4.2 always terminates.

In order to prove Theorem 4.2 for the object u, we first compute a presentation of $\mathcal{B}_{31}(u,u)$ by the Reidemeister-Schreier method for groupoids (cf. Appendix A.2). We start from a presentation of the groupoid $\mathcal{B}_{31}$, for instance that of Theorem 3.27.

As we want atomic loops to appear as generators, we need to choose the Schreier transversal accordingly. We use the following lemmas concerning atomic loops in the category $\mathcal{C}_{31}$.

Lemma 4.5. *Let s be an atom of $\mathcal{C}_{31}$, the atomic loop $\lambda(s)$ has length two in $\mathcal{C}_{31}$.*

Proof. Let $s := (a,b)$. We denote by u the source of s, and by v its target. The morphism $s^{\#}$ is given by $(a,b)^{\#} = (a^{c^8 b^{-1}}, b)$. We know that

$$a^{c^7} b^{c^7} ab = c \Rightarrow c^7 a^{c^7} b^{c^7} a = abc^7 a = c^8 b^{-1}$$

and so $a^{c^8 b^{-1}} = a^{abc^7 a} = a^{(ba^{c^8})c^7} = a^{vc^7}$. The morphism $s^{\#\#}$ is then given by $((a^{vc^7})^{avc^7}, b)$. Because of Lemma 2.8, a and a^{vc^7} commute, so $s^{\#\#} = (a^{(vc^7)^2}, b)$. Since $(vc^7)^2 = c^{15}$ is central, we have $s^{\#\#} = s$ as claimed. □

Lemma 4.6. *Let u be an object of $\mathcal{B}_{31}$. There is a Schreier transversal T rooted in u and containing all atoms with source u. In particular the atomic loops of u appear as generators of the presentation of $\mathcal{B}_{31}(u,u)$ induced by T.*

Proof. First, note that if a Schreier transversal T contains an atom s with source u, then Lemma 4.5 gives that $\gamma(s^{\#}) = ss^{\#} = \lambda(s)$ with the notation of Lemma A.12.

Now, thanks to Proposition 3.3, all atoms with source u have different targets. We can thus consider a Schreier transversal T rooted in u and containing all atoms with source u. □

Let T be a Schreier transversal rooted in u and containing all atoms with source u. Let $\langle S^* \mid R^* \rangle$ denote the presentation of $\mathcal{B}_{31}(u,u)$ obtained by the Reidemeister-Schreier method applied to T and to the presentation of Theorem 3.27. Of course, the presentation $\langle S^* \mid R^* \rangle$ is quite redundant. We first want to show that every element of S^* can be expressed as a word in the atomic loops. For this we repeatedly apply Tietze transformations, as in Algorithm 4.3.

Algorithm 4.3 Reduction of generators

Input: A group presentation $\langle S \mid R \rangle$ and a subset S' of S
Output: A group presentation $\langle S' \mid R' \rangle$ equivalent to the first by Tietze transformation, or no output.
while $S' \neq S$ **do**
  choose $r \in R$ a relator with only one letter a not belonging to $S' \cup S'^{-1}$
  replace in R every occurrence of a by its expression in S' using the relator r.
  remove the relator r from R
  remove the letter a from S.
end while
return the presentation $\langle S' \mid R \rangle$

The fact that this algorithm terminates for each object of $\mathcal{B}_{31}$, which is again checked by computer, proves the following result

Proposition 4.7. *Let u be an object of $\mathcal{B}_{31}$. The atomic loops of u generate the group $\mathcal{B}_{31}(u,u)$. In particular the natural morphism $H_u \to \mathcal{B}_{31}(u,u)$ is surjective.*

By applying Algorithm 4.3 to the presentation $\langle S^* \mid R^* \rangle$, we obtain a presentation $\langle X_u \mid R'_u \rangle$ of the group $\mathcal{B}_{31}(u,u)$. In order to prove Theorem 4.2 for the object u, it is sufficient to prove that every relator of R'_u is in fact trivial in H_u. This will prove that the morphism $H_u \to \mathcal{B}_{31}(u,u)$ is injective.

Because the defining presentation of H_u^+ is right-complemented, we can consider a ***right-reversing*** algorithm in the sense of [DDGKM, Definition 4.21]. The main idea is that the relation $a\theta(a,b) = b\theta(b,a)$ implies $a^{-1}b = \theta(a,b)\theta(b,a)^{-1}$. We can use these type of relation in order to simplify words in $X_u \cup X_u^{-1}$.

Algorithm 4.4 Right-reversing in H_u^+ ([DDGKM, Algorithm II.4.33])

Input: A word w in $X_u \cup X_u^{-1}$
Output: A fraction fg^{-1} with $f, g \in H_u^+$ representing w in H_u, or no output.
while there is some subword of the form $a^{-1}b$ in w **do**
  put j the position in w of the first subword of the form $a^{-1}b$ in w
  if $a = b$ **then**
    remove the subword $a^{-1}b$ from w
  else
    replace $a^{-1}b$ with $\theta(a,b)\theta(b,a)^{-1}$ in w at position j
  end if
end while
return w.

Algorithm 4.4 doesn't always terminate: it may loop indefinitely. If it terminates, its output is a fraction, and we should check that it is trivial. Of course this may be quite long as algorithm 4.2 is far from optimal. This process can be sped up by "partially simplifying" at each step. The solution to the word problem given by Algorithm 4.2 allows for the computation of longest common divisors of elements of H_u^+. A general word in $X_u \cup X_u^{-1}$ can be written as a product of (short) fractions. We can simplify these fractions at each step of Algorithm 4.4.

We checked by computer that Algorithm 4.4 terminates on every relator in R'_u, that is every relator can be expressed as a right-fraction of elements of H_u^+. Finally, we use the following algorithm to prove that every relator, written as a fraction, is trivial in H_u.

Algorithm 4.5 Partial solution to the word problem in H_u

Input: A fraction fg^{-1} with $f, g \in H_u^+$.
Output: `true` If there is some $n \in H_u^+$ such that $fn = gn$. No output otherwise.
put $i := 1$
Compute the set S_i of words of length i in X_u
while $fn \neq gn$ for all $n \in S_i$ **do**
  put $i := i + 1$
  Compute the set S_i of words of length i in X_u
end while
return true

This algorithm is useful since the monoid H_u^+ may not be cancellative: we can have $fn = gn$ (and thus $fg^{-1} = 1$ in H_u) without having $f = g$ in H_u^+.

The fact that Algorithm 4.5 returns `true` for all relators of the presentation of $\mathcal{B}_{31}(u, u)$ finally proves that Theorem 4.2 holds for u.

4.2. **Computational data.** We consider the following elements in $\mathbb{C}^8$

$$\begin{cases} \alpha_1 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1) \\ \alpha_2 = (1,1,0,0,0,0,0,0) \\ \alpha_3 = (-1,1,0,0,0,0,0,0) \\ \alpha_4 = (0,-1,1,0,0,0,0,0) \\ \alpha_5 = (0,0,-1,1,0,0,0,0) \\ \alpha_6 = (0,0,0,-1,1,0,0,0) \\ \alpha_7 = (0,0,0,0,-1,1,0,0) \\ \alpha_8 = (0,0,0,0,0,-1,1,0) \\ \alpha_9 = (1,0,1,0,0,0,0,0) \end{cases} \qquad \begin{cases} \alpha_{10} = (0,0,-1,0,1,0,0,0) \\ \alpha_{11} = (1,0,0,0,0,1,0,0) \\ \alpha_{12} = \frac{1}{2}(-1,-1,-1,-1,-1,1,-1,1) \\ \alpha_{13} = (0,1,0,0,0,1,0,0) \\ \alpha_{14} = \frac{1}{2}(-1,-1,-1,-1,-1,-1,1,1) \\ \alpha_{15} = (0,1,0,0,0,0,1,0) \\ \alpha_{16} = \frac{1}{2}(1,-1,1,-1,-1,-1,1,1) \\ \alpha_{17} = \frac{1}{2}(1,-1,-1,1,-1,-1,1,1) \\ \alpha_{18} = \frac{1}{2}(-1,1,-1,-1,1,-1,1,1) \end{cases}$$

And for each $i \in [\![1, 18]\!]$, we consider the reflection s_i of $\mathbb{C}^8$ given by $s_i(x) = x - 2\frac{\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i$ (where $\langle ., . \rangle$ is the usual hermitian product on $\mathbb{C}^8$). The set $s_1, \ldots, s_8$ generates a subgroup W of $\mathrm{GL}_n(\mathbb{C})$ isomorphic to the complex reflection group G_{37} and which contains all the s_i for $i \in [\![1, 18]\!]$. A Coxeter element of W is given by $c = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$.

A system of representatives of ϕ_2-orbits of objects of the category $\mathcal{C}_{31} = M(c)_2^{15}$ is given by the following elements of W.

$$\begin{cases} u_1 = s_{10} s_{12} s_{13} s_{18} \\ u_2 = s_9 s_{10} s_{12} s_{16} \\ u_3 = s_3 s_{11} s_{12} s_{16} \\ u_4 = s_7 s_8 s_{14} s_{15} \end{cases} \quad \begin{cases} u_5 = s_3 s_7 s_{11} s_{12} \\ u_6 = s_3 s_4 s_7 s_{11} \\ u_7 = s_3 s_4 s_7 s_{17} \\ u_8 = s_3 s_7 s_{12} s_{17} \end{cases}.$$

As we want links between the different presentations we obtain for each representative, we give explicit isomorphisms between the different groups $\mathcal{B}_{31}(u_i, u_i)$. For this we use the following graph in $\mathcal{B}_{31}$

$$\begin{tikzcd}
 & u_2 & u_8 & \\
u_3 & u_1 \arrow[l] \arrow[u] \arrow[d] \arrow[r] & u_5 \arrow[u] \arrow[d] \arrow[r] & u_7 \\
 & u_4 & u_6 &
\end{tikzcd}$$

where each arrow is a simple morphism in $\mathcal{C}_{31}$ (because of Proposition 3.3, a simple morphism is uniquely determined by its source and target). For each $i, j \in [\![1, 8]\!]$, this graph induces a well defined isomorphism $\varphi_{i,j} : \mathcal{B}_{31}(u_i, u_i) \to \mathcal{B}_{31}(u_j, u_j)$ which preserves braided reflections.

For $i, j \in [\![1, 8]\!]$, we have by definition $\varphi_{i,j} = \varphi_{i,1}\varphi_{1,j}$, so we only need to describe morphisms of the form $\varphi_{i,1}$ and $\varphi_{1,i}$ for $i \in [\![1, 8]\!]$.

In the case of the orbit of u_1, we give expressions of the atomic loops in the generators $\sigma_1, \ldots, \sigma_8$ of the Artin group associated to W. Replacing $\sigma_1, \ldots, \sigma_8$ with $s_1, \ldots, s_8$ gives a set of elements in W which generate a copy of G_{31}. We also give a family of vectors in $\mathbb{C}^4$ such that the 2-reflections associated to the orthogonal hyperplanes of these vectors (in the usual hermitian product) generate a group isomorphic to G_{31}.

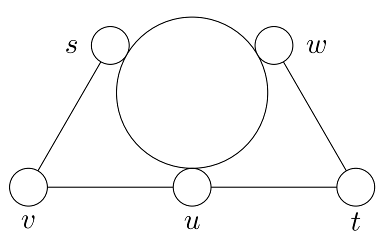
Futhermore, we know that the full-twist in $\mathcal{B}_{31}(u, u)$ is given by $\Delta^{60}(u)$ by Lemma 2.28. By [Gar23, Theorem 1.2 and Proposition 8.1], every root of the full-twist in $B(G_{31})$ is conjugate to a power of either a 20-th root or a 24-th root of the full-twist. Furthermore, the full-twist admits 20-th roots and 24-th roots. For each presentation we obtain, we give an explicit 20-th root (resp. 24-th root) of the full-twist as a word in the generators. If $\rho \in \mathcal{B}_{31}(u, u)$ is a 24-th root of the full-twist $\Delta^{60}(u)$, then ρ^6 is a 4-th root of $\Delta^{60}(u)$. By Proposition 1.27 and Theorem 4.1, we get that $\rho^6 = \Delta_2^{15}(u)$ is a generator of $Z(\mathcal{B}_{31}(u, u))$.

4.3. ***Presentation associated to representatives of the ϕ_2-orbits.***

4.3.1. *Orbit of u_1.* For the first orbit, we recover the presentation of $B(G_{31})$ conjectured in [BMR98, Table 3] and [BM04, Conjecture 2.4]. The object u_1 has 5 atomic loops s, t, u, v, w. The relations we obtain are as follows

$$\begin{cases} ts = st, \ vt = tv, \ wv = vw, \\ suw = uws = wsu, \\ svs = vsv, \ vuv = uvu, \ utu = tut, \ twt = wtw. \end{cases}$$

This presentation is usually represented by the following diagram (corresponding to the Broué-Malle-Rouquier diagram for the reflection group G_{31}).



In the Artin group associated to W, the atomic loops s, t, u, v, w can be expressed as

$$
\begin{aligned}
s &:= (\sigma_1\sigma_4)^{\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8} \\
t &:= (\sigma_4\sigma_2)^{\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7} \\
u &:= (\sigma_4\sigma_2)^{\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_5\sigma_7\sigma_6\sigma_8\sigma_7\sigma_6} \\
v &:= (\sigma_1\sigma_3)^{\sigma_3\sigma_4\sigma_5\sigma_4\sigma_6\sigma_7} \\
w &:= (\sigma_2\sigma_3)^{\sigma_1\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_6\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_7\sigma_6}
\end{aligned}
$$

In $\mathbb{C}^4$, the 2-reflections associated with the following roots (in the usual hermitian product) generate a subgroup of $\mathrm{GL}_4(\mathbb{C})$ which is isomorphic to G_{31}.

$$
\alpha_s := \frac{1}{2}\begin{pmatrix} 2 \\ 1+i \\ -1-i \\ 0 \end{pmatrix},\ \alpha_t := \begin{pmatrix} 1 \\ -1 \\ i \\ -1 \end{pmatrix},\ \alpha_u := \begin{pmatrix} 1 \\ 0 \\ -1-i \\ -i \end{pmatrix},\ \alpha_v := \frac{1}{2}\begin{pmatrix} 2 \\ 1-i \\ 1-i \\ 0 \end{pmatrix},\ \alpha_w := \begin{pmatrix} 1 \\ i \\ -1 \\ -1 \end{pmatrix}
$$

The monoid $H_{u_1}^+$ given by the above presentation is not cancellative: we have $tuwtuw \neq uwtuwt$ and $stuwtuw = suwtuwt$ in $H_{u_1}^+$. Thus $H_{u_1}^+$ cannot be a Garside monoid.

The submonoid $L_{u_1}^+$ of $\mathcal{C}_{31}(u_1, u_1)$ generated by s, t, u, v, w is cancellative, but it doesn't admit right-lcms: we know that $utu = tut$ is the shortest common multiple of t and u. If right-lcms exists in $L_{u_1}^+$, then tut must be the right-lcm of t and u. We would then have

$$tut \preceq tuwtuw \Rightarrow t \preceq wtuw$$

in $L_{u_1}^+$ by cancellativity. We can check that $t \preceq wtuw$ doesn't hold in $L_{u_1}^+$: the element $t^{-1}wtuw$ does lie in $\mathcal{C}_{31}(u_1, u_1)$, but not in the submonoid $L_{u_1}^+$.

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$stuvws$	$stuvw$

In particular, we get that $(stuvw)^6 = \Delta_2^{15}(u_1)$ generates $Z(\mathcal{B}_{31}(u_1, u_1))$.

4.3.2. *Orbit of* u_2. The object u_2 admits 7 atomic loops a, b, c, s, t, v, w, with relations as follows:

$$
\begin{cases}
sv = vb = bs,\ av = vc = ca, \\
wv = vw,\ st = ts,\ tv = vt,\ tb = bt, \\
was = swa = asw,\ wcb = bwc = cbw, \\
wtw = twt,\ ata = tat,\ aba = bab,\ tct = ctc, \\
swav = cabw.
\end{cases}
$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_2}^+$ given by this presentation is not cancellative: we have $bwab \neq abwa$ and $cbwab = cabwa$ in $H_{u_2}^+$. Thus $H_{u_2}^+$ cannot be a Garside monoid.

The submonoid $L_{u_2}^+$ of $\mathcal{C}_{31}(u_2, u_2)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $aba = bab$ is the shortest common multiple of a and b. If right-lcms exists in $L_{u_2}^+$, then aba must be the right-lcm of a and b. We would then have

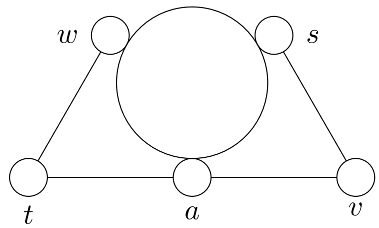
$$aba \preceq abwa \Rightarrow a \preceq wa$$

Which does not hold in $L_{u_2}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_2, u_2)$).

The relations defining $H_{u_2}^+$ give in particular $b = v^{-1}sv = s^v$ and $c = a^v$ in H_u. By deleting these generators, we get that $\mathcal{B}_{31}(u_2, u_2)$ is generated by a, s, t, v, w, with relations as follows:

$$\begin{cases} wv = vw,\ st = ts,\ vt = tv, \\ swa = was = asw, \\ twt = wtw,\ ata = tat,\ vav = ava,\ svs = vsv. \end{cases}$$

We recover the presentation of Section 4.3.1, summarized in the diagram



Again, we know that neither the monoid given by this presentation, neither the submonoid of $L_{u_2}^+$ generated by a, s, t, v, w are Garside monoids.

The morphisms $\varphi_{2,1}$ and $\varphi_{2,1}$ are given by

$$\varphi_{2,1} : \begin{cases} s \mapsto s & a \mapsto u^w \\ t \mapsto t & b \mapsto s^v \\ v \mapsto v & c \mapsto u^{wv} \\ w \mapsto w \end{cases} \text{ and } \varphi_{1,2} : \begin{cases} s \mapsto s \\ t \mapsto t \\ u \mapsto a^s \\ v \mapsto b \\ w \mapsto w \end{cases}$$

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$stwavs$	$stwav$

In particular, we get that $(stwav)^6 = \Delta_2^{15}(u_2)$ generates $Z(\mathcal{B}_{31}(u_2, u_2))$.

4.3.3. *Orbit of u_3.* The object u_3 admits 7 atomic loops d, e, f, t, u, v, w, with relations as follows:

$$\begin{cases} ue = ef = fu,\ wt = tf = fw, \\ df = fd,\ wv = vw,\ tv = vt,\ vf = fv, \\ udw = dwu = wud,\ dte = ted = edt, \\ utu = tut,\ uvu = vuv,\ dvd = vdv,\ eve = vev, \\ wtud = edwt. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_3}^+$ given by this presentation is not cancellative: we have $tudt \neq udtu$ and $wtudt = wudtu$ in $H_{u_3}^+$. Thus $H_{u_3}^+$ cannot be a Garside monoid.

The submonoid $L_{u_3}^+$ of $\mathcal{C}_{31}(u_3, u_3)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $utu = tut$ is the shortest common multiple of t and u. If right-lcms exists in $L_{u_3}^+$, then utu must be the right-lcm of u and t. We would then have

$$tut \preceq tudt \Rightarrow t \preceq dt$$

Which does not hold in $L_{u_3}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_3, u_3)$).

The relations defining $H_{u_3}^+$ give in particular $f := w^t$ and $e = f^u = w^{tu}$ in H_{u_3}. By deleting these generators, we get that $\mathcal{B}_{31}(u_3, u_3)$ is generated by d, t, u, v, w, with relations as follows:

$$\begin{cases} wv = vw,\ tv = vt, \\ udw = wud = dwu, \\ uvu = vuv,\ tut = utu,\ dvd = vdv,\ wtw = twt, \\ udtu = tudt. \end{cases}$$

The monoid given by this presentation is not cancellative: we have $tdwt \neq wtdw$ while $tdwtut = wtdwut$. The morphisms $\varphi_{3,1}$ and $\varphi_{1,3}$ are given by

$$\varphi_{3,1} : \begin{cases} t \mapsto t & d \mapsto s^u \\ u \mapsto u & e \mapsto w^{tu} \\ v \mapsto v & f \mapsto w^t \\ w \mapsto w & \end{cases} \text{ and } \varphi_{1,3} : \begin{cases} s \mapsto d^w \\ t \mapsto t \\ u \mapsto u \\ v \mapsto v \\ w \mapsto w \end{cases}$$

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$wtudvw$	$tudvw$

In particular, we get that $(tudvw)^6 = \Delta_2^{15}(u_3)$ generates $Z(\mathcal{B}_{31}(u_3, u_3))$.

4.3.4. *Orbit of* u_4. The object u_4 admits 12 atomic loops $g, h, k, l, m, n, o, p, s, t, u, v$, with relations as follows:

$$\begin{cases} gk = hg = kh,\ gs = lg = sl,\ gn = tg = nt,\ gp = vg = pv,\ ht = mh = tm, \\ kn = mk = nm,\ lo = tl = ot,\ so = ns = on,\ ut = tp = pu,\ uv = nu = vn, \\ gm = mg,\ go = og,\ hn = nh,\ st = ts,\ tv = vt,\ np = pn, \\ hus = shu = ush,\ hvo = ohv = voh,\ kpo = okp = pok, \\ lmv = mvl = vlm,\ smp = mps = psm, \\ hph = php,\ svs = vsv,\ mom = omo, \\ gnp = utv,\ htvl = lomv,\ khnps = lgomp,\ knps = somp, \\ khnps = utvsm,\ khn = tgm,\ khpo = vgok,\ lgomp = utvsm, \\ lgo = nst,\ lgmp = pvsm,\ usht = mhps,\ uvsh = ohuv. \end{cases}$$

The last three lines of relations can be omitted, as they are implied by the others. The monoid $H_{u_4}^+$ given by this presentation is not cancellative: we have $hpsh \neq pshp$ and $mhpsh = mpshp$ in $H_{u_4}^+$. Thus $H_{u_4}^+$ cannot be a Garside monoid.

The submonoid $L_{u_4}^+$ of $\mathcal{C}_{31}(u_4, u_4)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $hph = php$ is the shortest common multiple of h and p. If right-lcms exists in $L_{u_4}^+$, then hph must be the right-lcm of h and p. We would then have

$$hph \preceq hpsh \Rightarrow h \preceq sh$$

Which does not hold in $L_{u_4}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_4, u_4)$).The relations defining $H_{u_4}^+$ give in particular

$$g = n^t = u^{vt},\ k = h^g = h^{(u^{vt})},\ l = g^s = u^{tvs},\ m = h^t,\ n = u^v,\ o = n^s = u^{vs},\ p = u^t$$

in H_{u_4}. By deleting these generators, we get that $\mathcal{B}_{31}(u_4, u_4)$ is generated by h, s, t, u, v, with relations as follows:

$$\begin{cases} vt = tv,\ st = ts, \\ ush = shu = hus, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ tht = hth, \\ shvs = vshv. \end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have $shvs = vshv$ and $vsv = svs$ are common right-multiples of v and s, but their longest common divisor is vs,which is not a common righ-multiple of s and v. This also proves that the submonoid of $\mathcal{C}_{31}(u_4,u_4)$ generated by h,s,t,u,v does not admit right-lcms. The morphisms $\varphi_{4,1}$ and $\varphi_{1,4}$ are given by

$$\varphi_{3,1}:\begin{cases} g\mapsto u^{vt} & o\mapsto u^{vs}\\ h\mapsto w^{s} & p\mapsto u^{t}\\ k\mapsto u^{wtuwvsuvt} & s\mapsto s\\ l\mapsto u^{tvs} & t\mapsto t\\ m\mapsto w^{st} & u\mapsto u\\ n\mapsto u^{v} & v\mapsto v\end{cases}\text{ and }\varphi_{1,3}:\begin{cases} s\mapsto s\\ t\mapsto t\\ u\mapsto u\\ v\mapsto v\\ w\mapsto h^{u}\end{cases}$$

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$tuvshv$	$tuvsh$

In particular, we get that $(tuvsh)^6=\Delta_2^{15}(u_4)$ generates $Z(\mathcal{B}_{31}(u_4,u_4))$. We also have $((tuvsh)^t)^2=(uvsht)^2=\Delta^5(u_4)$ in this case.

4.3.5. *Orbit of* u_5. The object u_5 admits 10 atomic loops b,f,g,n,p,s,t,u,v,w, with relations as follows:

$$\begin{cases} ut=pu=tp,\ uv=nu=vn,\ gp=pv=vg,\\ gn=tg=nt,\ sv=vb=bs,\ wt=fw=tf,\\ st=ts,\ wv=vw,\ fv=vf,\ pn=np,\ tv=vt,\ tb=bt,\\ uws=suw=wsu,\ gfb=fbg=bgf,\ spf=fsp=pfs,\ wbn=nwb=bnw,\\ ufu=fuf,\ gsg=sgs,\ suw=wsu,\ sns=nsn,\ fnf=nfn\\ utv=gpn,\ uwsv=bsnw,\ gnfb=wtbg,\ wsut=pufs,\ pfsv=bsgf.\end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_5}^+$ given by this presentation is not cancellative: we have $ufsu\neq fsuf$ and $pufsu=pfsuf$ in $H_{u_5}^+$. Thus $H_{u_5}^+$ cannot be a Garside monoid.

The submonoid $L_{u_5}^+$ of $\mathcal{C}_{31}(u_5,u_5)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $ufu=fuf$ is the shortest common multiple of f and u. If right-lcms exists in $L_{u_5}^+$, then ufu must be the right-lcm of u and f. We would then have

$$ufu\preceq ufsu\Rightarrow u\preceq su$$

Which does not hold in $L_{u_5}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_5,u_5)$).

The relations defining $H_{u_5}^+$ give in particular

$$b=s^v,\ f=w^t,\ g=u^{vt},\ n=u^v,\ p=u^t$$

in H_{u_5}. By deleting these generators, we get that $\mathcal{B}_{31}(u_5,u_5)$ is generated by s,t,u,v,w, with the same relations as in $H_{u_1}^+$: we recover once again the obtained in Section 4.3.1. The morphisms $\varphi_{5,1}$ and $\varphi_{1,5}$ are given by

$$\varphi_{5,1}:\begin{cases} s\mapsto s & b\mapsto s^{v}\\ t\mapsto t & f\mapsto w^{t}\\ u\mapsto u & g\mapsto u^{vt}\\ v\mapsto v & n\mapsto u^{v}\\ w\mapsto w & p\mapsto u^{t}\end{cases}\text{ and }\varphi_{1,5}:\begin{cases} s\mapsto s\\ t\mapsto t\\ u\mapsto u\\ v\mapsto v\\ w\mapsto w\end{cases}$$

Maximal roots of the full-twist are given by the same expressions as for the object u_1.

4.3.6. *Orbit of u_6.* The object u_6 admits 10 atomic loops $b, f, g, n, o, p, q, r, s, v$, with relations as follows:

$$\begin{cases} vg = pv = gp,\ qn = fq = nf,\ qr = gq = rg,\ so = ns = on,\ vb = sv = bs, \\ vf = fv,\ qb = bq,\ sr = rs,\ np = pn,\ og = go, \\ vro = ovr = rov,\ qop = pqo = opq,\ spf = fsp = pfs, \\ bgf = fbg = gfb,\ brn = nbr = rnb, \\ vqv = qvq,\ vnv = nvn,\ qsq = sqs,\ sgs = gsg,\ ngn = gng,\ pqo = opq, \\ bsgf = pvfb,\ bsro = onvr,\ fqsp = onpf,\ fqbr = rgnb,\ pvqo = rovg \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_6}^+$ given by this presentation is not cancellative: we have $nvrn \neq vrnv$ and $onvrn = ovrnv$ in $H_{u_6}^+$. Thus $H_{u_6}^+$ cannot be a Garside monoid.

The submonoid $L_{u_6}^+$ of $\mathcal{C}_{31}(u_6, u_6)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $nvn = vnv$ is the shortest common multiple of n and v. If right-lcms exists in $L_{u_6}^+$, then nvn must be the right-lcm of n and v. We would then have

$$nvn \preceq nvrn \Rightarrow n \preceq rn$$

Which does not hold in $L_{u_6}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_6, u_6)$).

The relations defining $H_{u_6}^+$ give in particular

$$b = s^v,\ f = q^n,\ o = n^s,\ p = v^g,\ r = q^g$$

in H_{u_6}. By deleting these generators, we get that $\mathcal{B}_{31}(u_6, u_6)$ is generated by g, n, q, s, v, with relations as follows:

$$\begin{cases} nsn = sns,\ vgv = gvg,\ vsv = svs,\ qnq = nqn,\ vnv = nvn, \\ qgq = gqg,\ ngn = gng,\ qsq = sqs,\ sgs = gsg,\ vqv = qvq \\ gnvg = vgnv = nvgn,\ gqsg = qsgq = sgqs,\ nsgn = gnsg = sgns \\ vqsv = svqs = qsvq,\ qnvq = nvqn = vqnv,\ vgqnsv = svgqns \\ gqnsvgs = ngqnsvg \end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have that $vgqnsv = svgqns$ and $vsv = svs$ are common right multiples of s and v, but their longest common left-divisor is sv, which is not a common right-multiple of s and v. This also proves that the submonoid of $\mathcal{C}_{31}(u_6, u_6)$ generated by g, n, q, s, v does not admit right-lcms. The morphisms $\varphi_{6,1}$ and $\varphi_{1,6}$ are given by

$$\varphi_{6,1} : \begin{cases} b \mapsto s^v & p \mapsto u^t \\ f \mapsto w^t & q \mapsto t^{uwtv} \\ g \mapsto u^{vt} & r \mapsto u^{stuwtv} \\ n \mapsto u^v & s \mapsto s \\ o \mapsto u^{vs} & v \mapsto v \end{cases} \text{ and } \varphi_{1,6} : \begin{cases} s \mapsto s \\ t \mapsto p^{vn} \\ u \mapsto v^n \\ v \mapsto v \\ w \mapsto p^{vnf} \end{cases}$$

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$svgqns$	$svgqn$

In particular, we get that $(svgqn)^6 = \Delta_2^{15}(u_6)$ generates $Z(\mathcal{B}_{31}(u_6, u_6))$. We also have $svgqns = \Delta_2^3(u_6)$ in this case.

4.3.7. *Orbit of* u_7. The object u_7 admits 7 atomic loops g,k,m,n,o,p,s, with relations as follows:

$$\begin{cases} on=so=ns,\ kn=nm=mk,\\ go=og,\ gm=mg,\ pn=np,\\ pok=okp=kpo,\ psm=smp=mps,\\ omo=mom,\ gpg=pgp,\ gkg=kgk,\ gsg=sgs,\ gng=ngn,\\ kpon=somp.\end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_7}^+$ given by this presentation is not cancellative: we have $ompo\neq mpom$ and $sompo=smpom$ in $H_{u_7}^+$. Thus $H_{u_7}^+$ cannot be a Garside monoid.

The submonoid $L_{u_7}^+$ of $\mathcal{C}_{31}(u_7,u_7)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $omo=mom$ is the shortest common multiple of m and o. If right-lcms exists in $L_{u_7}^+$, then omo must be the right-lcm of m and o. We would then have

$$omo\preceq ompo\Rightarrow o\preceq po$$

Which does not hold in $L_{u_7}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_7,u_7)$). The relations defining $H_{u_7}^+$ give in particular $o=n^s$ and $k=n^m$ in H_{u_7}. By deleting these generators, we get that $\mathcal{B}_{31}(u_7,u_7)$ is generated by g,m,n,p,s, with relations as follows:

$$\begin{cases} gm=mg,\ pn=np,\\ smp=psm=mps,\\ sns=nsn,\ pgp=gpg,\ gsg=sgs,\ mnm=nmn,\ gng=ngn\\ sgns=nsgn=gnsg.\end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have that $sns=nsn$ and $sgns=nsgn$ are common right multiples of s and n, but their longest common left-divisor is ns, which is not a common right-multiple of s and n. This also proves that the submonoid of $\mathcal{C}_{31}(u_7,u_7)$ generated by g,m,n,p,s does not admit right-lcms. The morphisms $\varphi_{7,1}$ and $\varphi_{1,7}$ are given by

$$\varphi_{7,1}:\begin{cases} g\mapsto u^{vt} & p\mapsto u^t\\ k\mapsto u^{wtuwvsuvt} & s\mapsto s\\ m\mapsto w^{st} & \\ n\mapsto u^v & \\ o\mapsto u^{vs} & \end{cases} \text{ and } \varphi_{1,7}:\begin{cases} s\mapsto s\\ t\mapsto g^n\\ u\mapsto p^{gn}\\ v\mapsto p^g\\ w\mapsto p^{gkpn}\end{cases}$$

Maximal roots of the full-twist are given by

Maximal regular number d	20	24
d-th root of Δ^{60}	$gnmpsm$	$gnmps$

In particular, we get that $(gnmps)^6=\Delta_2^{15}(u_7)$ generates $Z(\mathcal{B}_{31}(u_7,u_7))$.

4.3.8. *Orbit of* u_8. The object u_8 admits 6 atomic loops g,m,n,p,s,t, with relations as follows:

$$\begin{cases} tg=nt=gn,\\ ts=st,\ pn=np,\ gm=mg,\\ psm=smp=mps,\\ tpt=ptp,\ tmt=mtm,\ pgp=gpg,\ sns=nsn,\ sgs=gsg,\ nmn=mnm.\end{cases}$$

The monoid $H_{u_8}^+$ given by this presentation is not cancellative: we have $psgpsg\neq gpsgps$ and $mpsgpsg=mgpsgps$ in $H_{u_8}^+$. Thus $H_{u_8}^+$ cannot be a Garside monoid.

The submonoid $L_{u_8}^+$ of $\mathcal{C}_{31}(u_8,u_8)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $pgp = gpg$ is the shortest common multiple of p and g. If right-lcms exists in $L_{u_8}^+$, then omo must be the right-lcm of m and o. We would then have

$$gpg \preceq gpsgps \Rightarrow g \preceq sgps$$

Which does not hold in $L_{u_8}^+$ (it does even hold in $\mathcal{C}_{31}(u_8,u_8)$, but $g^{-1}sgps$ is not generated positively by atomic loops). The relations defining $H_{u_8}^+$ give in particular $t = g^n$ in H_{u_8}. By deleting this generator, we get that $\mathcal{B}_{31}(u_8,u_8)$ is generated by g,m,n,p,s, with the same relations as in the end of Section 4.3.7. The morphisms $\varphi_{8,1}$ and $\varphi_{1,8}$ are given by

$$\varphi_{8,1}:\begin{cases} g\mapsto u^{vt}\\ m\mapsto w^{st}\\ n\mapsto u^{v}\\ p\mapsto u^{t}\\ s\mapsto s\\ t\mapsto t\end{cases} \quad\text{and } \varphi_{1,8}:\begin{cases} s\mapsto s\\ t\mapsto t\\ u\mapsto t^{p}\\ v\mapsto n^{tp}\\ w\mapsto m^{tmp}\end{cases}$$

Maximal roots of the full-twist are given by the same expressions as for the object u_7. Furthermore, we also have $(gnmps)^2 = \Delta_2^5(u_8)$ in this case.

Remark 4.8. As was pointed out to us by Jean Michel, the presentations with 5 generators that we give here are related by the Hurwitz action of the usual braid groups on the words giving 24-roots of the full-twist, as in [MM10, Section 6]. By identifying the atomic loops we consider with their image in $\mathcal{B}_{31}(u_1,u_1)$, we obtain for instance that (s,t,u,v,w), $(s,t,w,a,v), (t,u,d,v,w)$ and (t,u,v,s,h) lie in the same Hurwitz orbit. The words (s,u,v,w,t) and (s,v,g,q,n) also lie in the same Hurwitz orbit, as well as (u,v,w,t,s) and (g,n,m,p,s).

Appendix A. The Reidemeister-Schreier method for groupoids

A.1. **Presentations of categories and groupoids.**

Definition A.1. *([DDGKM, Definition II.1.4 and Definition II.1.32])*
An **oriented graph** *(or precategory) is a pair of sets $(\mathcal{O},\mathcal{S})$ endowed with two maps $s,t:\mathcal{S}\to\mathcal{O}$. The elements of $\mathcal{O}$ are called* **objects** *and those of $\mathcal{S}$ are called elements (or morphisms, or arrows). The maps s and t are called* **source** *and* **target**, *respectively.*

A **morphism** *between two oriented graphs $(\mathcal{O},\mathcal{S})$ and $(\mathcal{O}',\mathcal{S}')$ is given by two maps $\phi_0:\mathcal{O}\to\mathcal{O}'$, $\phi_1:\mathcal{S}\to\mathcal{S}'$ which preserve the source and target:*

$$\forall f\in\mathcal{S}, s(\phi_1(f)) = \phi_0(s(f)) \text{ and } t(\phi_1(f)) = \phi_0(t(f)).$$

Note that we make no assumption regarding the existence of loops or multiple arrows with the same source and target. In practice, we often amalgamate an oriented graph $(\mathcal{O},\mathcal{S})$ with its set of arrows $\mathcal{S}$. The set of objects will then be denoted by $\mathrm{Ob}(\mathcal{S})$ like for categories. We will sometimes denote by $\mathcal{S}(u,-)$ (resp. $\mathcal{S}(-,v)$, $\mathcal{S}(u,v)$) the set of arrows in $\mathcal{S}$ with source u (resp. with target v, with source u and target v).

Definition A.2. *([DDGKM, Definition II.1.28])*
Let $\mathcal{S}$ be an oriented graph. For $u,v\in\mathrm{Ob}(\mathcal{S})$, a **path** *of length $p\geqslant 1$ in $\mathcal{S}$ from u to v is a finite sequence $(g_1,\ldots,g_p)$ of elements of $\mathcal{S}$ such that*

$$s(g_1)=u,\ t(g_p)=v \text{ and } t(g_i)=s(g_{i+1}), \forall i\in [\![1,p-1]\!]$$

For $u\in\mathrm{Ob}(\mathcal{C})$, we also define an **empty path** *from u to itself, denoted by 1_u, and of length 0 by definition.*

Definition A.3. *([DDGKM, Definition II.1.28 and Proposition 1.33])*
Let $\mathcal{S}$ be an oriented graph. The **free category** *on $\mathcal{S}$, denoted by $\mathcal{S}^*$, is defined as follows*

- *The objects are the objects of $\mathcal{S}$.*
- *The morphisms $\mathcal{S}^*(u,v)$ are the paths from u to v in $\mathcal{S}$.*
- *Composition is given by concatenation of paths.*
- *The identity of some object u is the empty path 1_u.*

This category is free in the following sense: Let $\mathcal{S}$ be an oriented graph, and let $\mathcal{C}$ be a category. Any morphism of oriented graphs $\phi : \mathcal{S} \to \mathcal{C}$ induces a unique functor $\mathcal{S}^* \to \mathcal{C}$, sending a path $(g_1, \ldots, g_p)$ to the composition $\phi_1(g_1)\cdots\phi_1(g_p)$ in $\mathcal{C}$. In practice, a path $(f_1, \ldots, f_p)$ will often be denoted as a formal composition $f_1 \cdots f_p$ so that we have

$$\phi(f_1 \cdots f_p) = \phi_1(f_1)\cdots\phi_1(f_p).$$

This convenient definition of free category allows for defining relations and presentations of categories. Recall that a ***congruence*** on a category $\mathcal{C}$ is an equivalence relation $\equiv$ on $\mathcal{C}$ which is compatible with composition, that is, the conjunction of $f \equiv f'$ and $g \equiv g'$ implies $fg \equiv f'g'$ (if fg and $f'g'$ are defined of course). If $\equiv$ is a congruence on a category $\mathcal{C}$, one can form the quotient category $\mathcal{C}/\equiv$. It has the same objects as $\mathcal{C}$, and its morphisms are $\equiv$-equivalence classes of morphisms in $\mathcal{C}$.

Definition A.4. *([DDGKM, Definition II.1.36 and Lemma II.1.37])*
If $\mathcal{C}$ is a category, a **relation** *on $\mathcal{C}$ is a pair (g,h) of morphisms in $\mathcal{C}$ sharing the same source and the same target. If $\mathcal{C}$ is a category, and $\mathcal{R}$ is a family of relations on $\mathcal{C}$, there exists a smallest congruence $\equiv_{\mathcal{R}}^+$ of $\mathcal{C}$ which includes $\mathcal{R}$.*

The congruence $\equiv_{\mathcal{R}}^+$ is the reflexive-transitive closure of

$$\{(fgh, fg'h) \mid (g,g') \in \mathcal{R} \text{ or } (g',g) \in \mathcal{R}\}.$$

For readability purposes, it is convenient to write a relation (f,g) as an equality $f = g$ instead of a couple of paths. We use this convention from now on.

Definition A.5. *([DDGKM, Definition II.1.38])*
A **category presentation** *is a pair $(\mathcal{S},\mathcal{R})$, where $\mathcal{S}$ is an oriented graph, and $\mathcal{R}$ is a set of relations on $\mathcal{S}^*$. We call $\mathcal{S}$ the* **generators** *and $\mathcal{R}$ the* **relations**.
If $(\mathcal{S},\mathcal{R})$ is a category presentation, the quotient category $\mathcal{S}/\equiv_{\mathcal{R}}^+$ is denoted by $\langle \mathcal{S} \mid \mathcal{R}\rangle^+$.

Remark A.6. If we consider a graph $\mathcal{S}$ with one object, we recover the classical notion of monoid presentation.

Let $(\mathcal{S},\mathcal{R})$ be a categorical presentation, and let $\mathcal{C}$ be a category. Any morphisms of oriented graphs $\phi : \mathcal{S} \to \mathcal{C}$ such that

$$\forall f_1 \cdots f_p = g_1 \cdots g_q \in \mathcal{R}, \phi_1(f_1)\cdots\phi_1(f_p) = \phi_1(g_1)\cdots\phi_1(g_p) \in \mathcal{C}$$

induces a unique functor from the presented category $\langle \mathcal{S} \mid \mathcal{R}\rangle^+$ to $\mathcal{C}$. A first application of this property is the following lemma.

Lemma A.7. *Let $\mathcal{C} := \langle S \mid R\rangle^+$ be a presented category. If $\mathcal{R}$ consists of relations between paths of the same length in $\mathcal{S}^*$, then the category $\mathcal{C}$ is homogeneous. We then say that $\langle S \mid R\rangle^+$ is a* **homogeneous presentation**.

Proof. First, we note that the category $\mathcal{S}^*$ is homogeneous. The length functor is given by the length of the paths. By assumption, this functor induces a well-defined functor from $\mathcal{C}$ to $(\mathbb{Z}_{\geqslant 0}, +)$, such that elements of $\mathcal{S}$ are sent to 1. This functor is a length functor for $\mathcal{C}$ since it is generated by $\mathcal{S}$. $\square$

The notion of categorical presentation is useful for defining enveloping groupoids: let $\mathcal{C} = \langle \mathcal{S} \mid \mathcal{R}\rangle^+$ be a presented category. We consider $\overline{\mathcal{S}}$ a formal copy of $\mathcal{S}$, with source and target reversed. We consider the set $I(\mathcal{S})$ of relations on $\mathcal{S} \sqcup \overline{\mathcal{S}}$ defined by

$$\forall x \in \mathcal{S}, x\overline{x} = 1_{s(x)} \text{ and } \overline{x}x = 1_{t(x)}$$

Lemma A.8. *([DDGKM, Definition II.3.3 and Proposition II.3.5])*
The category $\mathcal{G}(\mathcal{C}) := \langle \mathcal{S} \sqcup \overline{\mathcal{S}} \mid \mathcal{R} \cup I(\mathcal{S})\rangle^+$ is a groupoid, called the **enveloping groupoid** *of $\mathcal{C}$. The inclusion map $\mathcal{S} \hookrightarrow \mathcal{S} \cup \overline{\mathcal{S}}$ induces a functor $\iota : \mathcal{C} \to \mathcal{G}(\mathcal{C})$. Every functor $\phi : \mathcal{C} \to \mathcal{G}$ where $\mathcal{G}$ is a groupoid induces a unique functor $\widetilde{\phi} : \mathcal{G}(\mathcal{C}) \to \mathcal{C}$ such that $\widetilde{\phi} \circ \iota = \phi$.*

This universal property of the enveloping groupoids ensures that it depends only on $\mathcal{C}$ and not on its presentation. By convention, if $\mathcal{C} = \langle \mathcal{S} \mid \mathcal{R}\rangle^+$ is a presented category, the presentation of the enveloping groupoid of $\mathcal{C}$ will be denoted by $\langle \mathcal{S} \mid \mathcal{R}\rangle$.

Remark A.9. Every category $\mathcal{C}$ admits a standard presentation, where the generators are all the morphisms in $\mathcal{C}$, and the relations are all couples (fg, h) for $f, g, h \in \mathcal{C}$ satisfying $fg = h$. Thus the enveloping groupoid can be defined for any category.

A.2. **Schreier transversal and presentation.** Let $\mathcal{G} = \langle \mathcal{S} \mid \mathcal{R}\rangle$ be a presented connected groupoid. We denote by $\mathcal{F}(\mathcal{S}) := \langle \mathcal{S} \mid \varnothing\rangle$ the free groupoid on the graph $\mathcal{S}$. The identity $\mathcal{S} \to \mathcal{S}$ induces a quotient map $\varphi : \mathcal{F}(\mathcal{S}) \to \mathcal{G}$. Just like for free groups, a morphism in $\mathcal{F}(\mathcal{S})$ is represented by a unique ***reduced path***, that is a path comporting no subpath of the form ss^{-1} or $s^{-1}s$ for $s \in \mathcal{S}$.

Definition A.10. *Let u be an object of $\mathcal{F}(\mathcal{S})$. A* **Schreier transversal** *of $\mathcal{F}(\mathcal{S})$ rooted in u is a family of reduced paths $\{m_v\}_{v \in \mathrm{Ob}(\mathcal{F}(\mathcal{S}))}$ satisfying:*
- For all object v of $\mathcal{S}$, the path m_v has source u and target v.
- The family $\{m_v\}$ is stable under prefix. It particular $m_u = 1_u$.

Remark A.11. Since $\mathcal{G}$ is connected, it is also the case of $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$. In particular, a Schreier transversal of $\mathcal{F}(\mathcal{S})$ rooted in u exists for all object u of $\mathcal{F}(\mathcal{S})$.

Let u_0 be an object of $\mathcal{F}(\mathcal{S})$, and let $\{m_v\}$ be a Schreier transversal in $\mathcal{F}(\mathcal{S})$ rooted in u_0. For $s \in \mathcal{S}(u, v)$, we define $\gamma(s) := m_u s m_v^{-1} \in \mathcal{F}(\mathcal{S})(u_0, u_0)$. Let S_1 be the set of all elements $\gamma(s) \neq 1_{u_0}$ for $s \in \mathcal{S}$.

Lemma A.12. *The group $\mathcal{G}(u_0, u_0)$ is generated by the $\varphi(\gamma(s))$ for $\gamma(s) \in S_1$.*

Proof. Let $g \in \mathcal{G}(u_0, u_0)$. Since $\mathcal{G}$ is generated by $\mathcal{S}$, we can write

$$g = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}$$

with $s_i \in \mathcal{S}$ for $i \in [\![1, k]\!]$ and $\varepsilon_i \in \{\pm 1\}$ for $i \in [\![1, k]\!]$. We denote by u_i the target of $s_i^{\varepsilon_i}$ for $i \in [\![1, k-1]\!]$. In $\mathcal{F}(\mathcal{S})$ we have

$$\begin{aligned} s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} &= s_1^{\varepsilon_1} m_{u_1}^{-1} m_{u_1} \cdots m_{u_{k-1}}^{-1} m_{u_{k-1}} s_k^{\varepsilon_k} \\ &= m_{u_0} s_1^{\varepsilon_1} m_{u_1}^{-1} m_{u_1} \cdots m_{u_{k-1}}^{-1} m_{u_{k-1}} s_k^{\varepsilon_k} m_{u_0}^{-1} \\ &= \gamma(s_1)^{\varepsilon_1} \cdots \gamma(s_k)^{\varepsilon_k}. \end{aligned}$$

Thus, we have $g = \varphi(\gamma(s_1))^{\varepsilon_1} \cdots \varphi(\gamma(s_k))^{\varepsilon_k}$ in $\mathcal{G}(u_0, u_0)$. □

As $\mathcal{G}$ is a groupoid, every relation defining $\mathcal{G}$ can be rewritten as a relation of the form

$$s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = 1_u$$

where $s_i \in \mathcal{S}$ and $\varepsilon_i \in \{\pm 1\}$ for $i \in [\![1, k]\!]$, and u is the source of $s_1^{\varepsilon_1}$.

Proposition A.13. *(Reidemeister-Schreier method for groupoids)*
Let S^* be a set of elements $\gamma(s)^*$ in one-to-one correspondence with those of S_1. Let also R^* be the set of all relations

$$\gamma^*(s_1)^{\varepsilon_1} \cdots \gamma^*(s_k)^{\varepsilon_k} = 1$$

where $s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = 1_u$ is in $\mathcal{R}$. The map $S^* \to \mathcal{G}(u_0, u_0)$ sending $\gamma(s)^*$ to $\varphi(\gamma(s))$ induces an isomorphism of groups between $G^* := \langle S^* \mid R^*\rangle$ and $\mathcal{G}(u_0, u_0)$.

Proof. First, we prove that the map $\gamma(s)^* \mapsto \varphi(\gamma(s))$ is compatible with the set of relations R^*. Let $\gamma^*(s_1)^{\varepsilon_1} \cdots \gamma^*(s_k)^{\varepsilon_k} = 1$ be in R^*. We have an equality $s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = 1_u$ in $\mathcal{G}$, and we have

$$\varphi(\gamma(s_1))^{\varepsilon_1} \cdots \varphi(\gamma((s_k))^{\varepsilon_k} = \varphi(m_u)s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}\varphi(m_u)^{-1} = 1_{u_0}.$$

Let $\pi : G^* \to \mathcal{G}(u_0, u_0)$ be the morphism induced by $\gamma(s)^* \mapsto \varphi(\gamma(s))$. We know that π is surjective by Lemma A.12.

Conversely, the map $\mathcal{S} \to S^*$ sending s to $\gamma(s)^*$ induces a functor $\phi : \mathcal{F}(\mathcal{S}) \to G^*$. Let v be an object of $\mathcal{F}(\mathcal{S})$. We show that ϕ sends m_v to 1 by induction on the length of m_v as an $\mathcal{S}$-path. First if $m_v = s_1^{\varepsilon_1}$ has length 1, we have $\phi(s_1^{\varepsilon_1}) = m_{u_0}m_v m_v^{-1} = 1$. Now if $m_v = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}$ is a decomposition of m_v on $\mathcal{S}$, we denote by v' the source of $s_k^{\varepsilon_k}$. By definition of a Schreier transversal we have $m_{v'} = s_1^{\varepsilon_1} \cdots s_{k-1}^{\varepsilon_{k-1}}$ and $\phi(m_{v'}) = 1$ by induction hypothesis. As we also have $\phi(s_k^{\varepsilon_k}) = m_{v'}s_k^{\varepsilon_k}m_v^{-1} = m_v m_v^{-1} = 1$, we get $\phi(m_v) = 1$.

By definition of the set R^*, ϕ induces a functor $\overline{\phi} : \mathcal{G} \to G^*$. We call ι the restriction of this functor to $\mathcal{G}(u_0, u_0)$. Let $\gamma(s)^*$ be a generator of G^*, with $s \in \mathcal{S}(u, v)$. We have

$$\begin{aligned}
\iota(\pi(\gamma(s)^*)) &= \iota(\varphi(\gamma(s))) \\
&= \iota(\varphi(m_u)s\varphi(m_v)^{-1}) \\
&= \overline{\phi}(\varphi(m_u))\overline{\phi}(s)\overline{\phi}(\varphi(m_v))^{-1} \\
&= \phi(m_u)\gamma(s)^*\phi(m_v)^{-1} = \gamma(s)^*
\end{aligned}$$

So $\iota \circ \pi$ induces the identity on the generators of G^*. We get $\iota \circ \pi = 1_{G^*}$ and π is injective. $\square$

Corollary A.14. *If $\mathcal{G}$ is a finitely presented groupoid, then for every object u of $\mathcal{G}$, the group $\mathcal{G}(u, u)$ is finitely presented.*

A.3. **The particular case of a subgroup of a presented group.** We explain rapidly how our result on groupoids can be used to recover the classical result of Reidemeister and Schreier about presentation of subgroups.

Definition A.15. *Let G be a group, with H a subgroup. The* **groupoid of cosets** *G_H of G and H is the groupoid defined by*
- *The objects are the right-cosets of H in G.*
- *Morphisms between two cosets Hg and Hg' are elements x of G such that $Hgx = Hg'$.*
- *Composition is given by the product in G.*

The underlying oriented graph of the groupoid of cosets is simply the graph of the action of G on the right-cosets of H in G. For each coset Hg and each element x of G, we denote by $x[Hg]$ the unique morphism $Hg \to Hgx$ corresponding to the action of the element x.

By definition of G_H, there is a natural functor $\pi : G_H \to G$, sending a morphism $x[Hg]$ to x. For two cosets Hg and Hg', the set of morphisms between Hg and Hg' is given by

$$G_H(Hg, Hg') = \{x \in G \mid Hgx = Hg'\} = g^{-1}Hg'.$$

In particular, we obtain that $G_H(H, H) = H$ and H is an automorphism group in G_H.

Since the natural action of G on $H \setminus G$ is transitive, the groupoid G_H is connected. In particular we can apply the Reidemeister-Schreier method to obtain a presentation of H from one of G_H.

Now, if $\langle X, R\rangle$ is a presentation of the group G. We denote by $F(X)$ the free group over the set X. We call $\mathcal{X}$ the subgraph of G_H made of those morphisms in G_H whose image under π lie in X. The restriction of π to $\mathcal{X}$ induces a functor $\widetilde{\pi} : \mathcal{F}(\mathcal{X}) \to F(X)$. The inclusion $\mathcal{X} \to G_H$ (resp. $X \to G$) induces a functor $\varphi : \mathcal{F}(\mathcal{X}) \to G_H$ (resp. $\phi : F(X) \to$

G). We have the following commutative square

$$\begin{tikzcd} \mathcal{F}(\mathcal{X}) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\varphi"'] & F(X) \arrow[d, "\phi"] \\ G_H \arrow[r, "\pi"'] & G \end{tikzcd}$$

Let $m := x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k}$ be a word in $F(X)$, and let Hg be a right-coset. There is a unique path in $\mathcal{F}(\mathcal{X})$ which starts at Hg and whose image under $\widetilde{\pi}$ is m. This path is given by

$$x_1[Hg]^{\varepsilon_1} x_2[Hgx_1^{\varepsilon_1}]^{\varepsilon_2} \cdots x_k[Hgx_1^{\varepsilon_1} \cdots x_{k-1}^{\varepsilon_{k-1}}]^{\varepsilon_k}$$

it will be denoted by $m[Hg]$. Note that $\varphi(m[Hg]) = \phi(m)[Hg]$. In particular we see that $\mathcal{X}$ generates G_H since X generates G.

Lemma A.16. *The groupoid of cosets G_H admits the presentation $\langle \mathcal{X} \mid \mathcal{R} \rangle$, where the set $\mathcal{R}$ consists of relations $r[Hg] = 1_{Hg}$, where $r = 1$ lies in R and Hg lies in $H \setminus G$.*

Proof. Let $\mathcal{G}$ be the groupoid presented by $\mathcal{X}$ and $\mathcal{R}$. Since the graph $\mathcal{X}$ generates G_H, the natural functor $\mathcal{G} \to G_H$ is surjective on morphisms. Consider now two paths in $\mathcal{F}(\mathcal{X})$

$$m := x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \text{ and } m' := y_1^{\eta_1} y_2^{\eta_2} \cdots y_m^{\eta_m}.$$

By definition, saying that these two paths induce the same morphism in G_H amounts to say that they share the same source and that the two words $\widetilde{\pi}(m)$ and $\widetilde{\pi}(m')$ induce the same element in G.

Suppose that m and m' induce the same element in G_H. We have $\widetilde{\pi}(m) = \widetilde{\pi}(m')$ in G, and there is a finite sequence of words $m_1, \ldots, m_p$ in $F(X)$ such that

- $m_1 = \widetilde{\pi}(m), m_p = \widetilde{\pi}(m')$
- For $i \in [\![1, p-1]\!]$, m_i is equivalent to m_{i+1} by the use of one relation in R.

Let Hg be the common source of m and m' in $\mathcal{F}(\mathcal{X})$. We consider the paths $m_i[Hg]$ in $\mathcal{F}(\mathcal{X})$. We have $m_1[Hg] = m, m_p[Hg] = m'$. For $i \in [\![1, p-1]\!]$, the path $m_i[Hg]$ is equivalent to $m_{i+1}[Hg]$ by the use of one relation in $\mathcal{R}$ by definition.

Thus m and m' induce the same element in $\mathcal{G}$, and the natural functor $\mathcal{G} \to G_H$ is then injective. $\square$

Let now $\widetilde{H} = \phi^{-1}(H) \subset F(X)$ be the preimage of H under ϕ. Recall from [LS01, Proposition I.3.8] that a Schreier transversal (in the classical sense) for $\widetilde{H}$ in $F(X)$ is a set of words T such that

- The map $t \mapsto Ht$ is a bijection between T and $H \setminus G$.
- The set T is stable under prefix (in particular, the empty word lies in T).

Let T be a Schreier transversal in the classical sense for $\widetilde{H}$ and $F(X)$. The set of paths $\{t[H]\}_{t \in T}$ is a Schreier transversal rooted in H in $\mathcal{F}(\mathcal{X})$ in the sense of Definition A.10. Conversely, if $\{m_{Hg}\}_{Hg \in \mathrm{Ob}(\mathcal{F}(\mathcal{X}))}$ is a Schreier transversal rooted in H in the sense of Definition A.10, then the set of words $\{\widetilde{\pi}(m_{Hg})\}_{Hg \in H \setminus G}$ is a Schreier transversal for $\widetilde{H}$ and $F(X)$ in the classical sense.

Let T be a Schreier transversal for $\widetilde{H}$ in $F(X)$, and let $\{m_{Hg}\}_{Hg \in \mathrm{Ob}(\mathcal{F}(\mathcal{X}))}$ be the associated Schreier transversal of $\mathcal{F}(\mathcal{X})$ rooted in H. For $g \in G$, let $\overline{g}$ denote the element of T such that $Hg = H\overline{g}$.

The elements of $\mathcal{X}$ are given by

$$\mathcal{X} = \{x[Hg] \mid x \in X, Hg \in \mathrm{Ob}(G_H)\}.$$

The set of generators of $G_H(H,H) = H$ we obtain by our method is

$$\begin{aligned}&\{\gamma(x[Hg]) = m_{Hg}xm_{Hgx}^{-1} \mid x \in X, Hg \in \mathrm{Ob}(G_H)\}\\ =&\{\gamma(x[Ht]) = m_{Ht}xm_{Htx}^{-1} \mid x \in X, t \in T\}\\ =&\{tx(\overline{tx})^{-1} \mid x \in X, t \in T\}\end{aligned}$$

which is the same set as given in [LS01, Proposition 4.1]. Following [LS01, Proposition 4.1], we denote $\gamma(t,x) := tx(\overline{tx})^{-1}$. This element is equal to what we denoted by $\gamma(x[Ht])$.

Now for the relators. Let $r = 1$ be a relation of G. It induces the following family of relation on G_H:

$$\{r[Hg] = 1_{Hg} \mid Hg \in \mathrm{Ob}(G_H)\} = \{r[Ht] = 1_{Ht} \mid t \in T\}$$

Each relation $r[Ht] = 1_{Ht}$ induces a relation on $G_H(H,H)$, given by

$$m_{Ht}r[Ht]m_{Ht}^{-1} = 1$$

If $r = y_1y_2y_3\cdots y_k$ is expressed as a word in $X \cup X^{-1}$, we have

$$\begin{aligned}m_{Ht}r[Ht]m_{Ht}^{-1} &= \gamma(y_1[Ht])\gamma(y_2[Hty_1])\cdots\gamma(y_k[Hty_1\cdots y_{k-1}])\\ &= \gamma(t,y_1)\gamma(\overline{ty_1},y_2)\cdots\gamma(\overline{ty_1\cdots y_{k-1}},y_k)\\ &= \gamma(1,t)\gamma(t,y_1)\gamma(\overline{ty_1},y_2)\cdots\gamma(\overline{ty_1\cdots y_{k-1}},y_k)\gamma(\overline{ty_1\cdots y_k},t^{-1})\end{aligned}$$

since both $\gamma(1,t)$ and $\gamma(\overline{ty_1\cdots y_k},t^{-1})$ are trivial. Thus the relations we obtain with our method are the same as those given in [LS01, Proposition 4.1], and Proposition A.13 applied to a groupoid of cosets gives a new proof of [LS01, Proposition 4.1].

Remark A.17. Let X be a transitive G-set, and let $x \in X$. There is a natural isomorphism between the category of cosets for the stabilizer of x in G and the graph of the action of G on X. In particular, our method gives a way to compute a presentation of the stabilizer starting from the graph of the action.

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