PROOF OF SHVARTSMAN'S CONJECTURE ON BRAID GROUPS OF PROJECTIVE COMPLEX REFLECTION GROUPS

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Abstract. The purpose of this note is to prove a conjecture of Shvartsman relating a complex projective reflection group with the quotient of a suitable complex braid group by its center. Shvartsman originally proved this result in the case of real projective reflection groups, and we extend it to all complex projective reflection groups. Our study also allows us to correct a result of Broué, Malle, Rouquier on projective reflection groups.

Introduction

Let V be a finite dimensional complex vector space, and let $\mathbb{P}(V)$ be the associated projective space. The image of a subset $X \subset V$ in $\mathbb{P}(V)$ will be denoted by $\widehat{X}$. For $x \in V$, the line spanned by x will be denoted by $[x]$. For k a positive integer, the group of k-th complex roots of unity will be denoted by μ_k.

Consider a complex reflection group $W \subset \mathrm{GL}(V)$. In order to define the braid group B attached to W, one first considers the subset X of V consisting of elements with a trivial stabilizer under the action of W, and then take the fundamental group of the quotient space X/W.

Similarly, if $G \subset \mathrm{PGL}(V)$ is a *projective reflection group* (i.e. a finite group which is generated by images of reflections in $\mathrm{PGL}(V)$), then we can consider the subset $\widehat{X}$ of $\mathbb{P}(V)$ consisting of elements with a trivial stabilizer under the action of G, and then take the fundamental group of the quotient space $\widehat{X}/G$. This gives a reasonable definition of a braid group attached to the projective reflection group G.

In his article [Shv96], Shvartsman describes the braid group attached to a real projective reflection group as the quotient of a spherical Artin group by its center. More precisely, if $G \subset \mathrm{PGL}(V)$ is a real projective reflection group, then there is a maximal real reflection group $W \subset \mathrm{GL}(V)$ which is a lift of G. The braid group attached to G is then isomorphic to the quotient $A/Z(A)$, where A denotes the Artin group attached to W [Shv96, Theorem A]. The goal of Shvartsman is to use this realization in order to compute the possible orders of torsion elements in $A/Z(A)$, which is done in [Shv96, Theorem B].

In [Shv96, Section 4], Shvartsman conjectures that his results could be adapted in the case of complex projective reflection groups, where the Artin groups should be replaced by a complex braid group. The second result [Shv96, Theorem B] about the possible orders of torsion elements was generalized to complex braid groups in [Bes15, Theorem 12.4] and in [Gar23, Proposition 8.2]. We give in this note an extension to all complex projective reflection groups of [Shv96, Theorem A]:

Theorem 1 (Theorem 2.7). *Let $G \subset \mathrm{PGL}(V)$ be an irreducible projective reflection group, and let $\widehat{X}$ be the subset of $\mathbb{P}(V)$ consisting of elements with a trivial stabilizer under the action of G. Let also $W \subset \mathrm{GL}(V)$ be the maximal reflection group which is a lift of G.*
The fundamental group of $\widehat{X}/G$ is isomorphic to the quotient $B/Z(B)$, where B denotes the complex braid group attached to W.

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In Section 1, we give preliminary results about complex (projective) reflection groups. In particular, we prove that a projective complex reflection group admits a unique maximal lift in $\mathrm{GL}(V)$ which is a reflection group. In Section 2, we prove Theorem 1 by introducing the *enlarged braid group* attached to a complex reflection group W. This group is an extension of the more classical complex braid group attached to W, and we give some of its properties. Lastly, in Section 2.3, we give some corrections on a result of Broué, Malle, Rouquier in [BMR98], which aims to relate braid groups of projective reflections groups with complex braid groups. As pointed out in [DMM11], their result is false in general, and we give a complete description of the cases for which it holds (see Proposition 2.10).

1. Preliminaries

1.1. **Reminders and generalities on complex reflection groups.**

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V. We mostly follow [LT09] for classical results on complex reflection groups.

Recall that a *complex reflection group* W is a finite subgroup of $\mathrm{GL}(V)$ which is generated by *reflections*, that is finite order linear automorphisms of V which pointwise fix some hyperplane.

A complex reflection group $W \subset \mathrm{GL}(V)$ is *irreducible* if there are no W-invariant subspaces in V apart from $\{0\}$ and V itself. Every complex reflection group decomposes as a direct product of irreducible complex reflection group and we will thus restrict our attention to irreducible groups from now on. Irreducible complex reflection groups were classified by Shephard and Todd in [ST54], and we freely use the notation of [LT09, Theorem 8.29] regarding this classification.

To an irreducible complex reflection group $W \subset \mathrm{GL}(V)$, one can attach the sequence $d_1 \leqslant \ldots \leqslant d_n$ of its *degrees*, and the sequence $d_n^* \geqslant \ldots \geqslant d_1^* = 0$ of its *codegrees* [LT09, Proposition 3.25 and Definition 10.27]. The degrees of W are by definition the degrees of a *system of basic invariants* of W [LT09, Theorem 3.20], that is, a family $(f_1, \ldots, f_n)$ of homogeneous elements of $S(V^*)$ which freely generate $S(V^*)^W$. Such a sequence always exists by the Chevalley-Shephard-Todd Theorem [LT09, Theorem 3.20]. Moreover, a system of basic invariants $(f_1, \ldots, f_n)$ for W induces a homeomorphism $V/W \simeq \mathbb{C}^n$, sending an orbit $W.v$ to the n-tuple $(f_1(v), \ldots, f_n(v))$ [LT09, Proposition 9.3].

To a complex reflection group W one can also attach the complement X in V of the union of the reflecting hyperplanes attached to the reflections of W. The *braid group* $B(= B(W))$ of W is then defined as the fundamental group of X/W, while the *pure braid group* $P(= P(W))$ of W is defined as the fundamental group of X [BMR98, Section 2.B]. The projection map $X \twoheadrightarrow X/W$ is a covering map by Steinberg's Theorem [LT09, Theorem 9.44], and it induces a short exact sequence

$$1 \to P \to B \to W \to 1.$$

Let $W \subset \mathrm{GL}(V)$ be a complex reflection group, and let $\zeta \in \mathbb{C}^*$. An element $g \in W$ is said to be ζ-*regular* if it admits a ζ-eigenvector which lies in X. In other words, $g \in W$ is ζ-regular if $V(g,\zeta) \cap X \neq \varnothing$, where $V(g,\zeta)$ denotes the ζ-eigenspace of g. The eigenspace $V(g,\zeta)$ is then called a *regular eigenspace* for W. An integer $k \geqslant 0$ is said to be *regular* for W if W contains some ζ_k-regular element, where $\zeta_k := \exp(\frac{2i\pi}{k})$.

If $g \in W$ is ζ-regular, then g^k is ζ^k regular for all integer k. In particular, if $\zeta \in \mathbb{C}^*$ has order m, then W contains a ζ-regular element if and only if m is regular for W. We also have that regular eigenspaces for W can all be written as $V(g, \zeta_m)$, where g is a ζ_m-regular element for some integer m.

An important criterion for regularity is given in [LT09, Theorem 11.28], stating that a positive integer k is regular for W if and only if it divides as much degrees of W as it does codegrees (both counted with multiplicity).

1.2. **Projective reflection groups.**

In this section, we fix a finite dimensional complex vector space V of dimension $n > 1$.

We define a *projective reflection group* as a finite subgroup of $\mathrm{PGL}(V)$ which is generated by images of reflections in $\mathrm{PGL}(V)$. Imitating the case of linear reflection group, we say that a projective reflection group G is *irreducible* if the only G-invariant subspace of $\mathbb{P}(V)$ is $\mathbb{P}(V)$ itself. We say that G is *imprimitive* if there is a direct sum $V = V_1 \oplus \cdots \oplus V_m$ with $m \geqslant 2$ such that the action of G on $\mathbb{P}(V)$

permutes the subspaces $\widehat{V_1}, \dots, \widehat{V_m}$ among themselves (we call $\{V_1, \dots, V_m\}$ a *system of imprimitivity* for G).

If $W \subset \mathrm{GL}(V)$ is an irreducible complex reflection group, then the image $\widehat{W}$ of W in $\mathrm{PGL}(V)$ is a projective complex reflection group, which is imprimitive if and only if W itself is imprimitive. The kernel the projection map $W \twoheadrightarrow \widehat{W}$ is the set of scalar matrices lying in W. Since W is irreducible, this set coincides with the center $Z(W)$ of W by Schur's lemma, and we identify $\widehat{W}$ with the quotient group $W/Z(W)$. An elementary result is that every projective reflection group is the image in $\mathrm{PGL}(V)$ of some linear reflection group:

Lemma 1.1. *Let $G \subset \mathrm{PGL}(V)$ be an irreducible projective reflection group. There is an irreducible complex reflection group $W \subset \mathrm{GL}(V)$ such that $\widehat{W} = G$.*

Proof. This result is shown in [ST54, Section 1.3 and 1.4] in the case where G is primitive, so that we can assume from now that G is imprimitive. Let us consider a system of imprimitivity $\{\widehat{V_1}, \dots, \widehat{V_k}\}$ for G. By following the proof of [LT09, Lemma 2.12], we obtain that all the V_i have dimension 1. We can then identify V with $\mathbb{C}^n$ and G with a subgroup of $\mathrm{PGL}_n(\mathbb{C})$ with the canonical basis as a system of imprimitivity. Under this identification, we see that any lift of an element of G in $\mathrm{GL}_n(\mathbb{C})$ is a monomial matrix. Consider the finite set $PR \subset G$, consisting of the elements of G which are images of reflections in $\mathrm{GL}_n(\mathbb{C})$. For any $r \in PR$, there is a finite number of reflections in $\mathrm{GL}_n(\mathbb{C})$ with image r in $\mathrm{PGL}_n(\mathbb{C})$. Thus we can consider the finite set $R \subset \mathrm{GL}_n(\mathbb{C})$ containing all the possible lifts of elements of PR which are reflections. The elements of R are all monomial reflections, and thus the subgroup W of $\mathrm{GL}_n(\mathbb{C})$ generated by R is contained in some group $G(m, 1, n)$ for m big enough. In particular, W is a finite reflection group in $\mathrm{GL}_n(\mathbb{C})$ with image G in $\mathrm{PGL}_n(\mathbb{C})$. □

Remark 1.2. Let $W \subset \mathrm{GL}_1(\mathbb{C}) \simeq \mathbb{C}^*$ be a complex reflection group of rank 1. In this case W is a cyclic group and the quotient $\widehat{W}$ is the trivial group acting on $\mathbb{P}^0(\mathbb{C}) = \{*\}$. This situation is not very rich and Theorem 1 is immediate in this case. This is why we assume that $\dim V > 1$ when considering projective reflection groups.

In order to describe the action of $\widehat{W}$ on the projective space $\mathbb{P}(V)$, we first consider the inverse image $\widetilde{W}$ of $\widehat{W}$ in $\mathrm{GL}(V)$. We can describe the group $\widetilde{W}$ explicitly: Let Z denote the center of $\mathrm{GL}(V)$, that is the subgroup of scalar multiples of the identity. The groups W and Z normalize each other, and we can consider the product group ZW (not to be confused with the center $Z(W)$ of W). Since Z is also the kernel of the natural morphism $\mathrm{GL}(V) \to \mathrm{PGL}(V)$, we have $ZW = \widetilde{W}$. This group contains both Z and W as normal subgroups, and we have :

$$\begin{cases} \widetilde{W}/Z \simeq W/(Z \cap W) = \widehat{W}, \\ \widetilde{W}/W \simeq Z/(Z \cap W) \simeq \mathbb{C}^*/\mu_{|Z(W)|} \simeq \mathbb{C}^*. \end{cases}$$

By Lemma 1.1, we know that every projective reflection group is the image in $\mathrm{PGL}(V)$ of at least one linear reflection group in $\mathrm{GL}(V)$. However, a given projective reflection group may have distinct lifts W, W'. The groups $\widetilde{W}$ and $\widetilde{W'}$ help us to relate W and W' in this case:

Lemma 1.3. *Let $W, W' \subset \mathrm{GL}(V)$ be two irreducible complex reflection groups. We have $\widetilde{W} = \widetilde{W'}$ if and only if $\widehat{W} = \widehat{W'}$, and in this case, W and W' normalize each other. Moreover, if $W \subset W'$, then $\widetilde{W} = \widetilde{W'}$ is also equivalent to $[W : Z(W)] = [W' : Z(W')]$.*

Proof. The projection map $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ induces a bijection between the subgroups of $\mathrm{PGL}(V)$ and the subgroups of $\mathrm{GL}(V)$ which contain Z. Since $\widetilde{W}$ and $\widetilde{W'}$ both contain Z, they are equal if and only if their images in $\mathrm{PGL}(V)$ are equal. Since the image of $\widetilde{W^{(\prime)}}$ in $\mathrm{PGL}(V)$ is $\widehat{W^{(\prime)}}$, we have the first claim. Now, if $\widetilde{W} = \widetilde{W'}$, then W is normal in $\widetilde{W} = \widetilde{W'}$, which contains W'. Thus W and W' normalize each other.

Lastly, if $W \subset W'$ then $\widetilde{W} \subset \widetilde{W'}$ and $\widehat{W} \subset \widehat{W'}$. Since the cardinality of $\widehat{W}$ is $[W : Z(W)]$, we obtain that $[W : Z(W)] = [W' : Z(W')]$ if and only if $\widehat{W} = \widehat{W'}$. □

Using this result, we are able to show that any projective reflection group admits a lift in $\mathrm{GL}(V)$ which is a reflection group and which is maximal relative to inclusion. By Lemma 1.1, this is equivalent to the following proposition:

Proposition 1.4 (Full reflection group). *Let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group. The family of complex reflection groups $W' \subset \mathrm{GL}(V)$ such that $\widetilde{W} = \widetilde{W'}$ admits a maximum W_f (for inclusion). We call W_f the* full reflection group *associated to W.*

1. *If W is a primitive group and $\dim(V) = 2$, then W_f has type G_7, G_{11}, G_{19} depending on whether W is a tetrahedral, octahedral or icosahedral group.*
2. *If W is a primitive group and $\dim(V) > 2$, then $W_f = W$, except when W has type G_{25}, in which case W_f has type G_{26}.*
3. *If W is an imprimitive group of type $G(m,p,2)$, then W_f has type $G(\frac{2m}{p\wedge 2},2,2)$,*
4. *If W is an imprimitive group of type $G(m,p,n)$ with $n > 2$, then W_f has type $G(m, p\wedge n, n)$.*

Proof. The case where W is primitive is studied in [ST54, Section 1.3 and 1.4], where the authors classify all the possible lifts of $\widehat{W}$. Assume now that W is imprimitive, and let W' be such that $\widetilde{W} = \widetilde{W'}$. In this case, the group $\widehat{W}$ is imprimitive, and W' shares a system of imprimitivity with W. It is then sufficient to consider the case where $W = G(m,p,n)$ and $W' = G(m',p',n')$. The group $G(m,p,n)$ has order $\frac{m^{n-1}n!}{p\wedge n}$, and the index of its center is $\frac{m}{p\wedge n}$.

(3) First, $W := G(m,p,2)$ is a reflection subgroup of $W' := G(\frac{2m}{p\wedge 2},2,2)$. Moreover, the indices of the centers are equal, and $\widetilde{W} = \widetilde{W'}$ by Lemma 1.3. Now, let $W'' = G(m',p',2)$ be such that $\widetilde{W''} = \widetilde{W}$. The indices of the centers of W and of W'' are equal, that is, we have $\frac{2m}{p\wedge 2} = \frac{2m'}{p'\wedge 2}$.

- If p and p' have the same parity, then we deduce that $m = m'$ and $W'' = G(m,p',2)$.
- If p is even and p' is odd, then we deduce that $m = 2m'$ and $W'' = G(\frac{m}{2},p',2)$.
- If p is odd and p' is even, then we deduce that $2m = m'$ and $W'' = G(2m,p',2)$.

In each case, W'' is a reflection subgroup of W' and we have the result.

(4) Assume now that $n > 2$. In this case, $G(m,p,n)$ contains the reflection

$$M := \begin{pmatrix} 0 & \zeta_m & 0 \\ \zeta_m^{-1} & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

If $W' = G(m',p',n')$ is such that $\widehat{W'} = \widehat{W}$, there is some $\lambda \in \mathbb{C}^*$ such that $\lambda M \in W'$. By definition of $G(m',p',n)$, this implies that $\lambda, \lambda\zeta_m \in \mu_{m'}$ (note that λ is a coefficient of λM only because $n > 2$). We then have $\zeta_m \in \mu_{m'}$ and m divides m'. By exchanging W and W', we obtain that m' divides m and $m = m'$. We then have

$$[W : Z(W)] = [W' : Z(W')] \Leftrightarrow p\wedge n = p'\wedge n,$$

and W' ranges among the family of groups $\{G(m,p',n) \mid p'\wedge n = p\wedge n\}$. All the groups in this family are included in $W_f = G(m, p\wedge n, n)$. □

Remark 1.5. Note that Proposition 1.4 fails if $n = 1$. Indeed, if W is a complex reflection group of rank 1, then W is cyclic, and $\widetilde{W} = \mathbb{C}^*$ does not depend on W. This is another reason why we assume $n > 1$ when considering projective reflection groups.

We finish this section by proving that full reflection groups admit no regular hyperplanes. This result will be used in Section 2.2.

Lemma 1.6. *Let $W \subset \mathrm{GL}(V)$ is a complex reflection group. If $W = W_f$ is its own full reflection group, then W admits no regular hyperplanes in V.*

Proof. If W has rank 2, then in each case in Proposition 1.4, all regular elements of W_f are central, and W_f admits no proper regular eigenspace, in particular no regular hyperplanes.

Now, assume that W has rank $n \geqslant 3$. We denote by D the gcd of all the degrees (that is, the order of $Z(W)$) and by D_j the gcd of all the degrees but the j-th one. If k is a regular number for W whose associated regular eigenspaces are hyperplanes, then k divides exactly $n-1$ degrees of W. In other words, k divides some D_j while not dividing D, and D_j does not divide D.

- If W is an exceptional group, then we can explicitly compute both D and the D_j. We see that $D_j = D$ for all $j \in [\![1,n]\!]$, except in the case where W has type G_{25}, but $W \neq W_f$ in this case.
- If W is a symmetric group $\mathfrak{S}_{n+1}$ (acting on a space of dimension n) with $n \geqslant 4$, then the degrees of W are $2, 3, \ldots, n+1$. Since $n \geqslant 4$, the prime numbers $2, 3, 5$ appear in the list of degrees of W. We then have $D_j = 1 = D$ for all $j \in [\![1,n]\!]$.
- If $W = W_f$ has type $G(m,p,n)$ with $n \geqslant 3$, then $p|n$ by Proposition 1.4. In this case, the degrees of W are $m, 2m, \ldots, (n-1)m, \frac{mn}{p}$, and $D = m \wedge (m\frac{n}{p}) = m$. For $j \in [\![1,n]\!]$, since $n \geqslant 3$, we have
$$D_j | m \text{ or } D_j | \left(m(n-1) \wedge m\frac{n}{p} \right).$$
Since $n-1$ and $\frac{n}{p}$ are coprime, we have $m(n-1) \wedge m\frac{n}{p} = m$ and $D_j = m$ for all $j \in [\![1,n]\!]$, thus $D_j = m = D$ in each case.

□

2. Main results

In this section, we fix a finite dimensional complex vector space V of dimension $n \geqslant 1$. We also fix an irreducible complex reflection group $W \subset \mathrm{GL}(V)$. We otherwise keep the notation from Section 1.

2.1. Actions of $\widetilde{W}$, $\widehat{W}$ and $Z/Z(W)$.

In order to prove Theorem 1, we need to study the action of $\widehat{W}$ on $\mathbb{P}(V)$. In this section, we relate this action with the action of $\widetilde{W}$ on V, and with the action of $\widetilde{W}/W$ on V/W.

Since Z is a normal subgroup of $\widetilde{W}$, there is an action of $\widetilde{W}/Z \simeq \widehat{W}$ on V/Z, which we can restrict to an action on $(V \setminus 0)/Z = \mathbb{P}(V)$. This action coincides with the natural action of $\widehat{W}$ on $\mathbb{P}(V)$. Similarly, since W is a normal subgroup of $\widetilde{W}$, there is an action of $\widetilde{W}/W \simeq Z/Z(W)$ on V/W.

Note that the faithful action of $\widehat{W}$ on $\mathbb{P}(V)$ can also be seen as an action of W on $\mathbb{P}(V)$ with kernel $Z(W)$. In particular, we will sometimes denote $\mathbb{P}(V)/W$ instead of $\mathbb{P}(V)/\widehat{W}$ to alleviate notation. Similarly, we can see the action of $Z/Z(W)$ on V/W as an action of $Z \simeq \mathbb{C}^*$. We have the following commutative square of topological spaces:

$$\begin{array}{ccc} V \setminus 0 & \xrightarrow{/W} & (V \setminus 0)/W \\ {\scriptstyle /Z}\downarrow & \searrow^{/\widetilde{W}} & \downarrow{\scriptstyle /Z} \\ \mathbb{P}(V) & \xrightarrow[/W]{} & (V \setminus 0)/\widetilde{W} \end{array} \tag{2.1}$$

This square (or rather its restriction to a convenient subspace of V) will prove useful to us in the next section. Another relation between the actions of $\widetilde{W}, \widehat{W}$ and $Z/Z(W)$ is the computation of nontrivial stabilizers:

Lemma 2.1 (Stabilizers). *Let $x \in V$ be different from* 0.

(a) The stabilizer of $x \in V$ under the action of $\widetilde{W}$ is given by
$$\mathrm{Stab}_{\widetilde{W}}(x) = \{\zeta w \in \widetilde{W} \mid x \in V(w, \zeta^{-1})\}.$$
It is nontrivial if and only if x belongs to either a reflecting hyperplane of W or to a proper regular eigenspace of W.

(b) The stabilizer of $[x]$ under the action of $\widehat{W}$ is trivial if and only if $\mathrm{Stab}_{\widetilde{W}}(x)$ is trivial.

(c) The stabilizer of $W.x$ under the action of $Z/Z(W)$ is trivial if and only if $\mathrm{Stab}_{\widetilde{W}}(x) \subset W$.

Proof. (a) Let $\zeta w \in \widetilde{W}$. We have

$$(\zeta w).x = x \Leftrightarrow w.x = \zeta^{-1}x \Leftrightarrow x \in V(w,\zeta^{-1}).$$

Now, assume that $\mathrm{Stab}_{\widetilde{W}}(x)$ is nontrivial and that x does not belong to a reflecting hyperplane of W. By assumption, there is some $w \in W$, and some $\zeta \in \mathbb{C}^*$ such that $x \in V(w,\zeta)$ and that $w \neq \zeta$ (otherwise $\zeta^{-1}w$ is trivial). Since x does not belong to any reflecting hyperplane of W, $x \in V(w,\zeta)$ implies that w is ζ-regular. Then, $w \neq \zeta$ implies that w is noncentral, and thus $V(w,\zeta) \subsetneq V$ is a proper regular eigenspace.

Conversely, assume that x belongs to the reflecting hyperplane associated to a reflection $r \in W$. By definition, we have $x \in V(r,1)$, and thus $r \in \mathrm{Stab}_{\widetilde{W}}(x)$. Lastly, assume that g is a ζ-regular element such that $x \in V(g,\zeta) \subsetneq V$. We have $g \neq \zeta$ since $V(\zeta\,\mathrm{Id},\zeta) = V$, and thus $1 \neq \zeta^{-1}g \in \mathrm{Stab}_{\widetilde{W}}(x)$, which is nontrivial.

(b) Let $wZ(W) \in \widehat{W}$. By definition, we have $wZ(W).[x] = [w.x]$ and thus

$$\begin{aligned} wZ(W) \in \mathrm{Stab}_{\widehat{W}}([x]) &\Leftrightarrow [x] = [w.x] \\ &\Leftrightarrow \exists \zeta \in \mathbb{C}^* \mid w.x = \zeta^{-1}x. \\ &\Leftrightarrow \exists \zeta \in \mathbb{C}^* \mid (\zeta w).x = x \\ &\Leftrightarrow \exists w\zeta \in wZ \cap \mathrm{Stab}_{\widetilde{W}}(x). \end{aligned}$$

In particular, we obtain that $\mathrm{Stab}_{\widehat{W}}([x])$ is trivial if and only if Z is the only Z-coset in $\widetilde{W}$ which intersects $\mathrm{Stab}_{\widetilde{W}}(x)$ nontrivially. This is equivalent to $\mathrm{Stab}_{\widetilde{W}}(x) \subset Z$. However, since x is nonzero, $Z \cap \mathrm{Stab}_{\widetilde{W}}(x)$ is always equal to $\{1\}$, and $\mathrm{Stab}_{\widetilde{W}}(x) \subset Z$ is equivalent to $\mathrm{Stab}_{\widetilde{W}}(x) = \{1\}$.

(c) Let $\lambda W \in \widetilde{W}/W$. By definition, we have $\lambda Z(W).W.x = W.(\lambda.x)$ and thus

$$\begin{aligned} \lambda W \in \mathrm{Stab}_{Z/Z(W)}(W.x) &\Leftrightarrow W.(\lambda x) = W.x \\ &\Leftrightarrow \exists w \in W \mid w\lambda x = x \\ &\Leftrightarrow \exists w\lambda \in \lambda W \cap \mathrm{Stab}_{\widetilde{W}}(x). \end{aligned}$$

In particular, we obtain that $\mathrm{Stab}_{Z/Z(W)}(W.x)$ is trivial if and only if W is the only W-coset in $\widetilde{W}$ which intersects $\mathrm{Stab}_{\widetilde{W}}(x)$ nontrivially. This is equivalent to $\mathrm{Stab}_{\widetilde{W}}(x) \subset W$. □

2.2. **Enlarged complex braid groups.** Now that we understand the stabilizers under the action of $\widehat{W}$ on $\mathbb{P}(V)$, we are ready to introduce the enlarged (pure) braid group of a complex reflection group.

Definition 2.2 (Strongly regular vectors). The set of *strongly regular vectors* attached to W is defined as

$$X_S (= X_S(W)) := \{x \in V \mid \mathrm{Stab}_{\widetilde{W}}(x) = 1\}.$$

By Lemma 2.1, X_S is the complement in V of the union of both the reflecting hyperplanes of W and of its proper regular eigenspaces. We can now define the enlarged (pure) braid group as the fundamental group of X_S/W (of X_S). We fix a basepoint x_0 in X_S for the remainder of this section.

Definition 2.3 (Enlarged braid group). The *enlarged braid group* attached to W is defined as $B_S(= B_S(W)) := \pi_1(X_S/W, W.x_0)$ and the *enlarged pure braid group* attached to W is defined as $P_S(= P_S(W)) := \pi_1(X_S, x_0)$.

The inclusion map $X_S/W \hookrightarrow X/W$ (resp $X_S \hookrightarrow X$) induces a morphism $B_S \to B$ (resp. $P \to P_S$). The following result gives some general information on these two morphisms:

Proposition 2.4.

(a) The morphism $B_S \to B$ (resp. $P_S \to P$) induced by the inclusion $X_S/W \hookrightarrow X/W$ (resp. $X_S \hookrightarrow X$) is surjective. Moreover, if $W = W_f$, then it is an isomorphism.

(b) In general, the group B_S is the inverse image of W under the natural projection $B(W_f) \twoheadrightarrow W_f$. It is a normal subgroup of index $[W_f : W]$ of $B(W_f)$. In particular, B_S is torsion free.

Proof. (a) Among the proper regular eigenspaces of W, we distinguish between the regular hyperplanes and the regular eigenspaces of complex codimension $\geqslant 2$. Let X'_S be the space obtained from X by removing the regular hyperplanes of W. We have $X_S \subset X'_S \subset X$.

The space X'_S/W (resp. X'_S) is obtained from X/W (resp. from X) by removing an algebraic hypersurface. Since X/W and X are themselves complements of algebraic hypersurface in an affine space, [BMR98, Proposition A1] gives that the inclusion $X'_S/W \hookrightarrow X/W$ (resp. $X'_S \hookrightarrow X$) induces a surjective morphism between the associated fundamental groups. Then, the space X_S/W (resp. X_S) is obtained from the smooth complex manifold X'_S/W (resp. X'_S) by removing a subvariety of complex codimension $\geqslant 2$. By [God71, Theorem X.2.3], the inclusion $X_S/W \hookrightarrow X'_S/W$ (resp. $X_S \hookrightarrow X_S$) induces an isomorphism between the associated fundamental groups.

Now, if $W = W_f$, then W admits no regular hyperplanes by Lemma 1.6. In this case, we have $X'_S = X$ and the second part of the above argument gives the isomorphism $B_S \simeq B$.

(b) Consider W_f the full reflection group attached to W. Since $X_S(W)$ depends only on $\widetilde{W}$, and since $\widetilde{W} = \widetilde{W_f}$, we have $X_S = X_S(W) = X_S(W_f)$. Since W has finite index in W_f, the covering map $X_S \twoheadrightarrow X_S/W_f$ factors through the covering map $X_S \twoheadrightarrow X_S/W$ into a covering map $X_S/W \twoheadrightarrow X_S/W_f$ and B_S is a finite-index subgroup of $B_S(W_f)$. Moreover, by Lemma 1.3, W_f is a normal subgroup of W, and the covering map $X_S/W \twoheadrightarrow X_S/W_f$ is the quotient by the action of W_f/W on X_S/W. This covering map induces a short exact sequence

$$1 \to B_S \to B_S(W_f) \to W/W_f \to 1$$

which gives the desired result. □

Now, restricting the commutative square (2.1) to X_S yields a commutative square:

$$\begin{array}{ccc} X_S & \longrightarrow & X_S/W \\ \downarrow & & \downarrow \\ \widehat{X_S} & \longrightarrow & \widehat{X_S}/W \end{array}$$

in which all the maps are fibrations:

- The fiber bundle $V \twoheadrightarrow \mathbb{P}(V)$ restricts to a fiber bundle $X_S \to \widehat{X_S}$. We obtain a short exact sequence
$$1 \to \pi_1([x_0] \setminus 0, x_0) \to P_S \to \pi_1(\widehat{X_S}, [x_0]) \to 1.$$
In other words, the natural morphism $P_S \to \pi_1(\widehat{X_S}, [x_0])$ is surjective and its kernel is generated by the (homotopy class in X_S) of the loop $t \mapsto \exp(2i\pi t)x_0$, which we denote by π_S.
- The covering map $X \twoheadrightarrow X/W$ restricts to a covering map $X_S \twoheadrightarrow X_S/W$. We obtain a short exact sequence
$$1 \to P_S \to B_S \to W \to 1.$$
- The action of $\widehat{W}$ on $\widehat{X_S}$ is free by Lemma 2.1. Since $\widehat{W}$ is finite, it acts properly on $\widehat{X_S}$, which is locally compact. The projection $\widehat{X_S} \twoheadrightarrow \widehat{X_S}/W$ is then a covering map, and we have a short exact sequence
$$1 \to \pi_1(\widehat{X_S}, [x_0]) \to \pi_1(\widehat{X_S}/W, W.[x_0]) \to \widehat{W} \to 1.$$
- The action of $\mathbb{C}^* \simeq Z/Z(W)$ on X_S/W is free by Lemma 2.1. It is then a free and proper Lie group action of $\mathbb{C}^*$ on X_S/W, which is a smooth manifold (as an open subset of $V/W \simeq \mathbb{C}^n$). By [Lee13, Theorem 21.10], the quotient map $p : X_S/W \to (X_S/W)/\mathbb{C}^* \simeq \widehat{X_S}/W$ is a principal $\mathbb{C}^*$-bundle and we have a short exact sequence
$$1 \to \pi_1(p^{-1}(W.x_0), W.x_0) \to B_S \to \pi_1(\widehat{X_S}/W, W.[x_0]) \to 1.$$
In other words, the natural morphism $B_S \to \pi_1(\widehat{X_S}/W, W.[x_0])$ is surjective and its kernel is generated by the (homotopy class in X_S/W) of the loop $t \mapsto \exp(2i\pi t).(W.x_0)$. By construction of the action of $\mathbb{C}^* \simeq Z/Z(W)$ on V/W (see Section 2.1), this last loop is actually the loop $t \mapsto W.\left(e^{\frac{2i\pi t}{|Z(W)|}}x_0\right)$. We denote its homotopy class by β_S.

By construction, $\beta_S^{|Z(W)|}$ is the image in B_S of π_S, and the image of β_S in W is $e^{\frac{2i\pi}{|Z(W)|}}\,\mathrm{Id}$, which generates $Z(W)$. These elements allow us to describe the center of enlarged braid groups:

The center of complex braid groups was studied in [BMR98], [Bes15] and [DMM11]. In the space X, the path $t \mapsto \exp(\frac{2i\pi t}{|Z(W)|})x$ induces a well defined element $\beta \in Z(B)$. The element $\beta^{|Z(W)|} = \pi \in P$ is represented by the path $t \mapsto \exp(2i\pi t)x$. The main results of [DMM11] state that $Z(B)$ (resp. $Z(P)$) is cyclic and generated by β (resp. by π) when W is irreducible. Moreover, if $U \subset B$ is a finite index subgroup, then $Z(U) \subset Z(B)$. Using their results, along with Theorem 2.6, we are able to describe the center of enlarged braid groups.

Corollary 2.5 (Center of enlarged braid groups). *The center of B_S is infinite cyclic and generated by β_S. If $U \subset B_S$ is a finite index subgroup, then $Z(U) \subset Z(B_S)$. In particular, the center of P_S is infinite cyclic and generated by π_S.*

Proof. First, if $W = W_f$ has no regular hyperplane, then the natural morphism $B_S \to B$ (resp $P_S \to P$) is an isomorphism by Proposition 2.4. Since this morphism sends β_S to β (resp. π_S to π), the result is precisely [DMM11, Theorem 1.2 and Theorem 1.4].

In the general case, we consider W_f the full reflection group attached to W. We have $X_S = X_S(W) = X_S(W_f)$. By Proposition 2.4, B_S is a finite index subgroup of $B(W_f)$. This shows directly that if $U \subset B_S$ has finite index, then $Z(U) \subset Z(B_S)$.

From the first part of the proof, we know that the center of $B_S(W_f)$ is infinite cyclic and generated by $\beta_S(W_f)$ and that the center of B_S is infinite cyclic and generated by the smallest power of $\beta_S(W_f)$ which belongs to it.

For a positive integer k, $\beta_S(W_f)^k$ is the homotopy class of the image in X_S/W_f of the path in X_S given by

$$\gamma_k : t \mapsto e^{\frac{2ik\pi t}{|Z(W_f)|}} x_0.$$

We have $\beta_S(W_f)^k \in B_S$ if and only if the endpoint $e^{\frac{2ik\pi}{|Z(W_f)|}}$ of γ_k lies in $W.x_0$. Now, since $x_0 \in X_S(W)$, having $\lambda x \in W.x$ for some $\lambda \in \mathbb{C}^*$ is possible only if $\lambda\,\mathrm{Id} \in Z(W)$, in other words if λ is a power of $e^{\frac{2i\pi t}{|Z(W)|}}$. Thus, the endpoint of γ_k lies in $W.x_0$ if and only if k is a multiple of $[Z(W_f) : Z(W)]$. The center of B_S is then generated by the homotopy class of the image in X_S/W of the path

$$\gamma_{[Z(W_f):Z(W)]} : t \mapsto e^{\frac{2i\pi t}{|Z(W)|}} x_0,$$

which is precisely $\beta_S(W)$.

Lastly, $Z(P_S)$ is generated by the smallest power of $\beta_S(W)$ which lies in P_S. In other words, $Z(P_S)$ is generated by the homotopy class in X_S of the path $\gamma_{k[Z(W_f):Z(W)]}$, where k is the smallest integer such that the endpoint of $\gamma_{k[Z(W_f):Z(W)]}$ is x_0. Since the endpoint of $\gamma_{k[Z(W_f):Z(W)]}$ is $e^{\frac{2i\pi k}{|Z(W)|}}$, the smallest such integer is $k = |Z(W)|$, and $Z(P_S)$ is generated by the homotopy class in X_S of the path $\gamma_{|Z(W_f)|}$, which is π_S by definition. □

This corollary was the last argument needed, along with our commutative square of fibrations, to show the following result:

Theorem 2.6. *Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group. Let also $x_0 \in X_S$ be a basepoint. We have a commutative diagram, where all short sequences are exact:*

$$\begin{array}{ccccc}
\langle \pi_S \rangle & \hookrightarrow & \langle \beta_S \rangle & \twoheadrightarrow & Z(W) \\
\downarrow & & \downarrow & & \downarrow \\
P_S & \hookrightarrow & B_S & \twoheadrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\widehat{X_S}, [x_0]) & \hookrightarrow & \pi_1(\widehat{X_S}/W, W.[x_0]) & \twoheadrightarrow & \widehat{W}
\end{array}$$

In particular, the fundamental group of $\widehat{X_S}/W$ *(resp. of* $\widehat{X_S}$*) is isomorphic to* $B_S/Z(B_S)$ *(resp. to* $P_S/Z(P_S)$*).*

This theorem allows us to complete the proof of Theorem 1;

Theorem 2.7 (Computation of projective braid groups). *Let* V *be a finite dimensional complex vector space, and let* $G \subset \mathrm{PGL}(V)$ *be a nontrivial irreducible projective reflection group. Let also* $W \subset \mathrm{GL}(V)$ *be the maximal reflection group such that* $\widehat{W} = G$*, and let* $\widehat{X} := \{[x] \in \mathbb{P}(V) \mid \mathrm{Stab}_G([x]) = 1\}$*. The fundamental group of* $\widehat{X}/G$ *is isomorphic to* $B/Z(B)$*.*

Proof. By construction, we have that $W = W_f$ is its own full group. By Proposition 2.4, the morphism $B_S(W) \to B(W)$ induced by the inclusion $X_S/W \hookrightarrow X/W$ is an isomorphism. Now, by Lemma 2.1, we have $\widehat{X} = \widehat{X_S}$, thus $\widehat{X}/G = \widehat{X_S}/\widehat{W}$, and the fundamental group of this space is isomorphic to $B_S/\langle \beta_S \rangle \simeq B/\langle \beta \rangle = B/Z(B)$ by Theorem 2.6. □

Theorem 2.6 only exists at the level of enlarged braid groups, and cannot be extended to braid groups in every case (see Section 2.3 for more details). However, it can be extended in the case where all regular elements of W are central.

Corollary 2.8. *If all regular elements of* W *are central, then we have a commutative diagram, where all short sequences are exact:*

$$\begin{array}{ccccc}
\langle \pi \rangle & \hookrightarrow & \langle \beta \rangle & \twoheadrightarrow & Z(W) \\
\downarrow & & \downarrow & & \downarrow \\
P & \hookrightarrow & B & \twoheadrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\widehat{X}, [x_0]) & \hookrightarrow & \pi_1(\widehat{X}/W, W.[x_0]) & \twoheadrightarrow & \widehat{W}
\end{array}$$

In particular, the fundamental group of $\widehat{X}/W$ *(resp. of* $\widehat{X}$*) is isomorphic to* $B/Z(B)$ *(resp. to* $P/Z(P)$*).*

Proof. The only statement which is not directly implied by Theorem 2.6 is the fact that $\langle \beta \rangle = Z(B)$ (resp. $\langle \pi \rangle = Z(P)$) , which is known for all irreducible complex braid groups [DMM11, Theorem 1.1 and Theorem 1.2]. □

2.3. Correction of a result of Broué, Malle, Rouquier.

The natural commutative square

$$\begin{array}{ccc}
X & \longrightarrow & X/W \\
\downarrow & & \downarrow \\
\widehat{X} & \longrightarrow & \widehat{X}/\widehat{W}
\end{array}$$

Induces a commutative diagram of groups

$$\begin{array}{ccccc}
\langle \pi \rangle & \longrightarrow & \langle \beta \rangle & \longrightarrow & Z(W) \\
\downarrow & & \downarrow & & \downarrow \\
P & \longrightarrow & B & \longrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\widehat{X}, [x]) & \longrightarrow & \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x]) & & \widehat{W}
\end{array} \tag{2.2}$$

The second row of this diagram is a short exact sequence (see Section 1). The first row is a short exact sequence by [DMM11, Theorem 1.3]. The first column is a short exact sequence, induced by the $\mathbb{C}^*$-bundle $X \twoheadrightarrow \widehat{X}$ (restriction of the $\mathbb{C}^*$-bundle $V \setminus 0 \twoheadrightarrow \mathbb{P}(V)$). The third column is a short exact sequence by construction.

Contrary to the projection map $X_S/W \twoheadrightarrow \widehat{X_S}/\widehat{W}$ studied in Section 2.2, the projection map $X/W \twoheadrightarrow \widehat{X}/\widehat{W}$ may not be a fibration. In particular, we cannot use the same argument as in the proof of Theorem 2.6 to obtain that the second column of Diagram (2.2) is a short exact sequence. However, we still have the following partial result:

Proposition 2.9. *The morphism $B \to \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ induced by the projection map $X/W \to \widehat{X}/\widehat{W}$ is surjective. Furthermore, if $b \in B$ admits a nontrivial power in $Z(B)$, then the image of b in $\pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ is trivial.*

Proof. Let $f = (f_1, \ldots, f_n)$ be a system of basic invariants for W. Under the homeomorphism $V/W \simeq \mathbb{C}^n$ induced by f, the action of $Z \simeq \mathbb{C}^*$ is given by

$$\zeta.(x_1, \ldots, x_n) = (\zeta^{d_1} x_1, \ldots, \zeta^{d_n} x_n),$$

where $d_1, \ldots, d_n$ are the degrees of W (since f_i is homogeneous of degree i). The quotient space $(V \setminus 0/W)/(Z/Z(W)) \simeq (V \setminus 0)/\widetilde{W}$ is then isomorphic to the weighted projective space $\mathbb{P}(d_1, \ldots, d_n)$. In particular, it is an irreducible normal algebraic variety [Dol82, Proposition 1.3.3], and $\widehat{X}/\widehat{W}$ is an irreducible normal algebraic variety, since it is an algebraic open subset of $\mathbb{P}(d_1, \ldots, d_n)$.

Now, consider the open set $X_S \subset X$ studied in Section 2.2. For $x \in X_S$, we have a commutative diagram

$$\begin{array}{ccc} B_S & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_1(\widehat{X_S}/\widehat{W}, \widehat{W}.[x]) & \longrightarrow & \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x]) \end{array} \tag{2.3}$$

We are going to show that all the morphisms in this diagram are surjective. Let us recall that, if Y is an irreducible complex normal algebraic variety, and if $U \subset Y$ is a nonempty (algebraic) open subset of Y, then the natural morphism from the fundamental group of U to that of Y is surjective (see [ADH16, Theorem 2.1] and the references there).

- By Proposition 2.4, the morphism $B_S \to B$ is surjective.
- Since $\widehat{X}/\widehat{W}$ is a normal variety, and since $\widehat{X_S}/\widehat{W}$ is an analytic open subset of $\widehat{X}/\widehat{W}$, the morphism $\pi_1(\widehat{X_S}/\widehat{W}, \widehat{W}.[x]) \to \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ is surjective.
- The morphism $B_S \to \pi_1(\widehat{X_S}/\widehat{W}, \widehat{W}.[x])$ was proven to be surjective in Theorem 2.6.

Since three out of four arrows in Diagram (2.3) are surjective, then so is the fourth one, which is what we wanted to show.

Lastly, let d be a regular number for W, let $g \in W$ be a d-regular element, and let $x \in V(g, \zeta_d) \cap X$. Fixing x as a basepoint, we can consider the path $\gamma : [0,1] \to X$ sending t to $\exp(\frac{2i\pi t}{d})x$. The endpoint of γ is $\zeta_d x = g.x \in W.x$. Thus γ induces a well defined element $\widetilde{g}$ of B which is a lift of g under the projection map $B \twoheadrightarrow W$. By construction, $\widetilde{g}$ belongs to the kernel of the morphism $B \to \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ since the image of the path γ remains in $\widetilde{W}.x$ at all times.

Now, the element $\widetilde{g}$ is a d-th root of the full-twist π. Moreover, by [Gar23, Theorem 1.2 and Proposition 8.1], an element $b \in B$ admits a central power if and only if it is conjugate to a power of an element of the form $\widetilde{g}$ (for some regular element $g \in W$). Thus, any element of B admitting a central power belongs to the kernel of the morphism $B \twoheadrightarrow \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$. □

Note that the above result applies to β, which admits itself as a central power in B. In particular, the composition of the second column of Diagram (2.2) is trivial, even if the sequence is not exact.

Now, it is claimed in [BMR98, Proposition 2.23] that Diagram (2.2) can always be completed by a morphism $\pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x]) \to \widehat{W}$ into a commutative diagram in which every row and every column is a short exact sequence. It is pointed out in [DMM11] that this result is false in general. We give in a complete description of the cases in which it holds:

Proposition 2.10. *The following statements are equivalent:*

(i) All regular elements in W are central.

(ii) The second column in Diagram (2.2) is a short exact sequence.

(iii) There is a morphism $\pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x]) \to \widehat{W}$ which completes Diagram (2.2) into a commutative diagram.

Moreover, if these statements hold, then all the rows and columns in the completed diagram are short exact sequences.

Proof. Let $K \subset B$ denote the kernel of the natural morphism $B \to \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$. By Proposition 2.9, the group $\pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ is isomorphic to the quotient B/K. Point (iii) is then equivalent to stating that the kernel of the morphism $B \to \widehat{W}$ contains K. The kernel of the morphism $B \to \widehat{W}$ is the preimage of $Z(W)$ under the morphism $B \twoheadrightarrow W$: it is the subgroup of B generated by P and β.

$(i) \Rightarrow (ii)$ is proven in Corollary 2.8.

$(ii) \Rightarrow (iii)$: If the second column of Diagram (2.2) is a short exact sequence, then $K = \langle \beta \rangle \subset \langle P, \beta \rangle$, which proves (iii).

$(iii) \Rightarrow (i)$: we prove the contrapositive. Assume that W admits a noncentral regular element g. By [Gar23, Theorem 1.2], there is some root of π in B which is a lift of g under the projection map $B \twoheadrightarrow W$. By Proposition 2.9, ρ belongs to the kernel of the projection map $B \twoheadrightarrow \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$. However, since g is not central in W, the image of ρ in $\widehat{W}$ is not trivial. The kernel K of the morphism $B \to \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x_0])$ is then not included in $\langle P, \beta \rangle$ and (iii) is false. Lastly, we saw in Corollary 2.8 that, if all regular elements in W are central, then we can complete Diagram (2.2) into a commutative diagram in which all rows and all columns are short exact sequences. □

Remark 2.11. In the general case, Proposition 2.9 only gives that we have a surjective morphism $B/K \twoheadrightarrow \pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$, where K is the subgroup of B generated by the elements in B which admit a central power. We do not know whether this morphism is an isomorphism in the general case. However, preliminary computations seem to imply that in most cases, B/K is either trivial, or a cyclic group, which would in turn imply that $\pi_1(\widehat{X}/\widehat{W}, \widehat{W}.[x])$ is either trivial or cyclic, at least in a number of cases. We chose not to include those rather long computations here, as they only give partial results.

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