REGULAR THEORY IN COMPLEX BRAID GROUPS

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Abstract. In his seminal paper [Bes15], Bessis introduces a Garside structure for the braid group of a well-generated irreducible complex reflection group. Using this Garside structure, he establishes a strong connection between regular elements in the reflection group, and roots of the "full twist" element of the pure braid group.

He then suggests that it would be possible to extend the conclusion of this theorem to centralizers of regular elements in well-generated groups. In this paper we give a positive answer to this question and we show moreover that these results hold for an arbitrary reflection group.

Contents

1. Introduction

1.1. **Reflection groups, braid groups.** We refer to [LT09] for classic results on complex reflection groups. Let V be a complex vector space. An element $s \in \mathrm{GL}(V)$ is called a ***(pseudo-)reflection*** if $\mathrm{Ker}\,(s - \mathrm{Id}_V)$ is a hyperplane of V. We call $\mathrm{Ker}\,(s - \mathrm{Id}_V)$ the ***reflecting hyperplane*** of s. A subgroup $W \leqslant \mathrm{GL}(V)$ is called a ***complex reflection group*** if it is finite and generated by reflections of V. We say that $n = \dim(V)$ is the ***rank*** of W.

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We say that a complex reflection group $W \leqslant \mathrm{GL}(V)$ of rank n is ***well-generated*** if it can be generated by a set of n reflections. Otherwise, we say that W is ***badly-generated***.

To a complex reflection group is associated a hyperplane arrangement $\mathcal{A}$, defined as the set of reflecting hyperplanes of the reflections of W. It is a well-known fact that W acts freely on $X = X(W) := V \setminus \bigcup \mathcal{A}$, where $\bigcup \mathcal{A}$ denotes the reunion of the hyperplanes belonging to $\mathcal{A}$. We can then define $P(W) = \pi_1(X)$ the ***pure braid group*** of W, and $B(W) = \pi_1(X/W)$ the ***braid group*** of W. The action of W on X gives a covering map $X \twoheadrightarrow X/W$, which in turn induces a short exact sequence.

$$1 \to P(W) \to B(W) \to W \to 1$$

To simplify the notations, the projection of some $b \in B(W)$ in W will be denoted by $\overline{b}$.

Finite irreducible complex reflection groups were classified by Shephard and Todd:

- A general infinite family $G(de, e, n)$, defined for positive integral parameters d, e, n, that we call the infinite series.
- 34 exceptional groups $G_4, \cdots, G_{37}$.

These groups may be separated in two different classes, depending on wether or not they are well generated. In his article [Bes15], Bessis extensively studied the case of braid groups of well-generated irreducible reflection groups. In particular, [Bes15, Theorem 12.4] establishes a link between regular elements of a well-generated group W and the roots of a particular element in its braid group. Our main goal is to extend this theorem to all irreducible complex reflection groups.

1.2. **Regular elements.** The theory of regular elements of complex reflection groups has been introduced in [Spr74]. Let d be a positive integer. We denote by μ_d the set of d-th roots of unity in $\mathbb{C}$, and $\zeta_d := e^{\frac{2i\pi}{d}}$.

Definition 1.1. *Let $W \leqslant \mathrm{GL}(V)$ be a complex reflection group and d a positive integer. An element $g \in W$ is called ζ_d-**regular** if it admits regular eigenvectors for the eigenvalue ζ_d, that is if* $\mathrm{Ker}\ (g - \zeta_d) \cap X \neq \varnothing$.

One of the main results of this theory is that ζ_d-regular elements, should they exist, are all conjugate in W. Furthermore, for g a ζ_d-regular element, the centralizer $G := C_W(g)$ of g in W is again a complex reflection group acting on the eigenspace $\mathrm{Ker}\ (g - \zeta_d)$.

In particular, this allows the study of some badly-generated complex reflection groups by means of finding a regular embedding in a well-generated reflection group. A good example of this technique is the embedding of the exceptional group G_{31} in the complexified Coxeter group G_{37}, as the centralizer of a 4-regular element (cf [Bes15, Example 1.10]).

In their article [BM97, Section 3.B], Broué and Michel predicted the existence of an analogue of Springer regular elements in braid groups: such elements would in turn allow the study of braid groups of some badly-generated reflection groups. A major step towards finding such an analogue is achieved in [Bes15, Theorem 12.4], for the case of well-generated reflection groups. This paper extends the conclusions of Bessis to all complex reflection groups.

In [BMR98, Notation 2.3], Broué-Malle-Rouquier consider a particular central element in the pure braid group $P(W)$, represented by the loop $t \mapsto e^{2i\pi t}x_0$ (where $x_0 \in X$ is a basepoint for the pure braid group). It was later shown that, as they conjectured, the center of P is cyclic and generated by this element, that we will denote z_P from now on. Our main theorem is

Theorem 1.2. *Let W be a complex reflection group, and d be a positive integer.*

(A) There exists d-th roots of z_P if and only if d is regular.

(B) When d is regular, the d-th roots of z_P form a conjugacy class of $B(W)$, and they are mapped to ζ_d-regular elements in W.

(C) Let ρ be a d-th root of z_P, and w its image in W. The centralizer $C_{B(W)}(\rho)$ is isomorphic to the braid group $B(W')$ of the centralizer $W' := C_W(w)$.

A more precise version of property (C) will be given in Proposition 2.3 below. In the following we will show that properties $(A), (B)$ and (C) hold for different classes of groups.

We already know from [Bes15, Theorem 12.4] that $(A), (B)$ and (C) hold for a well-generated irreducible reflection group. So it only remains to show that $(A), (B), (C)$ hold for badly-generated reflection groups. Our proof consists of the following steps:

After recalling known useful results in Sections 2 and 3, we prove in Section 4 that properties $(A), (B), (C)$ hold for the members of the infinite family that are badly-generated (that is the $G(de, e, n)$ where $d \geqslant 2$ or $e \geqslant 2$). We then show in Section 5 that, if W is a reflection group for which $(A), (B), (C)$ hold, then $(A), (B), (C)$ also hold for every centralizer of a regular element of W. In Section 6, we show that if W and W' are reflection groups with the same (reflection) degrees and (reflection) codegrees, then properties $(A), (B), (C)$ for W implies properties $(A), (B), (C)$ for W'. Lastly, the two remaining groups, G_{12} and G_{13}, are studied separately in Section 7.

This work will be a part of my PhD thesis, done under the supervision of Ivan Marin. I thank him very much for his precious help, especially in Section 4.

2. Preliminary results

From now on, W will denote an irreducible complex reflection group, $\mathcal{A}$ its hyperplane arrangement, and d a positive integer. We also choose a basepoint x_0 in X and we set

$$P(W) := \pi_1(X, x_0),\ B(W) := \pi_1(X/W, \overline{x_0})$$

If x_1 is another basepoint, then since X is path-connected, we can always consider a path from x_0 to x_1. Such a path induces an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P(W) & \longrightarrow & B(W) & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow \simeq & & \| & & \\ 1 & \longrightarrow & \pi_1(X, x_1) & \longrightarrow & \pi_1(X/W, \overline{x_1}) & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

which shows that the change of basepoint in X yields an isomorphism of groups over W between $\pi_1(X/W, \overline{x_1})$ and $B(W) = \pi_1(X/W, \overline{x_0})$.

2.1. Centers of complex braid groups.

A long-standing task in braid group theory has been to determine the centers of both $B(W)$ and $P(W)$. A final statement has been obtained in [DMM11], completing previous results and conjectures of [BMR98] and [Bes15]:

Theorem 2.1. *([DMM11, Theorems 1.1, 1.2, 1.3])*
Assume that W is irreducible. The centers of $P(W), B(W)$ and W are cyclic, respectively generated by elements $z_P, z_B, \overline{z_B}$ satisfying $z_B^{|Z(W)|} = z_P$. In particular, we have a short exact sequence

$$1 \to Z(P(W)) \to Z(B(W)) \to Z(W) \to 1.$$

The element z_P is called the ***full-twist*** of W, and it has a topological origin: it is the homotopy class of the loop $z_{x_0} : t \mapsto e^{2i\pi t} x_0$. If x_1 is another basepoint, and γ is a path from x_0 to x_1, then it is easy to show that the concatenated paths $\gamma * z_{x_1}$ and $z_{x_0} * \gamma$ are homotopic. This means that the isomorphism between $B(W) = \pi_1(X/W, \overline{x_0})$ and $\pi_1(X/W, \overline{x_1})$ sends z_P to the homotopy class of z_{x_1}. This allows us to define z_P without mentioning the basepoint.

This full-twist will play a key role in defining analogues to regular elements, which will be the roots of z_P. The first important task is to define, for regular elements, lifts which are roots of z_P. We recall this construction now.

Let $g \in W$ be a ζ_d-regular element and $x \in X$ be a ζ_d-regular eigenvector for g. We define a path in X by

$$\widetilde{g} : t \mapsto e^{\frac{2i\pi t}{d}} x$$

Since this path ends at $\zeta_d x = g.x$, it induces a well-defined element of $\pi_1(X/W, \overline{x})$. This element is a d-th root of z_P and a lift of g. Now if $y \in X$ is another basepoint, then we

choose a path from x to y in X. We get an isomorphism $\pi_1(X/W, \overline{x}) \simeq \pi_1(X/W, \overline{y})$, which sends the homotopy class $[\widetilde{g}]$ of $\widetilde{g}$ to an element of $\pi_1(X/W, \overline{y})$, which is also a d-th root of z_P and a lift of g.

In general, a lift of g in $\pi_1(X/W, \overline{y})$ which is a d-th root of z_P will be called a ***regular lift*** of g (in $\pi_1(X/W, \overline{y})$).

2.2. **A precise rephrasing of property** (C)**.** The isomorphism whose existence is asserted by property (C) is explicit, and can be constructed as follows:

Let g be a ζ_d-regular element in W. The hyperplane arrangement for $G = C_W(g)$ is $\{H \cap V_g \mid H \in \mathcal{A}\}$ where $\mathcal{A}$ is the hyperplane arrangement for W. We deduce that

$$X_g := V_g \setminus \bigcup \mathcal{A}_g = V_g \setminus \left(\left(\bigcup \mathcal{A}\right) \cap V_g\right) = V_g \cap X$$

The embedding $X_g \to X$ induces a homeomorphism $p : X_g/G \to (X/W)^{\mu_d}$: Denoting $\widehat{x} = G.x$ and $\overline{x} := W.x$, we have $p(\widehat{x}) = \overline{x}$. The inverse bijection of p is a little bit harder to compute. Let $\overline{x} \in (X/W)^{\mu_d}$, this means that $\zeta_d \overline{x} = \overline{\zeta_d x} = \overline{x}$, so there is some $g' \in W$ such that $g'(x) = \zeta_d x$. Since $x \in X$, we get that g' is also a ζ_d-regular element of W. We obtain that $w^{-1}gw = g'$ for some $w \in W$, so

$$w^{-1}gw.x = \zeta_d x \Rightarrow g.(w.x) = \zeta_d(w.x)$$

this proves $w.x \in X_g$, and so $p^{-1}(\overline{x}) = \widehat{w.x}$. The following lemma is well-known (see for instance [Bro00, Chapter II]). We give a detailed proof here for the sake of clarity.

Lemma 2.2. *The map $X_g/G \to (X/W)^{\mu_d} \hookrightarrow X/W$ induces a group morphism*

$$B(G) = \pi_1(X_g/G, \widehat{x}) \to \pi_1(X/W, \overline{x})$$

whose image lies inside $C_{\pi_1(X/W,\overline{x})}([\widetilde{g}])$, where $[\widetilde{g}]$ is the regular lift of g in $\pi_1(X/W, \overline{x})$ constructed above.

Proof. Let $x \in X_g$ be our basepoint. Let us recall that, since we have a covering map $X_g \twoheadrightarrow X_g/G$, an loop γ from $\overline{x}$ to $\overline{x}$ in X_g/G induces a unique well-defined path $\widetilde{\gamma}$ in X_g, starting from x, and ending at some $w.x$. Furthermore the homotopy class $[\widetilde{\gamma}]$ depends only on $[\gamma] \in \pi_1(X_g/G, \overline{x})$. The path $w.\widetilde{\gamma}$ is then the unique lift of γ starting at $w.x$.

We know that g acts on V_g by multiplication by ζ_d, the path $\widetilde{g}$ defined above is a path inside X_g, and it induces an element $[\widetilde{g}]$ in $\pi_1(X_g/G, \widehat{x})$ (which is by construction a regular lift of g). Let now $[\gamma]$ be another element of $\pi_1(X_g/G, \widehat{x})$, the loop γ lifts to a unique path $\widetilde{\gamma}$ in X_g from x to some $w.x$ (with $w \in G$). The product $[\gamma][\widetilde{g}]$ is defined using the lift

$$\widetilde{\gamma} * (w.\widetilde{g}) : t \mapsto \begin{cases} \widetilde{\gamma}(2t) & \text{si } t \leqslant 1/2 \\ w.\widetilde{g}(2t-1) = e^{\frac{2i\pi(2t-1)}{d}} w.x & \text{si } t \geqslant 1/2 \end{cases}$$

And the product $[\widetilde{g}][\gamma]$ is defined using the lift

$$\widetilde{g} * (g.\widetilde{\gamma}) : t \mapsto \begin{cases} \widetilde{g}(2t) = e^{\frac{4i\pi}{d}}.x & \text{si } t \leqslant 1/2 \\ g.\widetilde{\gamma}(2t-1) = \zeta_d.\widetilde{\gamma}(2t-1) & \text{si } t \geqslant \frac{1}{2} \end{cases}$$

Both of these paths are homotopic to

$$f : t \mapsto e^{\frac{2i\pi t}{d}} \widetilde{\gamma}(t)$$

therefore $[\gamma][\widetilde{g}] = [\widetilde{g}][\gamma]$ in $\pi_1(X_g/G, \widehat{x})$. □

Composing the morphism $B(G) \to C_{\pi_1(X/W,\overline{x})}([\widetilde{g}])$ with an isomorphism $\pi_1(X/W, \overline{x}) \to B(W)$ coming from a path from x to x_0 in X yields a morphism $B(G) \to C_{B(W)}([\widetilde{g}])$, where $[\widetilde{g}]$ is a regular lift of g in $B(W)$. This is the morphism we need in order to properly state property (C).

Proposition 2.3. *Let W be a complex reflection group for which properties (A) and (B) hold. Let also ρ be a d-th root of z_P, let $g = \overline{\rho}$ be its image in W, and let $G := C_W(g)$. Then the morphism $B(G) \to C_{B(W)}([\widetilde{g}])$ defined above is an isomorphism. Since $[\widetilde{g}]$ and ρ are conjugate in $B(W)$ by property (B), we deduce an isomorphism of groups over G between $B(G)$ and $C_{B(W)}(\rho)$.*

This proposition is what we will call point (C) from now on, we will prove it for all badly-generated reflection groups in the subsequent sections.

2.3. **Garside theory.** As we said, a key ingredient in [Bes15] is the dual braid monoid, which is a Garside monoid for the braid group of a well-generated complex reflection group. Our approach will also use Garside structures and tools from Garside theory. We give a quick introduction here, in order to settle notations and to state the main results we are going to use. Our main reference is [DDGKM].

Definition 2.4. *Let M be a monoid. Assume that*

- *M is* **homogeneous***, that is if there is some monoid morphism $\ell : M \to \mathbb{N}$ such that M is generated by elements of positive length.*
- *M is left and right cancellative*
- *the posets $(M, \prec)$ and $(M, \succ)$ of left and right divisibility are both lattices (that is every pair of elements admits left and right lcms and gcds)*

On such a monoid, a **Garside structure** *is given by an element Δ, called a* **Garside element***, such that the set of left and right divisors of Δ are equal, finite, and generates M. This set of divisors will be noted $\mathcal{S}$, and its elements will be called* **simples** *of the Garside structure.*

This definition is far from minimal, but it is sufficient for the study of most braid groups. One can consult [DDGKM] for a more general setup.

We recall three important properties of Garside monoids:

- A Garside monoid (M, Δ) embeds in its envelopping group $\mathcal{G}(M)$ (that can be accurately described as a group of fractions of M). We say that $\mathcal{G}(M)$, endowed with M and Δ, is a ***Garside group***.
- Conjugation by Δ in $\mathcal{G}(M)$ restricts to an automorphism ϕ of M, defined by $\Delta\phi(x) = x\Delta$ for $x \in M$. We call this automorphism the ***Garside automorphism*** of the Garside monoid (M, Δ).
- If (M, Δ) is a Garside monoid, then (M, Δ^k) is also a Garside monoid for $k \in \mathbb{N}^*$.

The feature of Garside structures on braid groups that we are going to use is that z_P is equal to some power of Δ. That way a root of z_P becomes a ***periodic element*** in a Garside structure, in the sense of [DDGKM, Definition VIII.3.2]. An element γ in a Garside group is (p, q)-periodic if

$$\gamma^p = \Delta^q$$

In general we will say that an element is periodic if it is (p, q)-periodic for some couple (p, q). It is obvious that a (p, q)-periodic element is also (np, nq)-periodic for every integer n. There is some kind of converse statement, which will be useful for reducing computations:

Proposition 2.5. *([DDGKM, Proposition 3.34])*
Let B be a Garside group, and let $\gamma \in B$ be a (p, q)-periodic element. We denote by $p \wedge q$ the gcd of p and q.

(a) Up to conjugacy, we can assume that $\gamma^{p'} = \Delta^{q'}$, where $p' = \frac{p}{p \wedge q}$ and $q' = \frac{q}{p \wedge q}$ and for all $(u, v) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\leqslant 0}$ such that $p'u + q'v = 1$, we have $\gamma^v \Delta^u \in \mathcal{S}$.
(b) Furthermore, the value of $g = \gamma^v \Delta^u$ does not depend on the choice of the pair (u, v) and we have $\gamma^{p'v}\Delta^{p'u} = \Delta$, $\gamma^{q'v}\Delta^{q'v} = \gamma$.
(c) In particular, if Δ and γ commute, we have $g^{p'} = \Delta$ and $g^{q'} = \gamma$.

The last tool we need to introduce is the ***divided category*** (see [DDGKM, Definition XIV.1.2]), which will allow us to explicitly compute presentations of centralizers of regular elements when needed. Let (M, Δ) be a Garside monoid with set of simples $\mathcal{S}$. If m, n are two integers, we set

$$D_m^n(\Delta) := \left\{ (a_1, \cdots, a_m) \in \mathcal{S}^m \mid \prod_{i=1}^{m} a_i = \Delta \text{ and } (a_1, \cdots, a_m)^{\phi^n} = (a_1, \cdots, a_m) \right\}$$

where ϕ acts on tuples by

$$(a_1, \cdots, a_m)^{\phi} := (a_2, \cdots, a_m, a_1^{\phi})$$

For instance, we have that $D_2^0(\Delta)$ is in bijection with the set $\mathcal{S}$, or that $D_2^1(\Delta)$ is in bijection with elements s such that $s^2 = \Delta$.

For a couple of integers (p, q), we define a category $\mathcal{C}_p^q$ by using the sets $D_p^q(\Delta), D_{2p}^{2q}(\Delta)$ and $D_{3p}^{3q}(\Delta)$, considered respectively as a set of objects, generating morphisms, and relations:

- An element $(a_1, \cdots, a_{2p}) \in D_{2p}^{2q}(\Delta)$ seen as a morphism has source and target as follows

  $$(a_1, \cdots, a_{2p}) : (a_1 a_2, \cdots, a_{2p-1} a_{2p}) \to (a_2 a_3, \cdots, a_{2(p-1)} a_{2p-1}, a_{2p} a_1^{\phi})$$

  one can easily check that both the source and the target are elements of $D_p^q(\Delta)$.
- An element $(a_1, \cdots, a_{3p}) \in D_{3p}^{3q}(\Delta)$ induces the following relation

$$(a_1, a_2 a_3, \cdots, a_{3p-2}, a_{3p-1} a_{3p})(a_2, a_3 a_4, \cdots, a_{3p-1}, a_{3p} a_1^{\phi}) = (a_1 a_2, a_3, \cdots, a_{3p-2} a_{3p-1}, a_{3p})$$

  between three elements of $D_{2p}^{2q}(\Delta)$.

Theorem 2.6. *([DDGKM, Proposition XIV.1.8])*

(a) The application $(a_1, \cdots, a_{2p}) \mapsto a_1$ induces a functor from $\mathcal{C}_p^q$ to M. It will be called the **collapse functor** *(following [Bes15, Definition B.23]).*

(b) The (undirected) connected components of $\mathcal{C}_p^q$ are in one to one correspondance with conjugacy classes of (p, q)-periodic elements.

(c) Consider the enveloping groupoid $\mathcal{G}_p^q$ of $\mathcal{C}_p^q$, and x an object in this groupoid. The collapse functor induces a functor $\mathcal{G}_p^q \to \mathcal{G}(M)$, which sends $\mathcal{G}_p^q(x, x)$ to the centralizer of some (p, q)-periodic element of $\mathcal{G}(M)$.

In order to prove points (B) and (C) of our main theorem, it is enough to compute directly a presentation of the category $\mathcal{C}_p^q$. We can then see that this category is connected, which will prove point (B), and use its presentation to deduce a presentation of the centralizer of some (p, q)-periodic element in order to prove point (C). Here we see a first use of Proposition 2.5: replacing (p, q) by $\left(\frac{p}{p \wedge q}, \frac{q}{p \wedge q}\right)$ makes the set $D_{kp}^{kq}(\Delta)$ a lot easier to compute in practice.

3. First reductions for the proof

We begin our proof by noticing that some implications in the theorem are obvious:

- If d is a regular number for W, then there exists d-th roots of z_P. Indeed we have seen that for instance, a regular lift $\widetilde{g}$ of a regular element $g \in W$ is such a root.
- If the d-th roots of z_P are conjugate, and if d is regular, then all d-th roots of z_P are conjugate to every regular lift of a regular element. So they are mapped to ζ_d-regular elements of W.

Thanks to this, in practice we will only need to show that, if d-th roots of z_P exist, then d is regular, that d-th roots of z_P are conjugate, and that the morphism of Lemma 2.2 is an isomorphism.

There are 2 cases (namely those of G_{12} and G_{13}) that we will solve through explicit computations. In order to reduce these computations to a minimum, we give here some

results on regular elements and regular numbers. These results hold in a general setting, but we will mostly use them on groups of rank 2.

It is known that the existence of regular elements for a complex reflection group is conditioned by the degrees and codegrees of W (see for instance [LT09, Theorem 11.28]). When W is irreducible, we define $A(d)$ (resp. $B(d)$) as the degrees (resp. codegrees) of W that are divisible by d. The number d is regular for W if and only if $|A(d)| = |B(d)|$. When it is the case, the centralizer of a ζ_d-regular element is a reflection group, whose degrees (resp. codegrees) are the elements of $A(d)$ (resp. of $B(d)$).

This characterization in terms of divisibility has a first obvious consequence. If d is a regular number for W, then we have $|A(d)| = |B(d)|$. Moreover, if we define d' as the gcd of the reunion $A(d) \cup B(d)$, then we get that $d|d'$ (by definition of a gcd), and that d' is regular for W, with $A(d') = A(d)$ and $B(d') = B(d)$ (and d' is the greatest integer with this property). The number d' will be called the ***fundamental regular number*** associated to d. From this we get a bijection between conjugacy classes of centralizers of regular elements in W and fundamental regular numbers of W.

Consider $d_1 < \cdots < d_k$ the fundamental regular numbers of W. The regular numbers of W are exactly the integers that divide one of the d_i. The fundamental regular number d' associated to d is the smallest d_i such that $d|d_i$. We get

$$\forall d \in \mathbb{N}, i \in [\![2, k]\!],\ (A(d) = A(d_i) \text{ and } B(d) = B(d_i)) \Leftrightarrow (d|d_i \text{ and } d \nmid d_{i-1})$$

We write R_i the set of integers d satisfying those conditions (we set for R_1 the divisors of $|Z(W)|$, the fundamental regular number associated to 1).

Consider now $d \in R_i$ some regular number, we could look for a smallest number $d'' \in R_i$ dividing d. This exists by definition of R_i, but it depends on d. Consider for instance the dihedral group of order 24 $W = G(12, 12, 2)$. Its degrees and codegrees are $2, 12$ and $0, 10$. So 3 and 4 are coprime regular numbers, both with fundamental regular number 12, which is not the fundamental regular number for 1. However, it is easy to check by computer that the R_i are lattices when W is an exceptional group.

Lemma 3.1. *Let $g \in W$ be a ζ_d-regular element of W, d' the fundamental regular number associated to d, and let d'' be its smallest associated regular number. There exists some $\zeta_{d'}$-regular element g' such that $g'^{d'/d} = g$, and with*

$$C_W(g') = C_W(g) = C_W(g^{\frac{d}{d''}})$$

Proof. Let g be a ζ_d-regular element, and g_d be a $\zeta_{d'}$-regular element. We know that $g_d^{d'/d}$ is ζ_d-regular, so there is some $a \in W$ such that $a g_d^{d'/d} a^{-1} = g$. The element $g' := a g_d a^{-1}$ is $\zeta_{d'}$-regular and we have $g'^{d'/d} = g$ by definition. Moreover, the equality $C_W(g') = C_W(g)$ is obvious since we have $C_W(g') \subset C_W(g)$, and they are both reflection groups with the same degrees. The second equality of centralizers is obvious for the same reasons. □

Remark 3.2. *(Fundamental roots)*
We assume that properties (A) and (B) hold for W. Let ρ be a d-th root of z_P, and $g = \overline{\rho}$. We have that g is ζ_d-regular, so there is some $\zeta_{d'}$-regular element g' that is a root of g by Lemma 3.1. A regular lift of g' in $B(W)$ is by construction a $\frac{d'}{d}$-th root of some regular lift $\widetilde{g}$ of g. Since $\widetilde{g}$ and ρ are conjugate, we get that there is some ρ' such that $\rho'^{d'} = z_P$ and $\rho'^{d'/d} = \rho$. We call ρ' a **fundamental root** *of z_P, associated to ρ.*

Remark 3.3. *(Property (C) for groups of rank two)*
Let W be of rank 2 and such that properties (A) and (B) hold. We denote by d_1, d_2 the degrees of W and by d^* its non zero codegree. If $g \in W$ is a ζ_d-regular element, then we have either

- *d divides d_1, d_2, d^*, in which case $C_W(g) = W$, $V_g = V$, $X_g = X$ and property (C) is obvious since X/W and X_g/G are naturally homeomorphic.*

- *d divides d_1 and doesn't divide d_2 and d^*. Then d_1 doesn't divide d^*, and it is a regular number: the fundamental regular number associated to d. If ρ is a d-th root of z_P, and ρ' a fundamental root associated to ρ, then we have to show that $C_{B(W)}(\rho) = C_{B(W)}(\rho') = \langle\rho'\rangle$ to prove property (C). Thanks again to Lemma 3.1, we can always assume that d is equal to its own smallest associated number.*
- *d divides d_2 and doesn't divide d_1 and d^*. This corresponds to the above case when swapping d_1 and d_2 (as we did not assume that $d_1 < d_2$ here).*

4. Infinite series

This section consists in showing that properties (A), (B) and (C) hold for $W = G(de, e, n)$ a member of the infinite series. We need not consider the case where $d = 1$ or $e = 1$, as they give rise to well-generated reflection groups.

As is often the case, $G(de, e, n)$ will be tackled by tacking advantage of the natural embedding $G(de, e, n) \hookrightarrow G(de, 1, n)$ making $G(de, e, n)$ a finite index subgroup of $G(de, 1, n)$ (which is well-generated).

The first two points of our theorem for the infinite series are already considered in [CLL15, Section 4.3]. We borrow some of their notations and reproduce a synthetic proof for the sake of clarification.

We recall from [BMR98] that the group $B(de, 1, n) \simeq B(2, 1, n)$ is generated by $b_1, \cdots, b_r$ with the relations

(B_1) $b_1b_2b_1b_2 = b_2b_1b_2b_1$
(B_2) $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$ for $i \in [\![2, n-1]\!]$
(B_3) $b_ib_j = b_jb_i$ for $|i - j| > 1$

And that $B(de, e, n)$, seen as a subgroup of $B(de, 1, n)$, is generated by

$$z := b_1^e \quad t_i := b_1^{-i}b_2b_1^i (i \in \mathbb{Z}) \quad s_j = b_j (j \geqslant 3)$$

We consider the morphism[1] $\mathrm{wd} : B(de, 1, n) \to \mathbb{Z}$ defined by

$$\mathrm{wd}(b_1) = 1 \text{ and } \mathrm{wd}(b_i) = 0 \text{ for } i \in [\![2, n]\!]$$

We have that b is in $B(de, e, n)$ if and only if $\mathrm{wd}(b) \equiv 0[e]$. We will denote wd_e the morphism $b \mapsto \mathrm{wd}(b)[e]$, so we have $\mathrm{Ker}\ \mathrm{wd}_e = B(de, e, n)$.

We first notice that, as the hyperplane arrangements for $G(de, e, n)$ and $G(de, 1, n)$ in $\mathbb{C}^n$ are the same (since $d \geqslant 2$), the element z_P is the same for $B(de, e, n)$ and $B(de, 1, n)$. So there is no conflict of notation between $z_{P(de,e,n)}$ and $z_{P(de,1,n)}$.

The element $\varepsilon := b_n \cdots b_2b_1$ is a Garside element defining a dual structure of $B(de, 1, n)$. We have

$$\varepsilon^n = z_{B(de,1,n)} \text{ and } \varepsilon^{den} = z_P$$

On the other hand, we know that the degrees and codegrees of $B(de, 1, n)$ are

$$\begin{array}{cccc} de & 2de & \cdots & den \\ 0 & de & \cdots & de(n-1) \end{array}$$

So a number k is regular for $G(de, 1, n)$ if and only if it divides den. Applying point (B) to the well-generated group $G(de, 1, n)$, we get

Lemma 4.1. *Every periodic element in $B(de, 1, n)$ is conjugate to some power of ε.*

Proof. Let $\gamma \in B(de, 1, n)$ be such that $\gamma^p = \varepsilon^q$. Up to conjugacy, we can assume in particular that p and q are coprime (thanks to Proposition 2.5 applied to the dual Garside structure for $B(de, 1, n)$). Let k be the gcd of q and den, we have

$$\gamma^{\frac{pden}{k}} = \varepsilon^{\frac{qden}{k}} = z_P^{\frac{q}{k}}$$

[1] wd stands for "winding number", a notation we borrow from [CLL15, Definition 2.2]

We can then consider $z_P = \varepsilon^{den}$ as a Garside element for another Garside structure on $B(de, 1, n)$. Using again Proposition 2.5, γ is conjugate to some $\widetilde{\gamma}$ such that

$$\widetilde{\gamma}^{\frac{pden}{ka}} = z_P^{\frac{q}{ka}}$$

where a is the gcd of $\frac{q}{k}$ and $\frac{pden}{k}$ (so $ka = q \wedge (pden)$). Another use of Proposition 2.5, gives us a $\frac{pden}{ka}$-th root of z_P (since z_P is central). By [Bes15, Theorem 12.4] we obtain $\frac{pden}{ka} | den \Rightarrow p | ka$.

But since ka divides q, we get that p divides q, and since they are coprime, we have $p = 1$. Up to conjugacy, we can assume that a periodic element γ is such that $\gamma = \varepsilon^q$, which is what we wanted. □

This generalizes property (B) for $B(de, 1, n)$ to all periodic elements instead of just roots of z_P, which are a certain kind of periodic elements.

Since $\mathrm{wd}(\varepsilon) = 1$, we have $\lambda := \varepsilon^e \in B(de, e, n)$, this element is a nd-th root of z_P. The following proposition has been proven in [CLL15, Theorem 4.14], using the tools they introduce there. We provide a more direct proof here:

Proposition 4.2. *Every element in $B(de, e, n)$ that admits a central power is conjugate in $B(de, e, n)$ to some power of λ.*

Proof. Thanks to [DMM11, Theorem 1.4], since $B(de, e, n)$ is a finite index subgroup of $B(de, 1, n)$, we have $Z(B(de, e, n)) \subset Z(B(de, 1, n))$. So if $\rho \in B(de, e, n)$ admits a central power in $B(de, e, n)$, then it also does in $B(de, 1, n)$. Since the center of $B(de, 1, n)$ is generated by ε^n, we deduce that ρ is periodic in $B(de, 1, n)$, the last proposition then implies that ρ is conjugate in $B(de, 1, n)$ to some ε^r.

Since $B(de, e, n)$ is normal, we must have $\varepsilon^r \in B(de, e, n)$, that is $\mathrm{wd}(\varepsilon^r) = r \equiv 0[e]$, so $r = pe$ for some integer p and we have

$$\exists g \in B(de, 1, n) \mid \rho^g = \varepsilon^{pe} = \lambda^p$$

so ρ is conjugate to some power of λ, but the conjugating element g need not be in $B(de, e, n)$. But ε is an element of winding number 1 that centralizes λ, so assuming $i = \mathrm{wd}(g)$, we get

$$\rho^{g\varepsilon^{-i}} = (\lambda^{\varepsilon^{-i}})^p = \lambda^p$$

and $g\varepsilon^{-i} \in B(de, e, n)$ is an element of $B(de, e, n)$ that conjugates ρ to λ^p. □

Corollary 4.3. *The properties (A), (B) and (C) are true for $B(de, e, n)$ when $d, e \geqslant 2$.*

Proof. Let $\rho \in B(de, e, n)$ be such that $\rho^k = z_P$. By the last proposition ρ is conjugate to some λ^r and we have $\lambda^{rk} = z_P = \lambda^{dn}$. Since $B(de, e, n)$ has no torsion, we have $rk = dn$ and k divides dn, which gives point (A).

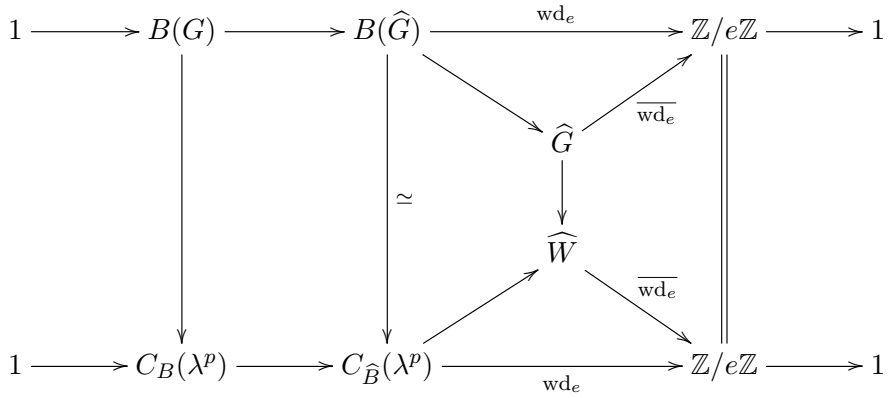
For point (B), any two k-th roots of z_P are conjugate to the same power of λ, and so they are conjugate. As powers of λ are mapped in $G(de, e, n)$ to some conjugate of a regular element, it is also the case of any k-th root of z_P.

For the last point, we only have to show the result for powers of λ. Set $W = G(de, e, n)$, $\widehat{W} = G(de, 1, n)$, B and $\widehat{B}$ their respective braid groups, $G = C_W(\overline{\lambda}^p)$ and $\widehat{G} = C_{\widehat{W}}(\overline{\lambda}^p)$. We have

$$C_B(\lambda^p) = C_{\widehat{B}}(\lambda^p) \cap B = C_{\widehat{B}}(\lambda^p) \cap \mathrm{Ker}\ (\mathrm{wd}_e) = \mathrm{Ker}\ (\mathrm{wd}_{e|C_{\widehat{B}}(\lambda^p)})$$
$$G = \widehat{G} \cap W = \widehat{G} \cap \mathrm{Ker}\ (\overline{\mathrm{wd}_e}) = \mathrm{Ker}\ (\overline{\mathrm{wd}_e}_{|\widehat{G}})$$

We already know that the morphism $B(\widehat{G}) \to C_{\widehat{B}}(\lambda^p)$ is an isomorphism. Furthermore, since ε commutes with λ and has winding number 1, the morphisms $\overline{\mathrm{wd}_e}_{|C_{\widehat{W}}(\overline{\lambda}^p)}$ and $\mathrm{wd}_{e|C_{\widehat{B}}(\lambda^p)}$

are surjective, and we have short exact sequences



We deduce from this that the morphism $B(G) \to C_B(\lambda^p)$ is an isomorphism. This proves property (C) for $B(de, e, n)$. □

5. Centralizers of regular elements

We now turn to the case of reflection groups that appear as centralizers of regular elements. More precisely we show that if properties $(A), (B), (C)$ hold for a given group W, then they also hold for centralizers of regular elements in W (this actually follows from [Bes15, Remark 12.6]).

Let W be a reflection group for which properties $(A), (B)$ and (C) are true, and let δ be a r-root of z_P in $B(W)$. We set $G = C_W(\overline{\delta})$, we know by hypothesis that $B(G)$ identifies to $C_{B(W)}(\delta)$, identification which sends z_P on the full twist defined for W. Let d be another integer, and set $k = d \wedge r$, $d = d'k$ and $r = r'k$.

Lemma 5.1. *Let $\rho \in B(G)$ be a d-th root of z_P. The value of $\rho^v \delta^u$ does not depend on the choice of (u, v) such that $d'u + r'v = 1$, we denote it by $q(\rho)$. It is an element of $B(W)$ such that*

(a) $q(\rho)^{d \vee r} = z_P$, where $d \vee r$ is the lcm of d and r.
(b) $q(\rho)^{d'} = \delta$
(c) $q(\rho)^{r'} = \rho$

Proof. This result is analogous to Proposition 2.5 but here, instead of considering the connection between a periodic element and a Garside element, we consider the connection between two periodic elements. This Lemma could be proved using Proposition 2.5 in a divided category but here, it is easy to give a more direct proof:

Let (u, v) be such that $d'u + r'v = 1$, we have

$$(\rho^v \delta^u)^{d \vee r} = \rho^{dr'v} \delta^{d'ru} = z_P^{r'v + d'u} = z_P$$

Let now (u', v') be another pair such that $d'u' + r'v' = 1$. It is known that $v' = v + d'q$, $u' = u - r'q$ for some integer q. We obtain

$$(\rho^{v'} \delta^{u'}) = \rho^v \rho^{d'q} \delta^u \delta^{-r'q} = (\rho^v \delta^u)(\rho^{d'} \delta^{-r'})^q$$

and since $(\rho^{v'} \delta^{u'})^{d \vee r} = z_P$, we have $(\rho^{d'} \delta^{-r'})^{q(d \vee r)} = 1$. Since $B(W)$ is without torsion, we have $\rho^{v'} \delta^{u'} = \rho^v \delta^u$ as we claimed.

Then, since d' and r are also coprime, we can consider (a, b) such that $d'a + rb = 1$, we then get

$$q(\rho)^{d'} = (\rho^{kb} \delta^a)^{d'} = \rho^{db} \delta^{ad'} = z_P^b \delta^{1-rb} = \delta$$

Likewise, by taking x, y such that $dx + r'y = 1$, we get $q(\rho)^{r'} = \rho$. □

This Lemma is the key ingredient of this section:

(*A*) If ρ is a d-th root of z_P in $B(G)$, then $q(\rho)$ is a $d \vee r$-th root of z_P in $B(G) \subset B(W)$. So $d \vee r$ is regular for W by point (*A*) for W: it divides as many degrees as codegrees of W. But by definition of the lcm, the (co)degrees of G divided by r are exactly the (co)degrees of W divided by d and r (i.e divided by $d \vee r$). So d is regular for G.

(*B*) If ρ and ρ' are two d-th roots of z_P in $B(G)$, then $q(\rho)$ and $q(\rho')$ are conjugate by some $g \in B(W)$. But then
- $g\delta g^{-1} = g(q(\rho)^{d'})g^{-1} = q(\rho')^{d'} = \delta$, so $g \in B(G) = C_{B(W)}(\delta)$
- $g\rho g^{-1} = g(q(\rho)^{r'})g^{-1} = q(\rho')^{r'} = \rho'$

and ρ and ρ' are conjugate by $g \in B(G)$.

(*C*) Let ρ be a d-th root of z_P in $B(G)$ and w its image in G. Again thanks to Lemma 5.1, we have

$$C_{B(G)}(\rho) = B(G) \cap C_{B(W)}(\rho) = C_{B(W)}(\rho) \cap C_{B(W)}(\delta) = C_{B(W)}(q(\rho))$$

The same reasoning gives $C_{B(G)}(\widetilde{w}) = C_{B(W)}(q(\widetilde{w}))$ and $G' = C_W(\overline{q(\widetilde{w})})$. The morphism $B(G') \to C_{B(G)}(\widetilde{w})$ is the same morphism as $B(C_W(\overline{q(\widetilde{w})})) \to C_{B(W)}(q(\widetilde{w}))$, which is known to be an isomorphism, this concludes the proof.

It is known that G_{31} and G_{22} respectively admits regular embeddings into G_{37} and G_{30}. Since both G_{37} and G_{30} are well-generated groups for which the theorem is already known to hold, we get that the theorem is also true for G_{22} and G_{31}.

6. Degrees, codegrees and isodiscriminantality

No exceptional badly-generated group other than G_{31} and G_{22} admit a regular embedding inside a well-generated group. However, the preceding sections can still be used for more than these two groups. For instance, G_7 has the same degrees and codegrees as the group $G(12, 2, 2)$, which is a member of the infinite series. By Section 4, we know that our theorem holds for $G(12, 2, 2)$. Here we show that this is sufficient to show that it also holds for G_7.

Proposition 6.1. *Let W, W' be two irreducible complex reflection groups having the same degrees and codegrees. If (A) (resp. (B)) holds for W, then it also holds for W'.*

Proof. The main tool for this proof is the classification of all the pairs irreducible groups having the same degrees and codegrees. This classification is easy by direct inspection of the degrees and codegrees of each irreducible group, and summarized by the following lemma:

Lemma 6.2. *The only pairs of irreducible complex reflection groups having same degrees and codegrees are*

$$G_5 \leftrightarrow G(6,1,2) \quad G_{10} \leftrightarrow G(12,1,2) \quad G_{18} \leftrightarrow G(30,1,2) \quad G_7 \leftrightarrow G(12,2,2)$$
$$G_{11} \leftrightarrow G(24,2,2) \quad G_{15} \leftrightarrow G(24,4,2) \quad G_{19} \leftrightarrow G(60,2,2) \quad G_{26} \leftrightarrow G(6,1,3)$$

We could also include $G(2,2,3) \leftrightarrow G(1,1,4)$, $G(3,3,2) \leftrightarrow G(1,1,3)$ and $G(2,1,2) \leftrightarrow G(4,4,2)$, but these pairs are moreover isomorphic as complex reflection groups.

Each of these pairs are pairs of isodiscriminantal groups (cf [LT09, Chapter 6.6] and [OS88]) having a center of the same order (this last point is obvious as the cardinality of the center is the gcd of the degrees).

The isomorphism $\varphi : B(W) \to B(W')$ induced by isodiscriminantality sends $z_{B(W)}$ to $z_{B(W')}$. As the centers of W and W' have the same size, it also sends $z_{P(W)}$ to $z_{P(W')}$. So the existence of d-roots for $z_{P(W)}$ implies the existence of d-th roots of $z_{P(W')}$. As the groups W and W' have the same degrees and codegrees, we get that properties (*A*) and (*B*) for W imply properties (*A*) and (*B*) for W'. □

In order to prove property (*C*), we are going to restrict to groups of rank 2 so that we can use Remark 3.3. The only pair of groups having identical degrees and codegrees and of rank more than 2 is $G_{26} \leftrightarrow G(6,1,3)$. Since both of these groups are well-generated, they already are known to satisfy the theorem.

Let d be a regular number giving a group of rank 1, d' its associated fundamental regular number. If ρ is an d-th root of $z_{P'}$ and ρ' is an associated fundamental root, then we already know that $C_{B(W)}(\varphi^{-1}(\rho)) = C_{B(W)}(\varphi^{-1}(\rho')) = \langle \varphi^{-1}(\rho') \rangle$, and the isomorphism φ gives the desired result.

This settles the cases of G_7, G_{11}, G_{19} and G_{15}, since $G(12,2,2), G(24,2,2), G(60,2,2)$ and $G(24,4,2)$ belong to the infinite series.

7. The two remaining exceptional groups

By now, the only remaining groups are G_{12} and G_{13}. We are going to study them case by case with similar Garside theoretic tools. We recall that B_{12} and B_{13} are Garside groups (see for instance [Pic00, Example 11 and 13]). The relevant data are tabulated as follows:

	G_{12}	G_{13}
Degrees	6 8	8 12
Codegrees	0 10	0 16
Presentation of B	$stus = tust = ustu$	$cabc = bcab,\ abcab = cabca$
Δ	$stus$	$(abc)^3$
z_B	$(stu)^4 = \Delta^3$	$(abc)^3 = \Delta$
z_P	$(stu)^8 = \Delta^6$	$(abc)^{12} = \Delta^4$

The two presentations given for B_{12} and B_{13} also provide the presentations for the associated Garside monoid, that we denote M and N respectively.

The Garside automorphisms for M and N are respectively given by

$$\phi : \begin{cases} s \mapsto t \\ t \mapsto u \\ u \mapsto s \end{cases} \quad \text{and } \psi = 1_N$$

We study the d-th roots of $z_P = \Delta^q$, with $q = 6$ or 4 respectively. We are going to compute a presentation of $\mathcal{C}_{p'}^{q'}$, and use Theorem 2.6.

We are also going to make use of Remark 3.3 to prove property (C). We will only have to explicitly compute a presentation of $\mathcal{C}_{p'}^{q'}$ when p' is its own smallest regular number. In the other cases, we will only need to check if $\mathcal{C}_{p'}^{q'}$ is connected, so we will only compute $D_{p'}^{q'}(\Delta)$ and $D_{2p'}^{2q'}(\Delta)$.

7.1. **Case of G_{12}.** Recall that M is the Garside monoid defined by the presentation $\langle s,t,u \mid stus = tust = ustu \rangle$. It admits B_{12} as its group of fractions. The full twist of B_{12} is $z_P = (stu)^8 = \Delta^6$, which has length 24 in M. A root ρ of z_P then has a length that divides 24, so the "candidate regular numbers" are $\{1,2,3,4,6,8,12,24\}$.

Besides, we know that the regular number for B_{12} are $1,2,3,4,6,8$, the fundamental regular numbers are $2,6,8$ and we have $R_1 = \{1,2\}$ $R_2 = \{3,6\}$, $R_3 = \{4,8\}$.

7.1.1. *Proof of property* (A). Here we only have to show that there are no d-th roots of z_P for $d \in \{12,24\}$. Since 12 divides 24, it is sufficient to show that there are no 12-th roots of z_P. Since $z_P = \Delta^6$, applying Proposition 2.5 we compute $D_1^2(\Delta)$. For $(x,y) \in \mathcal{S}^2$, we have

$$(x,y) = (x,y)^\phi = (y, x^\phi) \Leftrightarrow x = y = x^\phi$$

Therefore we have $D_1^2(\Delta) = \{x \in \mathcal{S} \mid x^2 = \Delta\} = \varnothing$. The category $\mathcal{C}_1^2$ is empty and there are no 12-th roots of z_P in B_{12}. This proves point (A).

In order to prove points (B) and (C), we need to prove the connectedness (and non-emptiness) of $\mathcal{C}_p^6$ for $p = \{2,6,8\}$, and to compute a presentation for $p = \{1,3,4\}$.

7.1.2. *Proof of connectedness for non-smallest regular numbers.*

- $p = 2$, we compute $\mathcal{C}_1^3 = M^{\phi^3} = M$, which is connected (as a monoid).
- $p = 6$, we compute $\mathcal{C}_1^1 = M^{\phi}$, which is connected (it is also a monoid).
- $p = 8$, we compute $\mathcal{C}_4^3$. For $(x, y, z, t) \in \mathcal{S}^4$, we have

$$(x, y, z, t) \in D_4^3(\Delta) \Leftrightarrow xyzt = \Delta \text{ and } (x, y, z, t) = (t, x^{\phi}, y^{\phi}, z^{\phi})$$

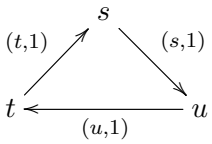
The second condition gives $y = x^{\phi}, z = x^{\phi^2}, t = x$, so we deduce that

$$D_4^3(\Delta) \approx \{x \in \mathcal{S} \mid xx^{\phi}x^{\phi^2}x = \Delta\} = \{s, t, u\}$$

the same reasoning gives us

$$D_8^6(\Delta) \approx \{(\alpha, \beta) \in \mathcal{S}^2 \mid (\alpha\beta)(\alpha\beta)^{\phi}(\alpha\beta)^{\phi^2}(\alpha\beta) = \Delta\}$$

with $(\alpha, \beta) : \alpha\beta \to \beta\alpha^{\phi}$, in particular we have the following subgraph of $\mathcal{C}_4^3$:



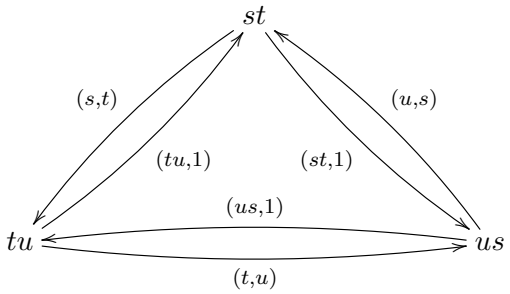
which proves that $\mathcal{C}_4^3$ is connected.

7.1.3. *Proof of connectedness and computation of the centralizer for the smallest regular numbers.*

- $p = 1$, we compute $\mathcal{C}_1^6 = M^{\phi^6} = M$, its enveloping group is B_{12}. We have that z_P is a 1-th root of z_P, and that Δ^3 is an associated fundamental root. As both z_P and Δ^3 are central in B_{12}, we have $C_{B_{12}}(\Delta^3) = C_{B_{12}}(z_P) = B_{12}$ as expected.
- $p = 3$, we compute $\mathcal{C}_1^2 = M^{\phi^2}$, which is generated by the simples $x \in \mathcal{S}$ such that $s^{\phi^2} = s$. The only such simple is Δ, so we have $M^{\phi^2} = \langle\Delta\rangle$, and Δ is a 6-th root of z_P.
- $p = 4$, we compute $\mathcal{C}_2^3$. We obtain

$$D_2^3(\Delta) \approx \{x \in \mathcal{S} \mid xx^{\phi^2} = \Delta\} = \{st, tu, us\}$$
$$D_4^6(\Delta) \approx \{(\alpha, \beta) \in \mathcal{S}^2 \mid (\alpha\beta)(\alpha\beta)^{\phi^2} = \Delta\}$$
$$D_6^9(\Delta) \approx \{(u, v, w) \in \mathcal{S}^3 \mid (uvw)(uvw)^{\phi^2} = \Delta\}$$

With $(\alpha, \beta) : \alpha\beta \to \beta\alpha^{\phi^2}$, and (u, v, w) inducing the relation $(u, vw)(v, wu^{\phi^2}) = (uv, w)$. We obtain that $\mathcal{C}_2^3$ is presented by the following graph



With the relations

$$(u, s) \circ (s, t) = (us, 1); (s, t) \circ (t, u) = (st, 1); (t, u) \circ (u, s) = (tu, 1)$$

From this we deduce that the automorphism group of the object st is cyclic and generated by $(s, t) \circ (t, u) \circ (u, s)$, which is sent by the collapse functor to stu, a 8-th root of z_P. Since 8 is the fundamental regular number associated to 4, this concludes the proof of our theorem for G_{12}.

7.2. **Case of G_{13}.** Recall that N is the Garside monoid defined by the presentation $\langle a, b, c \mid bcab = cabc,\ abcab = cabca\rangle$. It admits B_{13} as its group of fractions. The full twist of B_{13} is $z_P = (abc)^{12} = \Delta^4$, which has length 36 in N. A root of z_P then has a length that divides 36, so the "candidate regular numbers" are $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

Besides we know that the regular numbers for B_{13} are $1, 2, 3, 4, 6, 12$, the fundamental regular numbers are $4, 12$, and we have $R_1 = \{1, 2, 4\}$, $R_3 = \{3, 6, 12\}$.

7.2.1. *Proof of property* (A). Here we only have to show that there are no d-th roots of z_P for $d \in \{9, 18, 36\}$. Since 9 divides 18 and 36, it is sufficient to show that there are no 9-th roots of z_P. If ρ is such a root, then ρ^3 is a 3-th root of z_P. If we assume (for now) that point (B) holds for $d = 3$, then we get that $\overline{\rho}^3$ is a 3-regular element. In particular we have $\overline{\rho}^3 \neq 1$, thus $\overline{\rho}$ is of order 9. This contradicts the fact that 9 does not divide $|G_{13}| = 96$.

In order to prove points (B) and (C), we need to prove the connectedness (and non-emptiness) of $\mathcal{C}_p^4$ for $p = \{2, 4, 6, 12\}$ and to compute a presentation for $p = \{1, 3\}$

7.2.2. *Proof of connectedness for non-smallest regular numbers.*

- $p = 2$, we compute $\mathcal{C}_1^2 = N^{\psi^2} = N$, which is connected (as a monoid).
- $p = 4$, we compute $\mathcal{C}_1^1 = N^{\psi} = N$, which is also connected.
- $p = 6$, we compute $\mathcal{C}_3^2$, for $(x, y, z) \in \mathcal{S}^3$. We have

$$(x, y, z) \in D_3^2(\Delta) \Leftrightarrow xyz = \Delta \text{ and } (x, y, z) = (z, x, y)$$

  The second condition gives $x = y = z$, so we deduce that

$$D_3^2(\Delta) \approx \{x \in \mathcal{S} \mid x^3 = \Delta\} = \{abc, bca, cab\}$$

  The same reasoning gives us

$$D_6^4(\Delta) \approx \{(\alpha, \beta) \in \mathcal{S} \mid (\alpha\beta)^3 = \Delta\}$$

  with $(\alpha, \beta) : \alpha\beta \to \beta\alpha$, in particular we have the following subgraph of $\mathcal{C}_3^2$:

$$cba \xleftarrow{(ab,c)} abc \xrightarrow{(a,bc)} bca$$

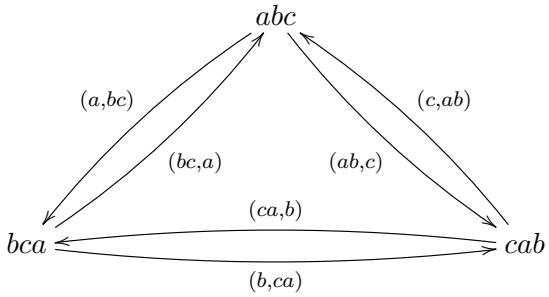
  which proves that $\mathcal{C}_3^2$ is connected.
- $p = 12$, we compute $\mathcal{C}_3^1$. The same reasoning as when $p = 6$ gives $D_3^1(\Delta) = \{abc, bca, cab\}$ and $D_6^2(\Delta) = D_6^4(\Delta)$. So $\mathcal{C}_3^1$ is also connected.

7.2.3. *Proof of connectedness and computation of the centralizer for the smallest regular numbers.*

- $p = 1$, we compute $\mathcal{C}_1^4 = N^{\psi^4} = N$. Its enveloping group is B_{13}. We have that z_P is a 1-root of z_P, and that Δ is an associated fundamental root. As both z_P and Δ are central in B_{13}, we have $C_{B_{13}}(\Delta) = C_{B_{13}}(z_P) = B_{13}$ as expected.
- $p = 3$, we compute $\mathcal{C}_3^4$. We have

$$D_3^4(\Delta) \approx \{x \in \mathcal{S} \mid x^3 = \Delta\} = \{abc, bca, cab\}$$
$$D_6^8(\Delta) \approx \{(\alpha, \beta) \in \mathcal{S}^2 \mid (\alpha\beta)^3 = \Delta\}$$
$$D_9^{12}(\Delta) \approx \{(u, v, w) \in \mathcal{S}^3 \mid (uvw)^3 = \Delta\}$$

With $(\alpha, \beta) : \alpha\beta \to \beta\alpha$, and (u, v, w) inducing the relation $(u, vw)(v, wu) = (uv, w)$. We obtain that $\mathcal{C}_2^1$ is presented by the following graph



With the relations

$(a, bc)(b, ca) = (ab, c)$ $(b, ca)(c, ab) = (bc, a)$ $(c, ab)(a, bc) = (ca, b)$
$(ab, c)(c, ab) = (a, bc)(bc, a)$ $(bc, a)(a, bc) = (b, ca)(ca, b)$ $(ca, b)(b, ca) = (c, ab)(ab, c)$

From this we deduce that the automorphism group of the object abc is cyclic and generated by $(ab, c)(c, ab)$, which is sent by the collapse functor do abc, a 12-th root of Δ. Since 12 is the fundamental regular number associated to 3, this concludes the proof.

8. A Kerékjártó type by-product

In [Bes15, Remark 12.5], Bessis suggests that our main theorem can be thought of as an analogue of the Kerékjártó Theorem, stating that every periodic homeomorphism of the disk is conjugate to a rotation. In this last section we establish a rephrasing of this analogy.

Proposition 8.1. *Let W be an irreducible complex reflection group. If $\gamma \in B(W)$ is an element admitting a central power, then we have $\gamma = \rho^p$ for ρ a fundamental root of z_P. In particular $\overline{\gamma}$ is a regular element in W.*

If $B(W)$ is a Garside group with a Δ such that z_B is a power of Δ, then an element of $B(W)$ admits a central power in $B(W)$ if and only if it is periodic in the Garside sense. So the two notions of periodic elements are the same. This also justify the connection between Proposition 8.1 and the Kerékjártó Theorem.

This statement is already known to hold for the infinite series by Lemma 6.1 and proposition 6.2. Indeed ε (resp. λ) is the only fundamental root of z_P in $B(de, 1, n)$ (resp. $B(de, e, n)$).

Let us first prove proposition 8.1 in the case where $B(W)$ is a Garside group, and $z_B = \Delta^n$. If $\gamma^p = \Delta^q$ in $B(W)$, then we have $\gamma^{pn} = \Delta^{qn} = z_B^q$.

If we note $k = (pn) \wedge q$, we have that γ is conjugate to some $\widetilde{\gamma}$ such that $\widetilde{\gamma}^{\frac{pn}{k}} = z_B^{\frac{q}{k}}$. But we know that $z_B = \Delta^n$ is also a Garside element, which is central. Using proposition 2.5, there is some ρ such that $\widetilde{\gamma} = \rho^{\frac{pn}{k}}$, where we have $\rho^{\frac{q}{k}} = z_B$. This settles the case of almost every irreducible group, the exceptions being G_{31} and G_{15}. For G_{15}, we use that it has the same degrees and codegrees as $G(24, 4, 2)$. The induced isomorphism between B_{15} and $B(24, 4, 2)$ preserves z_P and the notion of fundamental roots, so the proposition holds for B_{15}. Lastly, for B_{31}, let δ be a 4-th root of z_P in B_{37}. We know that B_{31} is isomorphic to $C_{B_{37}}(\delta)$. We have $Z(C_{B_{37}}(\delta)) = \langle\delta\rangle$ (see [Bes15, Corollary 12.7]), that is the isomorphism $C_{B_{37}}(\delta) \simeq B_{31}$ sends δ to z_B. The case of B_{31} is then an consequence of Lemma 5.1.

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