Thèse de Doctorat

Mention Mathématiques

présentée à ***l'École Doctorale en Sciences Technologie et Santé (ED 585)***

de l'Université de Picardie Jules Verne

par

Owen Garnier

pour obtenir le grade de Docteur de l'Université de Picardie Jules Verne

Garside groupoids and complex braid groups

Soutenue le 28 novembre 2024, après avis des rapporteurs, devant le jury d'examen :

M. Yves Guiraud	**CR INRIA, Université Paris Cité**	**Rapporteur**
M. Jon McCammond	**PR, UC Santa Barbara**	**Rapporteur**
M. Cédric Bonnafé	**DR, Université de Montpellier**	**Examinateur**
M. Adrien Brochier	**MCF, Université Paris Cité**	**Examinateur**
M. David Chataur	**PR, UPJV**	**Examinateur**
M. François Digne	**PR émérite, UPJV**	**Examinateur**
M. Eddy Godelle	**PR, Université de Caen Normandie**	**Président du Jury**
M. Ivan Marin	**PR, UPJV**	**Directeur de thèse**

À Charlie, Arthur et Delphine, pour tout.

Contents

Remerciements

Merci ! Merci ! Merci ! Comment commencer par un autre mot ? J'ai eu la chance immense d'être accompagné par des personnes très importantes pour moi durant tout le long chemin qui m'a mené à l'achèvement de cette thèse et il est temps pour moi de leur adresser les plus sincères remerciements, même si je sais que je leur dois bien plus encore.

Le difficile choix d'un ordonnancement m'est heureusement évité par la tradition. Je commence donc par remercier Ivan Marin d'avoir accepté de diriger cette thèse, ainsi que le mémoire de Master qui l'a précédée. Pendant ces quatre années de travail commun, tu n'as eu de cesse de m'amener a être plus rigoureux, autant en mathématique qu'à l'extérieur. Tu m'a appris a gérer les intimidantes questions administratives et à toujours prendre les devants. Tu as guidé mes travaux et tu m'as accompagné dans cette grande aventure tout en me laissant la liberté d'organiser mon travail comme je l'entendais. Je me souviens de cette première discussion où tu m'a expliqué l'interprétation du groupe de tresses usuel comme un groupe fondamental et je ne compte plus les éclairages qui, depuis, ont suivi sur les groupes de tresses complexes et tant d'autre choses. Si mes calculs sont justes, voilà 12 ans que tu t'occupes en continu d'une longue chaîne d'étudiant.e.s en thèse dont je suis -pour le moment au moins- le dernier maillon. Je n'ai donc pas la joie de transmettre quelque flambeau, mais je peux te remercier, Ivan, en notre nom à tous (je ne pense trahir personne) et te souhaiter la tranquillité de la vie sans doctorant.e.s dans tes pattes.

A warm thanks goes out to Jon McCammond and Yves Guiraud for having accepted the tedious task of refereeing this lengthy manuscript. Merci tout particulièrement à Yves pour avoir corrigé mon orthographe approximative et contrôlé les dégâts de mon intolérance chronique à la lettre s, and special thanks to Jon for accepting to endure my defense so early in the morning. Je remercie également Cédric Bonnafé, Adrien Brochier, David Chataur, François Digne et Eddy Godelle d'avoir accepté la responsabilité de faire partie de mon jury. Je suis également redevable à François et David d'avoir accepté de faire partie il y a déjà quelques années du jury de mon mémoire de Master. J'espère que ma soutenance fera bien montre de la progression de mes travaux.

Le travail de thèse ne se résume pas à la production d'un manuscrit : il consiste aussi à la découverte du monde de la Recherche et des esprits aimables et ouverts qui le peuplent. Je remercie donc Marithania Silvero, Bernhard Keller, Adrien Brochier, Alexis Virelizier, Yves Guiraud, Thomas Haettel et Hoel Queffelec, pour m'avoir permis de donner mes premiers exposés de séminaire loin de chez moi et m'avoir ainsi offert l'opportunité de découvrir quelques belles localités de ce vaste monde. Dans cette même logique, je remercie Juan González-Meneses de m'avoir accueilli à Séville pendant trois semaines fin 2022. Ce fut un séjour très fructueux pour moi et je suis très heureux d'avoir pu discuter longuement avec María Cumplido et Bruno Cisneros de recherche et de beaucoup d'autres choses. Plus généralement, je remercie tous les

collègues avec qui j'ai eu d'éclairantes discussions sur les mathématiques. Merci donc à Igor, Quentin, Hugo, Paul, David, Jean, Jean-Yves, Daniel, Ramla, Jacques et Serge.

Depuis mes débuts en Licence jusqu'à cette thèse, en passant par le Master et l'Agrégation, j'ai pu profiter de l'enseignement de nombreux.e.s professeur.e.s qui m'ont accompagné.e.s et ont participé a former le mathématicien que je suis devenu. En faire une liste exhaustive serait fastidieux et ne rendrait pas justice au travail accompli. Je remercie cependant Yann pour avoir guidé mes premiers pas vers les catégories, Élise, Lucie, Karine, Clément et Moussa pour l'algèbre linéaire, Alexander pour les représentations, Radu pour mes premières équations différentielles et pour les formes quadratiques, Abder et Clémence pour la géométrie différentielle, David pour les revêtements et les présentations, Emmanuel et Adrien pour l'homologie et Antoine pour les schémas. Même si je n'ai pas continué sur le chemin de l'analyse, je remercie également Benedetta pour les distributions, Stéphane pour les statistiques, Véronique, Marion, Jean-Paul, Olivier, Jérémy pour l'analyse numérique, Alberto pour les espaces de Hilbert et Claire pour les C^*-algèbres. Enfin, j'ai une pensée toute spéciale pour M. Hanotin et Mme. Godet, chez qui j'ai fait mes toutes premières armes mathématiques. Merci à toutes et à tous pour m'avoir donné l'envie d'en faire et d'en savoir plus !

Une autre opportunité de la thèse est celle de passer de l'autre côté du miroir de l'enseignement. Durant ces trois années, j'ai ainsi eu le plaisir d'enseigner des sujets variés à des étudiant.e.s remarquables et je remercie particulièrement Paul, Cyprien, Juliette, Mathilde, Julien, Léni, Georges, François, Lucie, Christophe, Anaëlle, Aurore et bien d'autres encore. Pour paraphraser Armstrong : Vous en apprendrez bien plus que je n'en saurai jamais.

En parlant de l'université, il serait injuste et impensable de ne pas remercier vivement nos membres non mathématicien.ne.s, qui m'ont si souvent sorti du pétrin et sans qui tout s'effondrerait sans le moindre doute. Un énorme merci donc à Mylène, Christelle, Étienne et Caroline pour le travail faramineux que vous faites concernant l'organisation, l'administration, les missions, les explications : vous méritez une ovation sans conditions !

Merci aussi à mes collègues doctorant.e.s. L'esprit d'équipe de cette petite troupe en fait un tout bien supérieur à la somme de ses parties ! Merci particulièrement à Damián pour tes conseils musicaux impeccables et pour toujours prendre de mes nouvelles. Merci à Tami pour ta gentillesse et pour cette satanée énigme des logiciens muets ! Plus généralement, merci à vous deux et à Bruno pour toutes ces sorties rafraîchissantes. Merci Michaël pour tes anecdotes savoureuses, ta bonne humeur communicative et toutes ces discussions gastronomiques ! Merci Marc pour ta culture bédéistique, tes boutades désopilantes et pour être le fournisseur officiel de jus de fruits de mon pot de thèse. Merci Maxime pour ta sympathie désarmante. Tu a été le premier à savoir que j'avais commencé la rédaction du manuscrit et conséquemment le premier à m'apporter ton soutien. Merci Nicolas pour ton rire communicatif et pour ton amour (presque) contagieux de Johnny : Je te promet que j'ai l'envie d'avoir envie d'écouter un de ces quatre. Dans l'espoir de n'oublier personne, merci aussi à Felipe, Karen, Bouasy, Gnord, Sheila et Ahmad. Parmi nos doctorant.e.s plus ancien.ne.s, merci au trio de choc Alice, Clément et Yohan : votre coaching impeccable m'a permis de survivre au débuts de la vie de doctorant ! Merci aussi à Ismaïl : ton coaching impeccable m'a permis de survivre à la suite de la vie doctorant, ainsi qu'aux affres de l'administratif de l'Éducation Nationale. Merci à Jihade, je me rappelle que c'est toi qui m'avait accueilli la première dans ce qui allait devenir mon bureau pour les trois prochaines années. Merci à Benjamin pour partager mon humour parfois douteux et pour ces nombreux exercices qui nous font garder l'esprit alerte ! Thanks also to Bastián, Mariem, Cheryl, Christopher, Afaf, Sebastian. Enfin, merci au petit.e.s nouvelle.aux de l'équipe :

Romain, Abigaël, Orlane, Pei, Simon, Adrien. Je sais que je laisse le Bureau entre de bonnes mains.

Une maxime sur-utilisée dit que le vrai trésor du voyage c'est les amis qu'on s'est fait en chemin. Si l'on accorde du crédit (sans mauvais jeu de mot) à cette maxime, alors je suis bien riche ! Un immense merci va aux ami.e.s qui m'ont soutenu et supporté durant ce long parcours. Merci à Elena pour ta culture politique et historique que tu partages sans compter. Merci à Kenza et Axelle. C'est un plaisir de pouvoir parler des universités avec une autre (futur) chercheuse. Merci à Mélanie pour m'avoir accompagné depuis si longtemps. L'époque du lycée parait bien lointaine et bien proche à la fois ! Merci également à Antoine pour ces retrouvailles régulières depuis 9 ans ! Merci Anne-Gaëlle pour les bons moment passés à Ambiani et ailleurs. Comme promis j'ai profité de ton cadeau d'anniversaire une fois ce manuscrit déposé. Merci à Akhane pour les discussions politiques et audiovisuelles ! Je vais regarder Cowboy Bebop et lire Damasio c'est promis ! Merci à Delphine et Arthur pour m'avoir si souvent accueilli à la maison et m'avoir écouté me plaindre de mes travaux qui n'avançaient pas assez vite. Merci aussi de m'avoir tant aidé à organiser le pot de thèse. Merci à Wolfgang et Ethan pour avoir toujours été si doux et compréhensifs avec moi. Vous m'avez aidé à découvrir plein de choses sur moi même. Merci à Ian, toi aussi pour ta grande culture bédéistique et artistique au sens large et pour me remettre à ma place quand je dis des âneries. Merci à Flo pour toutes ces tentaculaires discussions sur la vie l'univers et le reste. Merci à Lou pour l'aide que tu m'a apporté sur la traduction de l'introduction. Tu fais maintenant partie des rares personnes dont l'aventure mathématique a commencé avec la participation à une thèse de Doctorat !

Il est entendu que je ne saurais trop remercier ma famille pour tout : pour son amour sans limites naturellement, pour son soutien sans faille durant ces quatre dernières années et plus largement tout au long de mes études. Merci d'abord à mes grands parents ! Merci Mamie Danièle pour tout ce que tu nous a appris et pour toutes les merveilleuses vacances passées ensemble. Merci Papi Didier pour ces discussions professionnelles qui m'en ont beaucoup appris et pour ces fructueux séjours Leucatois. Je suis bien malheureux chère Françoise que tu n'ai pas pu assister à cette soutenance. Je ne peux que te remercier pour ces nombreux débats si enrichissants et pour m'avoir accueilli à Paris pendant mon Master. Merci Babeth pour m'avoir partagé tes lectures et m'avoir offert une partie de mes dix-neufs (sauf erreur) livres de Dune. Merci aussi à Régine, Jacques, Dédée, Rémi, sans oublier le jeune Robin ! Merci Papa pour ton soutien qui a été une grande source de force durant ce voyage. Merci pour ton savoir scientifique et pour m'avoir appris les rudiments de la géométrie. Merci à Laszlo pour ces déconnades et discussions interminables sur le lore de Skyrim, ce tacos/bière/café à Châtelet reste un excellent souvenir !

Malgré toute la littérature que tu m'as enseigné, je peine à trouver les mots pour te remercier ma chère petite Maman. Tu nous a tout donné à Arthur et moi depuis le début et tu t'es toujours donné tout le mal du monde pour nous transmettre l'envie d'apprendre et de progresser. De mes premiers mots jusqu'à cette soutenance, tu as accompagné ma découverte du monde, du savoir et des mathématiques. Depuis mon départ du nid, tes petites attentions constantes sont des points d'accroches qui m'ont permis de tenir le coup. Nos thèses sont aussi le résultats de tes sacrifices et cela s'étend jusqu'à l'organisation des pots ! J'espère que tu es fière de toi-même comme tu es fière de nous deux.

Ma très chère Delphine. On s'est rencontré il y a maintenant 12 ans, soit près de la moitié de ma vie ! Le moins que l'on puisse dire c'est que tu es devenue depuis une super frangine pour moi. C'est certainement toi qui a le plus eu a subir nos élucubrations mathématiques farfelues et tu t'en sors comme un charme. Merci pour ton indéfectible soutien si pragmatique, pour ton infatigable talent pour la logistique ainsi que pour tes critiques constructives sur mes blagues toujours bien trouvées.

Mon très cher Arthur, je ne sais pas par où commencer. Connaissant l'exercice, tu m'as suggéré de ne pas faire trop compliqué. Je commencerai donc par noter que c'est toi qui m'a le premier enseigné les définitions de groupes, de compacts, de groupes de réflexions complexes, de limites et ce n'est qu'un maigre échantillon des pantagruéliques connaissances mathématiques que tu partages avec moi sans compter. Nos longues parlottes m'ont tellement fait progresser sur mes capacités à faire des mathématiques, y compris sans papier crayon, que je ne sais pas où j'en serais sans ça. Même sans tenir compte des maths, tu as toujours su m'épauler et m'accompagner. Je suis fier de pouvoir dire que j'ai tenu la distance avec toi et je te souhaite paix et prospérité dans tes prochaines aventures.

Avant de clore ces remerciements, je me dois de remercier tous les artistes (et autres) dont les productions ont accompagné les heures de réflexions (complexes ?) et de rédaction qui ont meublé ces 1468 jours depuis le début de mon mémoire de Master (et depuis plus longtemps encore). Merci donc aux Smashing Pumpkins, Beatles, Siouxsie & the Banshees, Mylène Farmer, Billie Eilish, Caravan Palace, Bobby Lapointe, Zambla, Kristofer Maddigan, Lena Raine, Chopin, Rachmaninov, Beethoven, leurs interprètes, Ben Prunty, Edmund McMillen, Danielle Dax, Cure, Emily Remler, David Bowie, George Brassens, Queen et Tame Impala (entre autres !). Merci aussi à tous les créateurs de contenu dont j'ai pu profiter. Dans un ordre aléatoire, je remercie donc Mynthos, Christophe Michel, Baghera, Zerator, Ponce, Shisheyu, DFG, Ultia, MisterMV, JDay, Florence, JDG, Trivia, Karim Debbache, Defakator, Horty, Antoine Daniel, Étoiles, Michael Stevens, Julien Benoît, Benzaie et Schaffrillas.

Ça y est. C'est enfin ton tour. Je peux enfin essayer d'aligner quelques mots maladroits pour te rendre une infime partie du bonheur que tu me donnes au quotidien. Ton addition dans ma vie est récente, mais ça ne soustrait rien à l'amour que je te porte. Je veux multiplier les moments avec toi et que jamais les épreuves ne divisent notre équipe. J'espère que la suite de nos aventures ne sera ni exacte, ni courte. Merci de m'avoir si bien accompagné sur ce grand sprint final de la rédaction. Merci pour toutes ces attentions petites et grandes qui me vont toujours droit au cœur. Merci de toujours savoir comment me remonter le moral et me donner la force d'avancer. Le temps passé ensemble est toujours en caractéristique impaire, car être à deux avec toi, c'est jamais nul. Je t'aime Charlie.

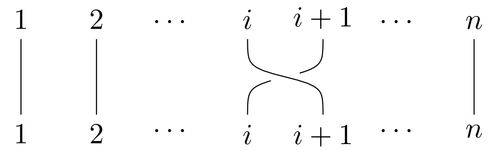
Introduction (en français)

Contexte et vue d'ensemble

Une très rapide histoire du groupe de tresses usuel

Les tresses mathématiques ont été formellement décrites pour la première fois par Artin dans [Art25], où elles sont définies comme des familles particulières de courbes ne s'intersectant pas dans l'espace euclidien (tresses géométriques), à une certaine relation d'équivalence près, obtenue en déformant les tresses géométriques. Bien-sûr, « déformer les tresses géométriques » peut avoir plusieurs sens a priori distincts. Cependant, il fut montré dans [Art47] que la plupart des notions naturelles de déformation donnent le même ensemble de classes d'équivalences (voir également [Deh19, Chapter 1]).

De plus, Artin a montré dans [Art25] que l'ensemble des tresses à n brins est naturellement équipé d'une loi de composition interne qui en fait un groupe, noté B_n (en anglais, le mot tresse se dit « braid », d'où la notation). Il en donne également une présentation par générateurs et relations : le groupe B_n est engendré par $n-1$ éléments $\sigma_1, \ldots, \sigma_{n-1}$ (les « générateurs d'Artin »), où σ_i est représentée par la tresse géométrique suivante :



Les relations sont données par $\sigma_i\sigma_j = \sigma_j\sigma_i$ quand $|i-j| > 1$ et par $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ pour $i \in [\![1, n-2]\!]$ (on s'avise facilement du fait que ces relations sont vérifiées en dessinant le diagramme associé).

Dans les années suivantes, quelques autres propriétés fondamentales des groupes de tresses ont été décrites : le pont entre tresses, nœuds et liens a été étudié par Alexander [Ale28] (article dans lequel le célèbre *polynôme d'Alexander* d'un nœud a été défini pour la première fois) et par Markov [Mar45] (voir également [Bir74, Chapter 2]). La théorie des représentations des groupes de tresses a été étudiée par Burau in [Bur35], où l'auteur décrit une représentation irréductible de dimension $n-1$ du groupe de tresses à n brins. Cette représentation (appelée de nos jours *représentation de Burau*) peut être utilisée pour construire le polynôme d'Alexander, mettant en évidence l'importance de la théorie des représentations de B_n, notamment en théorie des nœuds. Plus tard, dans les années soixante, Fox et Neuwirth ont réalisé dans [FN62b] le groupe B_n comme le groupe fondamental de l'espace des configurations de n points du plan complexe (voir également [FN62a]) et ils ont montré que ce dernier espace est un espace $K(\pi, 1)$ (ou espace d'Eilenberg-MacLane) (nous verrons dans la prochaine section que ceci sera un thème récurrent).

La cohomologie du groupe de tresses a été étudiée dans les années soixante-dix, d'abord par Arnold [Arn70], dans lequel l'auteur a montré en particulier des théorèmes de finitude, récurrence et stabilisation de cette cohomologie. Ces travaux ont été poursuivis par Fuks, qui a calculé la cohomologie du groupe de tresses modulo 2 dans [Fuk70]. Indépendamment et par des méthodes différentes, l'homologie de B_n a été étudiée par Cohen [Coh73a], [Coh73b], [CLM76] (pour plus de détails, voir [Ver98]).

Parmi les problèmes ouverts donnés par Artin dans [Art47] se trouvent le problème de conjugaison (existe-t-il un algorithme permettant de déterminer si deux tresses sont conjuguées ?), ainsi que la détermination du centralisateur d'une tresse donnée. Le premier de ces problèmes a été résolu par Garside dans [Gar69], où il introduit un sous-monoïde B_n^+ de B_n, ainsi qu'un élément particulier $\Delta \in B_n^+$. Il montre ensuite que le monoïde B_n^+ est simplifiable et admet à la fois des plus petits multiples communs (ppcm) et des plus grands diviseurs communs (pgcd). Ceci permet de décrire la totalité du groupe B_n en utilisant seulement B_n^+ et Δ : tout élément de B_n peut être écrit de façon unique comme un produit $\Delta^m b$, avec $m \in \mathbb{Z}$ maximal et $b \in B_n^+$. En utilisant cette description, Garside a ensuite défini un sous-ensemble caractéristique de toute classe de conjugaison dans B_n, donnant ainsi une solution au problème de conjugaison : le *summit-set* (« ensemble sommital » pour une traduction plus rigoureuse).

Les travaux de Garside ont été généralisés dans les années quatre-vingts en construisant des décompositions distinguées des éléments de B_n faisant apparaître l'élément Δ et le monoïde B_n^+. Même si de telles décompositions ne se trouvent pas explicitement dans [Gar69], le fait que les pgcds existent dans B_n^+, résultat démontré dans [Gar69], est une étape importante dans leur construction. Les premières apparitions explicites d'une telle décomposition, dite *forme normale*, se trouvent dans les travaux indépendants d'Adjan [Adj66], d'El-Rifai et Morton [EM94] et de Thurston [Thu88]. La forme normale est ensuite devenue un outil important dans l'étude des groupes de tresses. Par exemple, El-Rifai et Morton ont amélioré dans [EM94] la solution du problème de conjugaison donnée par Garside, en considérant un sous-ensemble caractéristique des classes de conjugaison strictement plus petit que les summit-sets de Garside, les *super summit-sets* (là encore, « ensemble super-sommital » serait une traduction plus rigoureuse).

Le deuxième problème (la détermination des centralisateurs) a été résolu théoriquement par Makanin dans [Mak71], dans lequel il donne une méthode pour calculer des générateurs du centralisateur d'une tresse arbitraire. Plus tard, un algorithme plus efficace a été proposé par Franco et González-Meneses dans [FG03a], là encore en utilisant des généralisations des travaux de Garside.

Une autre application de la forme normale sur B_n se trouve dans sa théorie des représentations. Il est connu que la représentation de Burau que nous avons mentionnée plus haut est fidèle pour $n \leqslant 3$, mais il a été prouvé que cette représentation n'est pas fidèle pour $n \geqslant 9$ par Moody [Moo91], puis pour $n \geqslant 6$ par Long et Paton [LP93], puis pour $n = 5$ par Bigelow [Big99]. La question de savoir si la représentation de Burau de B_4 est fidèle ou non est toujours ouverte. Là encore, un résultat récent sur ce problème [GWY23] utilise la forme normale sur B_4.

Dans le cas général, la non-fidélité de la représentation de Burau de B_n laisse ouverte la question de la *linéarité* de ce groupe : est-ce que B_n se plonge dans un groupe linéaire de dimension finie sur un corps ? Au début des années deux-mille, Krammer a construit dans [Kra00] une représentation du groupe de tresses B_n dans $\mathrm{GL}_{\frac{n(n-1)}{2}}(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$ et il a montré que cette représentation est fidèle dans le cas du groupe de tresses à quatre brins. Ensuite, Krammer [Kra02] et Bigelow [Big01] ont tous les deux prouvé que cette représentation de B_n est fidèle pour tout n.

Un autre aspect important du groupe de tresses est sa description en tant que *groupe modulaire* (« mapping class group » en anglais). Rappelons que si S est une surface compacte orientée (possiblement avec bord) avec un ensemble fini de points marqués $p := \{p_1, \ldots, p_n\} \subset S$, alors le groupe modulaire de S est défini comme l'ensemble des classes d'isotopie d'homéomorphismes de S qui préservent l'orientation, fixent globalement l'ensemble p et fixent le bord ∂S ponctuellement. Magnus a démontré dans [Mag34] que le groupe de tresses B_n se réalise comme le groupe modulaire du disque fermé avec n trous. Cette interprétation du groupe de tresses est utile car elle permet d'étendre la classification de Nielsen-Thurston des homéomorphismes des surfaces aux tresses. Une tresse est alors *périodique*, *réductible*, ou *pseudo-Anosov*. Cette trichotomie a été utilisée pour prouver plusieurs résultats sur les tresses. Par exemple, elle a été utilisée par González-Meneses dans [Gon03] pour prouver que deux racines k-ièmes d'un même élément de B_n sont toujours conjuguées. En outre, González-Meneses et Wiest ont décrit dans [GW04] le centralisateur d'une tresse en fonction de son type de Nielsen-Thurston.

Interpréter le groupe de tresses B_n comme groupe modulaire induit également des actions de B_n sur tout objet dans le disque percé qui soit défini à isotopie près. Par exemple, B_n agit sur l'ensemble des (classes d'isotopie de) courbes fermées simples et non-dégénérées tracées dans le disque percé. Ces courbes forment les sommets d'un complexe simplicial, appelé le *complexe de courbes* et B_n agit par isométries sur ce complexe. Il a été démontré par Masur et Minsky dans [MM99] que le complexe de courbes est δ-hyperbolique. On obtient alors une action de B_n sur un complexe simplicial δ-hyperbolique, ce qui induit des résultats de théorie géométrique des groupes.

Les groupes d'Artin comme première généralisation

Les groupes d'Artin (ou groupes d'Artin-Tits) ont été introduits par Tits dans [Tit66]. Ces groupes sont définis via une présentation similaire à la présentation donnée par Artin du groupe de tresses usuel. Cette première approche algébrique trouva vite un pendant topologique dans les travaux de Brieskorn [Bri71], où l'auteur réalise le groupe d'Artin $A(W)$ associé à un système de Coxeter (W, S) (avec W fini) comme le groupe fondamental d'un espace topologique attaché naturellement à W (nommément, l'espace des orbites complexes régulières). Dans le cas particulier où $W = \mathfrak{S}_n$ est le groupe symétrique, ce dernier espace est simplement l'espace des configurations de n points complexes. Le résultat de Brieskorn devient une généralisation du résultat de Fox et Neuwirth [FN62b] dans le cas du groupe symétrique.

Partant de là, il semble naturel de considérer les groupes d'Artin associés à des groupes de Coxeter finis (i.e. les *groupes d'Artin sphériques*) comme une généralisation du groupe de tresses usuel, ce qui a mené à plusieurs axes de recherche visant à généraliser les résultats connus des groupes de tresses usuels (par exemple, ceux mentionnés dans la section précédente) aux groupes d'Artin sphériques.

En premier lieu, Brieskorn, Saito [BS72] et Deligne [Del72] ont indépendamment résolu le problème de conjugaison et le problème du mot pour les groupes d'Artin sphériques. Ils ont également déterminé le centre de ces groupes. Les deux approches généralisent les travaux de Garside [Gar69] en introduisant un sous-monoïde particulier $A^+(W)$ d'un groupe d'Artin sphérique $A(W)$. Nous avons évoqué que l'espace de configurations de n points complexes est un espace classifiant pour le groupe de tresses à n brins. En suivant cette idée, Brieskorn a conjecturé dans [Bri73] que l'espace des orbites complexes régulières d'un groupe de Coxeter fini fournit aussi un espace classifiant pour le groupe d'Artin associé. Il a ensuite prouvé cette conjecture dans plusieurs, mais pas tous, les cas irréductibles. Rapidement, cette conjecture fut prouvée uniformément par Deligne dans [Del72], là encore en utilisant le sous-monoïde $A^+(W)$.

Plus tard, dans les années quatre-vingt-dix, la (co)homologie des groupes d'Artin sphériques fut calculée par Salvetti dans [Sal94] via une approche géométrique et par Squier dans [Squ94] via une approche plus algébrique (utilisant des généralisations des travaux de Garside, Brieskorn-Saito et Deligne).

Concernant la théorie des représentations, des travaux indépendants de Cohen, Wales [CW02] et Digne [Dig03] ont démontré que les groupes d'Artin sphériques sont linéaires. La représentation qu'ils utilisent est une généralisation de la représentation de Krammer du groupe de tresses usuel. Les preuves adaptent les arguments de Krammer, qui reposent eux-mêmes sur la forme normale définie dans le groupe de tresses usuel, en utilisant le monoïde introduit par Garside.

On constate que l'influence des travaux de Garside sur le groupe de tresses usuel est omniprésente dans l'étude des groupes d'Artin sphériques. Ceci est notamment dû au fait que le sous-monoïde $A^+(W)$ d'un groupe d'Artin sphérique $A(W)$ se comporte d'une manière très similaire au monoïde B_n^+ introduit par Garside. Vers la fin des années quatre-vingt-dix, cette réalisation a mené au développement de la notion de « monoïde de Garside », qui généralise les bonnes propriétés algébriques du monoïde B_n^+, mais nous différons cette histoire à la prochaine section.

Notons que ces généralisations laissent encore quelques questions sans réponse : par exemple, il n'y a pas d'interprétations générales des groupes d'Artin en tant que groupes modulaires. En particulier, il n'y a pas de généralisation géométrique de la classification de Nielsen-Thurston pour les éléments des groupes d'Artin sphériques (la notion d'élément périodique peut être généralisée algébriquement comme les éléments qui admettent une puissance centrale). Par conséquent, les arguments de González-Meneses [Gon03] prouvant que les racines sont uniques à conjugaison près dans le groupe de tresses usuel ne sont pas généralisables en l'état. Notons toutefois que le résultat d'unicité des racines à conjugaison près a tout de même été généralisé par Lee et Lee [LL10] au cas des groupes d'Artin associés aux groupes de Coxeter de type B, en les plongeant dans le groupe de tresses usuel.

Groupes et groupoïdes de Garside

Nous l'avons mentionné dans la section précédente : les travaux de Garside sur le groupe de tresses usuel ont été rapidement adaptés au contexte plus général des groupes d'Artin sphériques. Plus tard ces adaptations ont été formalisées dans une théorie algébrique générale, qui est maintenant appelée « théorie de Garside ». L'avènement de cette théorie est décrit comme un programme « naturel mais dont la mise en place fut longue » (« natural but slowly emerging ») dans l'ouvrage de référence [DDGKM], publié quarante-six ans après la thèse de Garside ! Il paraît alors raisonnable de découper cette riche histoire en plusieurs actes distincts, de sorte à mettre en exergue les nombreuses nouvelles idées et méthodes apparues dans ce long intervalle.

L'acte 1 peut être désigné comme la proto-histoire des structures de Garside. Il contient la plupart des travaux que nous avons évoqués dans les sections précédentes, c'est-à-dire les généralisations des travaux de Garside [Gar69] aux groupes d'Artin sphériques. Ce premier acte se termine à la fin des années quatre-vingt-dix avec l'émergence de nouvelles structures ressemblant aux groupes d'Artin sphériques (et à leurs sous-monoïdes positifs), tout en étant qualitativement différentes. Par exemple, Birman, Ko et Lee ont introduit dans [BKL98] un nouveau sous-monoïde du groupe de tresses usuel, qui donne lieu à de nouvelles solutions du problème du mot et du problème de conjugaison et qui n'est pas un monoïde d'Artin-Tits. Concomitamment, Dehornoy considère les interactions entre le groupe de tresses usuel et les systèmes auto-distributifs [Deh92], [Deh94]. En particulier, il introduit un monoïde M_{LD} dans lequel une forme normale évoquant la forme normale sur B_n existe. Cependant, le monoïde M_{LD} ne contient pas d'élément similaire à l'élément $\Delta \in B_n$, ce qui met en évidence le besoin de généraliser les travaux de Garside.

L'acte 2 commence au tournant du siècle avec l'article fondateur de Dehornoy et Paris [DP99], qui introduit les notions de monoïdes de Garside et de groupes de Garside, cependant pas tout à fait dans leurs formes modernes : les auteurs considèrent trois classes distinctes de monoïdes : les monoïdes Gaussiens, les petits monoïdes Gaussiens et les monoïdes de Garside (chaque classe contenant la suivante).

Pour faire court, un monoïde simplifiable M est *Gaussien* si la longueur d'une décomposition de $x \in M$ en un produit (sans terme non trivial) est bornée (la borne dépendant de x) et si les ppcms de deux éléments de M existent toujours. Un monoïde Gaussien est *petit* s'il admet un élément Δ dont les diviseurs à gauche et à droite coïncident, sont en nombre fini et engendrent M. Un petit monoïde Gaussien est *de Garside* si l'élément Δ peut être choisi comme le ppcm (à droite ou à gauche) des atomes de M. Un monoïde Gaussien (resp. petit Gaussien, de Garside) se plonge toujours dans son groupe enveloppant $G(M)$, qui peut être décrit comme un groupe de fractions. Le groupe $G(M)$ est alors appelé un groupe Gaussien (resp. petit Gaussien, de Garside). Plus généralement, on décrira un groupe Gaussien comme un triplet (G, M, Δ), où (M, Δ) est un petit monoïde Gaussien et où G est le groupe enveloppant de M. Par construction, le monoïde $A^+(W)$ associé à un groupe d'Artin sphérique $A(W)$ est un monoïde de Garside dans le sens de [DP99], de même que le monoïde défini par Birman, Ko et Lee dans [BKL98].

La terminologie de (petit) groupe Gaussien fut rapidement abandonnée au profit de celle de groupe de Garside. Dans la terminologie moderne, les groupes de Garside sont ceux que Dehornoy et Paris avaient originellement appelés petits groupes Gaussiens. Ce changement de terminologie est déjà présent dans [Deh02], où il est attribué par Dehornoy à l'influence de travaux de Bessis, Charney, Digne et Michel entre autres. Cela dit, plusieurs articles furent écrits avec la terminologie des groupes Gaussiens [Pic00], [Pic01a], [DL03], [Pic01b].

La formalisation algébrique des groupes de Garside est à l'origine de plusieurs axes de recherche dans les années deux-mille, visant à décrire à la fois les aspects théoriques et les aspects algorithmiques de ces groupes. Dans [DL03], Dehornoy et Lafont donnent deux méthodes algébriques pour calculer l'homologie des groupes de Garside. La deuxième méthode s'inspire des travaux de Squier [Squ94] sur l'homologie des groupes d'Artin. Une approche plus topologique est développée dans [CMW04], se basant sur des travaux de Bestvina [Bes99], là encore sur les groupes d'Artin sphériques.

La conjugaison dans les groupes de Garside fut largement étudiée : d'abord, l'algorithme d'El-Rifai et Morton [EM94] utilisant le super summit-set a été généralisé aux groupes de Garside dans [Pic01b] et amélioré dans [FG03b]. Un sous-ensemble particulier du super summit-set, appelé l'*ultra summit-set* fut introduit par Gebhardt dans [Geb05]. Cet ensemble a été étudié exhaustivement par Birman, Gebhardt et González-Meneses dans une série d'articles [BGG07a],[BGG08], [BGG07b], dans l'espoir de donner des algorithmes de complexité polynômiale pour résoudre le problème de conjugaison dans les groupes de tresses. Dans [GG10b] et [GG10a], Gebhardt et González-Meneses ont introduit une nouvelle opération pour résoudre le problème de conjugaison dans les groupes de Garside. Cette nouvelle opération, appelée *cyclic sliding*

(« glissement cyclique » est une traduction rigoureuse, mais inutilisée) induit des sous-ensembles caractéristiques des classes de conjugaison qui sont encore plus petits que les ultra summit-sets. Ces sous-ensembles sont appelés les *sliding circuits* (« circuits glissants »).

Parmi d'autres travaux, citons [FG03a], dans lequel Franco et González-Meneses donnent un algorithme pour calculer un ensemble de générateurs du centralisateur d'un élément dans un groupe de Garside. Aussi, l'extraction de racines d'un élément d'un groupe de Garside a été étudiée par Sibert dans [Sib02] et par Zheng dans [Zhe06].

Notons que l'approche par les monoïdes et les groupes de Garside reste insuffisante pour étudier le monoïde M_{LD} de façon adéquate, celui-ci n'étant pas Gaussien car deux éléments de M_{LD} peuvent ne pas admettre de multiples à droite (ceux qui en admettent admettent toujours un ppcm à droite). Cependant, Dehornoy fut tout de même en mesure d'étudier ce monoïde dans [Deh00] et de le connecter avec le groupe de tresses B_∞ sur un nombre infini de brins.

Le deuxième acte touche à sa fin dans la deuxième moitié des années deux-mille, avec la réalisation que le fonctionnement de la forme normale telle que construite pour les monoïdes de Garside peut facilement être généralisé aux catégories plutôt qu'aux monoïdes. L'acte 3 est ainsi marqué par l'avènement des catégories et des groupoïdes de Garside. On peut (plus ou moins arbitrairement) fixer le début de cet acte à la publication par Krammer de [Kra08], qui est le premier article mentionnant explicitement des structures de Garside sur des catégories (cet article est d'abord paru sous forme de pré-publication en 2005). Mentionnons également que des travaux antérieurs utilisent implicitement des structures semblables (voir par exemple [DL76], [Del72], [Deh94] ou [God01]).

Moins de deux années après la pré-publication de [Kra08], d'autres travaux de Digne, Michel [DM06] et Bessis [Bes07] ont également introduit des analogues catégoriques des monoïdes de Garside et des groupes de Garside. Ces travaux sont tous deux motivés par des situations dans lesquelles de tels analogues apparaissent spontanément et dans lesquelles les monoïdes de Garside ne sont pas suffisants : respectivement, l'étude des variétés de Deligne-Lusztig et la théorie des éléments périodiques dans un groupe de Garside. Déjà dans [DM06], les auteurs mentionnent le projet d'écrire un livre de référence sur le sujet des catégories de Garside.

Le troisième acte se termine (presque par définition) avec la publication de l'ouvrage de référence [DDGKM] par Dehornoy, Digne, Godelle, Krammer et Michel, qui est l'aboutissement du projet mentionné plus haut. Durant les sept années d'écriture de ce livre, les axiomes des structures de Garside ont été élargis et détendus pour obtenir l'approche finale présente dans [DDGKM]. Cette approche est centrée autour du concept de *décomposition gloutonne*, qui généralise la forme normale dans le monoïde de tresses B_n^+. Une *famille de Garside* sur une catégorie est alors définie comme une sous-famille de la catégorie qui donne lieu à une telle décomposition de chaque élément de la catégorie. Cette approche très générale s'applique dans une grande variété de contextes (voir [DDGKM, Part B]), mais elle donne des résultats nettement plus faibles que la théorie originale des groupes de Garside. Par exemple, l'étude d'un groupoïde partant d'une sous-catégorie n'est plus possible via la seule donnée d'une famille de Garside sur cette catégorie. Cela amène les auteurs à considérer des cas particuliers plus restrictifs de familles de Garside, notamment celles qui proviennent d'une *application de Garside* (qui généralise aux catégories la notion d'élément de Garside Δ dans un monoïde de Garside). Notons toutefois qu'aucune notion de « catégorie de Garside » n'est vraiment définie dans [DDGKM].

De nos jours (dans ce que l'on pourrait appeler l'acte 4), les groupes de Garside sont toujours étudiés pour eux-mêmes et ils n'ont pas été remplacés par les groupoïdes de Garside, mais plutôt complétés par eux. Des travaux récents de Paolini, Salvetti [PS21] et Delucchi, Paolini, Salvetti [DPS22] utilisent des ensembles ordonnés associés à des groupes d'Artin non sphériques qui ne sont pas des treillis et qui, par conséquent, ne donnent pas lieu à des structures de Garside. Néanmoins, ces ensembles ordonnés satisfont une importante propriété combinatoire (« EL-shellability ») et ils induisent également des modèles pour étudier des espaces classifiants de la même manière que les structures de Garside classiques. Ceci suggère la nécessité d'une nouvelle généralisation des structures de Garside.

Par ailleurs, il est dit dans [DDGKM] que le livre propose la « généralisation ultime » (« ultimate generalization ») dans la direction des décompositions gloutonnes (par définition, les familles de Garside sont exactement celles qui donnent lieu à une telle décomposition). Cependant, les auteurs soulignent également que d'autres directions de généralisations sont possibles. Par exemple, ils attribuent à Crisp, McCammond et Krammer l'approche consistant à voir un groupe de Garside essentiellement comme un groupe agissant sur un treillis dans lequel un certain intervalle joue un rôle prépondérant. Dans le même état d'esprit, Haettel et Huang ont récemment proposé dans [HH23] une caractérisation des groupes de Garside (faibles) comme des groupes agissant sur un certain complexes de drapeaux.

Groupes de tresses complexes et le programme de Broué-Malle-Rouquier

Les groupes d'Artin sphériques sont connus pour partager de nombreuses propriétés avec le groupe de tresses usuel. Partant de cette observation, il est naturel de chercher des généralisations des groupes d'Artin sphériques et de déterminer quelles propriétés ces généralisations partagent avec le groupe de tresses usuel. Dans la section précédente nous avons discuté d'une généralisation algébrique donnée par les groupes (et les groupoïdes) de Garside. Dans cette section, nous considérons une généralisation davantage topologique/géométrique, donnée par les groupes de tresses complexes.

Bien que les groupes d'Artin sphériques soient définis à partir des groupes de Coxeter finis, les travaux de Brieskorn [Bri71] considèrent des sous-groupes finis de $\mathrm{GL}_n(\mathbb{R})$ qui sont engendrés par des réflexions (des *groupes de réflexions réels*). Ces deux notions sont équivalentes dans le sens suivant : d'abord, si (W, S) est un système de Coxeter avec W fini, alors W admet une représentation fidèle en tant que groupe de réflexions réel (c'est la *représentation de Tits*). Réciproquement, tout groupe de réflexions réel $W \subset \mathrm{GL}_n(\mathbb{R})$ admet une présentation de Coxeter. Sous cette correspondance, l'espace des orbites régulières évoqué dans une section précédente est simplement l'espace des orbites de W agissant sur $\mathbb{C}^n$, où les complexifiés des hyperplans associés aux réflexions de W ont été retirés.

Une direction de généralisation possible est alors d'étendre le corps de base de $\mathbb{R}$ à $\mathbb{C}$ et de travailler avec des *groupes de réflexions complexes* (tous les groupes de réflexions seront supposés finis, sauf mention explicite du contraire). Par construction, les groupes de réflexions complexes sont en particulier une généralisation des groupes de réflexions réels, mais tout groupe de réflexions complexe ne provient pas d'un groupe de réflexions réel (par exemple, les réflexions complexes peuvent avoir un ordre arbitraire, tandis qu'une réflexion réelle est toujours involutive). On réduit facilement l'étude des groupes de réflexions complexes à celle des *groupes de réflexions complexes irréductibles*, c'est-à-dire les groupes de réflexions complexes $W \subset \mathrm{GL}_n(\mathbb{C})$ tels que $\mathbb{C}^n$ n'admet pas de sous-espace non trivial qui soit laissé globalement stable par W. Le *rang* d'un groupe de réflexions complexe irréductible $W \subset \mathrm{GL}_n(\mathbb{C})$ est défini comme étant l'entier n.

Les groupes de réflexions complexes ont été considérés de façon systématique à partir des années cinquante. Shephard a introduit la notion de *polytope complexe régulier* [She52] et a étudié une classe particulière de groupes de réflexions complexes [She53]. Les groupes de réflexions complexes irréductibles ont été rapidement classifiés par Shephard et Todd [ST54], en s'appuyant en particulier sur la littérature préexistante concernant les groupes engendrés par des collinéations sur les espaces projectifs. D'une part, il y a une « série infinie » de groupes irréductibles notés $G(de, e, n)$ qui dépendent de trois paramètres entiers d, e, n. D'autre part, il y a une suite finie de 34 groupes exceptionnels, notés $G_4, \ldots, G_{37}$ dans [ST54]. Notons que plusieurs de ces groupes apparaissent comme groupes de symétrie d'un polytope complexe régulier introduit dans [She52], ils sont nommés *groupes de Shephard* dans [OS88b].

En utilisant ce résultat de classification, il devient possible d'étudier les groupes de réflexions complexes au cas par cas. Un premier exemple de cette approche se trouve dans la preuve par Shephard et Todd [ST54] du fait que les groupes de réflexions complexes sont exactement les sous-groupes finis de $\mathrm{GL}_n(\mathbb{C})$ dont l'anneau des invariants dans l'algèbre polynomiale $\mathbb{C}[X_1, \ldots, X_n]$ est encore une algèbre polynomiale (notons que Chevalley a proposé une preuve conceptuelle de ce résultat dans [Che55]).

Il est bien connu que les groupes de réflexions *rationnels* (i.e. les groupes de Weyl) contrôlent de nombreux aspects de la théorie des représentations des groupes algébriques, entre autres. Ceci peut se comprendre en associant à un groupe de réflexions rationnel une *algèbre de Hecke* $\mathcal{H}(W)$, définie comme une déformation de l'algèbre de groupe $\mathbb{Z}W$. Dans le cas plus général où W est un groupe de réflexions réel, W admet une présentation de Coxeter, qui permet une définition directe de $\mathcal{H}(W)$ par générateurs et relations. Il est assez facile de montrer que $\mathcal{H}(W)$ est libre de rang $|W|$ sur son anneau de définition. Ces algèbres de Hecke associées aux groupes de Coxeter sont largement étudiées aujourd'hui (voir par exemple [GP00] ou [Eli+20]).

Dans les années quatre-vingt-dix, Broué, Malle, Michel [BMM93] et Broué, Malle [BM93] ont réalisé que les groupes de réflexions complexes jouent également un rôle dans la théorie des représentations des groupes réductifs finis. Il est alors naturel pour comprendre ce rôle de vouloir définir une algèbre de Hecke pour un groupe de réflexions complexe quelconque. Les premiers travaux dans ce sens par Ariki, Koike [AK94], Ariki [Ari95] et Broué, Malle [BM93], qui considèrent les groupes de la série infinie ainsi que certains groupes exceptionnels, utilisent une approche consistant à déformer une présentation du groupe de réflexions W en une présentation de l'algèbre de Hecke $\mathcal{H}(W)$. Cependant, généraliser cette approche aux autres cas pose le problème de la dépendance en la présentation choisie. En effet, il n'existe pas de bon analogue des présentations de Coxeter pour les groupes de réflexions complexes et donc le choix d'une présentation de W pourrait impacter la structure de l'algèbre $\mathcal{H}(W)$ associée. Ce fut une des principales motivations de Broué, Malle et Rouquier pour introduire dans [BMR98] les *groupes de tresses complexes* associés aux groupes de réflexions complexes.

À un groupe de réflexions complexe $W \subset \mathrm{GL}_n(\mathbb{C})$ on peut associer l'ensemble $\mathcal{A}$ des hyperplans associés aux réflexions de W, ainsi que le complémentaire X dans $\mathbb{C}^n$ de la réunion des éléments de $\mathcal{A}$. Le *groupe de tresses* $B(W)$ (resp. le *groupe de tresses pures* $P(W)$) est alors défini comme le groupe fondamental de X/W (resp. X). Un théorème classique de Steinberg implique que l'action de W sur X est libre, de sorte qu'on a un revêtement $X \twoheadrightarrow X/W$. Ce revêtement induit une suite exacte courte

$$1 \to P(W) \to B(W) \to W \to 1.$$

Dans le cas où W est un groupe de réflexions réel, Brieskorn [Bri71] a montré que le groupe de tresses $B(W)$ est isomorphe au groupe d'Artin $A(W)$ associé à W : les groupes de tresses complexes fournissent donc une nouvelle généralisation des groupes d'Artin sphériques.

Broué, Malle et Rouquier ont défini dans [BMR98] un ensemble infini de générateurs de $B(W)$, qu'ils appellent *générateurs de la monodromie.* Chaque tel élément $\sigma \in B(W)$ apparaît comme le relevé d'une réflexion de W par la projection $B(W) \twoheadrightarrow W$. Plus tard, ces éléments ont été appelés *« braid reflections »* (« réflexions-tresses ») dans [Bro01] et *« braided reflections »* (« réflexions tressées ») dans [Mar09]. Nous les appellerons *réflexions tressées* à partir de maintenant.

Les premières présentations de groupes de tresses complexes remontent aux travaux de Bannai [Ban76], qui a calculé les groupes de tresses associés aux groupes de réflexions complexes irréductibles de rang 2. Dans [OS88b], Orlik et Solomon ont montré qu'un groupe de Shephard W est toujours associé à un unique groupe de Coxeter W', tel que les espaces d'orbites régulières associés a W et à W' sont homéomorphes. Ceci entraîne en particulier que $B(W) \simeq B(W') \simeq A(W')$ est un groupe d'Artin sphérique.

Une fois que le groupe $B(W)$ est défini, il est possible de définir l'algèbre de Hecke $\mathcal{H}(W)$ comme quotient de l'algèbre de groupe de $B(W)$ sur un certain anneau de polynômes de Laurent. Cette construction est indépendante du choix d'une présentation de W, ce qui répond à la question de trouver une bonne (i.e. univoque) définition de l'algèbre de Hecke associée à un groupe de réflexions complexe. Cependant, étudier ces algèbres de Hecke s'est révélé bien plus délicat que dans le cas des groupes de Coxeter : le théorème que $\mathcal{H}(W)$ est libre de rang $|W|$ sur son anneau de définition fut prouvé en pas moins de vingt ans pour tous les cas [AK94], [Ari95], [BM93], [Mar14], [Mar12b], [MP17], [Cha18], [Mar19], [Tsu20].

De nombreuses propriétés importantes des groupes de tresses complexes (pures) sont énoncées dans [BMR98], généralisant des propriétés des groupes d'Artin sphériques. Cependant, la plupart sont prouvées uniquement pour les groupes de la série infinie, ou pour les groupes de rang 2, ce qui laisse de côté la famille de groupes $G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}$. Parmi eux, G_{25}, G_{26} et G_{32} sont des groupes de Shephard, ce qui permet souvent (mais pas toujours !) d'utiliser leurs groupes de Coxeter (et d'Artin) associés. Cela souligne la famille $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ comme une famille particulièrement problématique.

- (Centre) Si W est irréductible, alors le lemme de Schur entraîne que le centre $Z(W)$ de W est cyclique, disons d'ordre k. Broué, Malle et Rouquier définissent dans [BMR98] deux éléments $\pi \in Z(P(W))$ et $\beta \in Z(B(W))$. Ils montrent que $\beta^k = \pi$ et que l'image de β dans W engendre $Z(W)$. Ils conjecturent que le centre de $B(W)$ (resp. de $P(W)$) est cyclique et engendré par β (resp. par π) et que l'on a une suite exacte courte

$$1 \to Z(P(W)) \to Z(B(W)) \to Z(W) \to 1.$$

  Ils annoncent démontrer cette conjecture pour tous les groupes de réflexions complexes sauf $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ [BMR98, Theorem 2.24]. Cependant, comme indiqué dans [DMM11], il y a un défaut dans l'argument permettant de calculer le centre de $P(W)$ dans le cas des groupes de Shephard G_{25}, G_{26}, G_{32}.

- (Présentation) Si W est un groupe de réflexions réel, alors la présentation de Coxeter de W peut être résumée dans un graphe (le *diagramme de Coxeter*), qui induit également une présentation du groupe d'Artin $A(W)$. Dans [BMR98], Broué, Malle et Rouquier donnent des présentations des groupes de réflexions complexes irréductibles utilisant des diagrammes imitant les diagrammes de Coxeter [BMR98, Table 1 à 5] (plusieurs de ces présentations étaient déjà connues). On appellera ces diagrammes les *diagrammes de BMR*. Le diagramme de BMR d'un groupe de Coxeter W est égal à son diagramme de Coxeter. Le diagramme de BMR d'un groupe de Shephard coïncide avec le diagramme de Coxeter de son groupe de Coxeter associé. En particulier, le diagramme de BMR d'un groupe de Shephard induit une présentation de $B(W)$. Il est alors possible de conjecturer que tous les diagrammes de BMR donnent des présentations des groupes de tresses associés. Plus précisément, soit W un groupe de réflexions complexe irréductible. Le diagramme de BMR D associé à W induit une présentation de groupes, qui définit un groupe $G(D)$. On peut alors chercher à savoir s'il existe un isomorphisme $G(D) \simeq B(W)$ envoyant les générateurs de $G(D)$ sur des réflexions tressées dans $B(W)$. Ceci est démontré dans [BMR98, Theorem 2.27] pour tous les groupes de réflexions complexes irréductibles sauf ceux de la famille $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$. Broué, Malle, Rouquier conjecturent que le diagramme associé à G_{31} induit une présentation du groupe de tresses $B(G_{31})$, ils ne vont pas jusqu'à conjecturer la même chose pour les cinq autres groupes.

- (Monoïde) Soit W un groupe de réflexions complexe irréductible et soit D son diagramme de BMR associé. La présentation de groupe induite par D est toujours une présentation de groupes positive et homogène. En d'autres termes, les relations sont des égalités entre des mots positifs et de la même longueur entre les générateurs. En particulier, D peut être utilisé pour définir à la fois un groupe G et un monoïde M, G étant le groupe enveloppant de M. Dans les cas où il est connu que G est isomorphe à $B(W)$, Broué, Malle, Rouquier posent la question de savoir si le morphisme naturel $\iota : M \to G \simeq B(W)$ est injectif [BMR98, Question 2.28] et si le groupe $B(W)$ peut ou non être décrit comme $\{\pi^n \iota(b) \mid n \in \mathbb{Z}, b \in M\}$.

Toutes ces questions laissent penser à la possible existence d'une structure de groupe de Garside sur les groupes de tresses complexes (notons que les groupes de Garside n'avaient alors pas encore été définis). En effet, ces questions pourraient raisonnablement être étudiées dans un groupe de Garside : l'étude du problème de conjugaison dans un groupe de Garside permet souvent de déterminer son centre. Par construction, un groupe de Garside est donné avec une présentation, qui peut être manipulée via des transformations de Tietze. De plus, un groupe de Garside est donné avec un sous-monoïde remarquable pour lequel la description donnée plus haut existe toujours (en remplaçant π par l'élément de Garside Δ). Ainsi, si le monoïde M défini ci-dessus est en fait un monoïde de Garside, avec pour élément de Garside une racine de π, alors il est possible d'étudier ces conjectures de Broué, Malle, Rouquier.

Cependant, les présentations induites par les diagrammes de BMR pour les groupes G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34} ne donnent pas de structures de Garside en général. En particulier, le monoïde défini pour G_{31} ne se plonge pas dans le groupe $B(G_{31})$ ([Pic00, Example 14]). Les questions évoquées précédemment sont donc restées ouvertes pour $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ durant le début des années deux-mille.

Concernant les présentations des groupes de tresses complexes, Bessis a prouvé conceptuellement dans [Bes01] que les groupes de tresses complexes admettent toujours des présentations similaires à celles conjecturées dans [BMR98], dans le sens où elles sont positives, homogènes et admettent des réflexions tressées comme générateurs. Plus tard, Bessis et Michel [BM04] ont obtenu par ordinateur des présentations des groupes $B(G_{24})$ et $B(G_{27})$. En particulier, la présentation induite par le diagramme de BMR de G_{27} est bien une présentation du groupe de tresses associé. De plus, Bessis et Michel ont également proposé dans [BM04] des présentations conjecturales pour les groupes $B(G_{29}), B(G_{31}), B(G_{33}), B(G_{34})$, ces conjectures étant supportées par des calculs directs sur ordinateurs.

Une autre motivation pour chercher des structures de Garside sur les groupes de tresses complexes est que de telles structures donnent naturellement des moyens d'étudier des espaces d'Eilenberg-MacLane. Dans le cas d'un groupe de réflexions réel W (i.e. un groupe de Coxeter fini), nous avons mentionné que l'espace X (et donc X/W) est un espace d'Eilenberg-MacLane par [Del72]. C'est peut-être dans le livre de Orlik et Terao [OT92] qu'apparaît pour la première fois la conjecture que X est un espace classifiant si W est un groupe de réflexions complexe arbitraire. Au moment de la publication de [BMR98], en plus des travaux déjà effectués dans le cas réel, le résultat était connu pour les groupes de la série infinie grâce à Nakamura [Nak83] et pour les groupes de Shephard grâce à Orlik, Solomon [OS88a]. Comme le cas des groupes de rang 2 est immédiat (voir par exemple [OT92, Proposition 5.6]), la conjecture était toujours ouverte pour les groupes $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$.

L'absence de structures de Garside pour les groupes $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{31}), B(G_{33}), B(G_{34})$ a été comblée par la structure dite *monoïde dual*. Ce monoïde de Garside a d'abord été introduit par Bessis dans [Bes03] comme une structure de Garside alternative sur les groupes d'Artin sphériques, généralisant le monoïde défini par Birman, Ko et Lee [BKL98] pour le groupe de tresses usuel. Ensuite, il a été généralisé par Bessis et Corran [BC06] aux groupes de tresses complexes de la forme $B(G(e, e, n))$ pour $e, n > 1$. Ensuite, dans [Bes15], Bessis a généralisé la construction du monoïde dual à la classe des groupes de réflexions complexes *bien-engendrés*, qui inclut les groupes de réflexions réels et les groupes $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$ (seul G_{31} est manquant par rapport à la liste précédente).

Soit W un groupe de réflexions complexe bien-engendré. Dans une terminologie moderne, le monoïde dual est défini comme un *monoïde d'intervalle* associé à W et à un élément de Coxeter généralisé $c \in W$ (voir par exemple [GM22, Section 3.3] et [Nea18, Section 3.1.2]). Les différents choix possibles pour c induisent des monoïdes isomorphes. On peut alors noter $M(W)$ le monoïde dual associé à W et $G(W)$ le groupe enveloppant de $M(W)$. Cette définition semble uniforme, mais la preuve que $(G(W), M(W), \delta)$ est une structure de groupe de Garside sur $B(W)$ repose sur certains résultats fondamentaux sur les groupes bien-engendrés qui sont prouvés au cas par cas. L'utilisation de la classification de Shephard-Todd est alors limitée, mais cruciale. On appelle $G(W)$ le *groupe dual* de type W et $(G(W), M(W), \Delta)$ est la *structure duale* sur $B(W)$. Notons que l'isomorphisme $G(W) \simeq B(W)$ construit par Bessis envoie les générateurs de $M(W)$ sur des réflexions tressées dans $B(W)$.

En utilisant la structure duale, Bessis a montré que les espaces X et X/W sont des espaces d'Eilenberg-MacLane lorsque $W = G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$. Il a également prouvé que le centre des groupes de tresses associé est monogène et engendré par β, comme conjecturé par Broué, Malle et Rouquier. De plus, la présentation induite par le monoïde dual pour le groupe de tresses associé peut être utilisée pour démontrer la validité des présentations conjecturales de [BM04].

A priori, les conjectures énoncées plus haut restent alors ouvertes pour $W = G_{31}$. Cependant, ce groupe se plonge dans le groupe bien-engendré G_{37} (qui est un groupe de réflexions réel complexifié de type E_8) comme centralisateur d'un élément régulier au sens de Springer [Spr74]. Utilisant ceci, Bessis a défini un *groupoïde* de Garside pour le groupe de tresses complexe $B(G_{31})$. Il a ensuite utilisé ce groupoïde pour construire l'espace classifiant de $B(G_{31})$ et finalement pour prouver que l'espace associé X/W est un espace d'Eilenberg-MacLane. Ce groupoïde fait partie d'une famille plus générale de groupoïdes de Garside définis par Bessis pour les centralisateurs d'éléments réguliers dans les groupes de réflexions complexes bien-engendrés. Nous appelons ces groupoïdes les *groupoïdes de Springer*. Bessis a également pu montrer que le centre de $B(G_{31})$ est monogène et engendré par l'élément β défini plus haut. Pour ce qui est des présentations, contrairement à ce qui est annoncé dans [Bes15], il apparaît qu'il n'est pas immédiatement possible d'appliquer [Bes15, Corollaire 4.5] pour calculer une présentation de $B(G_{31})$ à partir des données de [BM04] : ces dernières ne satisfont pas toutes les hypothèses de [Bes15, Corollaire 4.5]. Cependant, Bessis mentionne également [Bes15, Remark 11.29] qu'il devrait être possible de déduire une présentation de $B(G_{31})$ en partant de celle de son groupoïde de Springer associé. Suivant cette remarque, nous proposerons dans cette thèse plusieurs présentations de $B(G_{31})$.

Enfin, la détermination du centre du groupe de tresses pures $P(W)$ a été complétée par Digne, Marin et Michel dans [DMM11]. Ils ont en fait démontré un résultat plus fort: si W est un groupe de réflexions complexe irréductible et si $U \subset B(W)$ est un sous-groupe d'indice fini, alors $Z(U) \subset Z(B(W))$. Ce résultat peut ensuite être appliqué à $P(W) \subset B(W)$ pour prouver la conjecture de Broué, Malle, Rouquier. La preuve donnée dans [DMM11] repose sur une utilisation limitée de la classification : les auteurs donnent un argument général de théorie de Garside, qu'ils appliquent ensuite à différents monoïdes de Garside permettant d'étudier tous les groupes de tresses complexes irréductibles sauf $B(G_{31})$. Le cas de $B(G_{31})$ est traité en utilisant la représentation linéaire fidèle de $B(G_{37}) \simeq A(E_8)$ définie dans [CW02] et [Dig03].

Pour ce qui est de la linéarité des groupes de tresses complexes, Marin a défini dans [Mar12a] une représentation de tout groupe de tresses complexe, qui généralise la représentation de Krammer du groupe de tresses usuel. Cette représentation avait déjà été étudiée par Marin dans le cas particulier des groupes de réflexions réels. En particulier, Marin a prouvé dans [Mar07] que cette représentation généralise la construction de Cohen, Wales dans le cas des groupes d'Artin de type A, D, E. Il est conjecturé que cette représentation est fidèle dans le cas général. Ce qui entraînerait que les groupes de tresses complexes sont tous linéaires, ce qui est toujours ouvert pour les groupes $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{33}), B(G_{34})$, ainsi que pour les groupes de la forme $B(G(e, e, n))$ (voir [Nea18, Chapter 5] pour davantage de détails sur cette dernière famille de groupes).

Développements récents : les sous-groupes paraboliques

Dans les sections précédentes, nous avons à dessein évité toute mention de la notion de sous-groupe parabolique. Nous choisissons de dédier cette section à ce sujet, étant donné que plusieurs des résultats principaux de cette thèse concernent ces sous-groupes particuliers et leur comportement dans les groupes de tresses complexes.

Soit (W, S) un système de Coxeter. Un *sous-groupe parabolique standard* de W est un sous-groupe W_I engendré par un sous-ensemble I de S. Le couple (W_I, I) est alors un système de Coxeter. Un *sous-groupe parabolique* de W est le conjugué dans W d'un sous-groupe parabolique standard. Comme le groupe d'Artin $A(W)$ associé à W est engendré par une copie formelle $\mathbf{S}$ de S, cette situation se transpose facilement dans $A(W)$: un sous-groupe parabolique standard de $A(W)$ est un sous-groupe de $A(W)$ engendré par un sous-ensemble $\mathbf{I}$ de $\mathbf{S}$ et un sous-groupe parabolique de $A(W)$ est un conjugué dans $A(W)$ d'un sous-groupe parabolique standard. Un théorème de van der Lek [Lek83] montre que, pour $\mathbf{I} \subset \mathbf{S}$ (associé à $I \subset S$), le sous-groupe parabolique standard $\langle \mathbf{I} \rangle$ de $A(W)$ est isomorphe à $A(W_I)$.

On peut énoncer deux conjectures importantes concernant les sous-groupes paraboliques des groupes d'Artin :

(1) Tout élément d'un groupe d'Artin est contenu dans un plus petit (par rapport à l'inclusion) sous-groupe parabolique.

(2) L'intersection de deux sous-groupes paraboliques d'un groupe d'Artin est encore un sous-groupe parabolique.

Bien-sûr, (2) implique (1), mais (2) est un énoncé plus fort a priori. Notons que (2) devient immédiat si l'on remplace « sous-groupe parabolique » par « sous-groupe parabolique standard ». Ces deux conjectures ont été prouvées pour les groupes d'Artin sphériques par Cumplido, Gebhardt, González-Meneses, Wiest dans [CGGW19]. Elles sont également prouvées pour d'autres familles particulières de groupes d'Artin [CMV23], [Mor21], mais le cas général est toujours ouvert.

Dans la première section, nous avons mentionné que le groupe de tresses usuel peut être interprété comme le groupe modulaire d'un disque percé. En particulier, il agit sur les classes d'isotopie de courbes tracées dans le disque percé. Ces classes d'isotopies de courbes forment les sommets d'un complexe simplicial, dit *complexe de courbes*, sur lequel le groupe de tresses usuel agit par isométries. Nous avons également mentionné que cette situation ne se transpose pas géométriquement aux autres groupes d'Artin sphériques. Cependant, Cumplido, Gebhardt, González-Meneses, Wiest suggèrent dans [CGGW19] que les sous-groupes paraboliques des groupes d'Artin sphériques fournissent un bon analogue algébrique des classes d'isotopie de courbes pour le groupe de tresses usuel. En particulier, ils forment également les sommets d'un complexe simplicial sur lequel le groupe d'Artin agit naturellement.

La notion de sous-groupe parabolique d'un groupe d'Artin a été généralisée au cas des groupes de Garside par Godelle dans [God07]. Ici encore, il y a une notion de sous-groupe parabolique standard et une notion de sous-groupe parabolique, défini comme le conjugué d'un sous-groupe parabolique standard. Soit (G, M, Δ) un groupe de Garside. Pour $s \in M$, on note M_s le sous-monoïde de M engendré par les diviseurs de s dans M et G_s le sous-groupe de G engendré par les diviseurs de s dans M. Un *sous-groupe parabolique standard* d'un groupe de Garside (G, M, Δ) est un sous-groupe de la forme G_s, où s est un diviseur de Δ dans M satisfaisant certaines hypothèses. Il est à noter que (G_s, M_s, s) est alors à son tour un groupe de Garside. Notons également que les sous-groupes paraboliques de (G, M, Δ) dépendent de (M, Δ) et pas seulement de G. En d'autres termes, la notion de sous-groupe parabolique d'un groupe de Garside G dépend de sa structure de Garside.

Soit $W \subset \mathrm{GL}_n(\mathbb{C})$ un groupe de réflexions complexe. Un *sous-groupe parabolique* de W est défini comme le fixateur ponctuel d'une partie de $\mathbb{C}^n$. Un théorème classique de Steinberg assure qu'un sous-groupe parabolique $W_0 \subset W$ est engendré par les réflexions de W qu'il contient. En particulier, $W_0 \subset \mathrm{GL}_n(\mathbb{C})$ est encore un groupe de réflexions complexe. Notons qu'il n'existe pas de notion de sous-groupe parabolique standard dans un groupe de réflexions complexe.

Soit $W_0 \subset W$ un sous-groupe parabolique. Soit également X_0 le complémentaire dans $\mathbb{C}^n$ des hyperplans de réflexions de W_0. On a $X \subset X_0$, ce qui induit un morphisme $P(W) \to P(W_0)$. Dans [BMR98, Section 2.D], les auteurs montrent que ce morphisme est un épimorphisme scindé et que l'on a un morphisme de suites exactes courtes :

$$\begin{tikzcd}
1 \arrow[r] & P(W_0) \arrow[r] \arrow[d, hook] & B(W_0) \arrow[r] \arrow[d, hook] & W_0 \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & P(W) \arrow[r] & B(W) \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

Il est raisonnable de définir l'image du morphisme du milieu comme un sous-groupe parabolique de $B(W)$. La construction donnée dans [BMR98, Section 2.D] dépend de plusieurs choix, mais elle est bien définie à conjugaison près. Cependant, il est à noter que cette construction a principalement lieu dans l'espace X. En particulier, si deux groupes de réflexions complexes W, W' sont tels que $X/W \simeq X'/W'$ (par exemple, un groupe de Shephard et son groupe de

Coxeter associé), alors on a $B(W) \simeq B(W')$, mais il n'est pas clair que les collections de sous-groupes paraboliques telles que définies ci-dessus sont les mêmes.

Pour gérer ce problème, González-Meneses et Marin introduisent dans [GM22] un concept purement topologique de sous-groupe parabolique dans un groupe de tresses complexe. Ce concept repose sur un concept plus général de groupe fondamental local défini pour une paire topologique. Dans le cas d'un groupe de réflexions complexe $W \subset \mathrm{GL}_n(\mathbb{C})$, González-Meneses et Marin montrent que les groupes fondamentaux locaux existent toujours pour la paire topologique $(X/W, \mathbb{C}^n/W)$. Les images de ces groupes dans le groupe fondamental ambiant $B(W)$ sont par définition les sous-groupes paraboliques de $B(W)$. Là encore, il n'y a pas de notion de sous-groupe parabolique standard a priori.

Quand $B(W)$ est équipé d'une structure de groupe de Garside, il est possible de chercher à identifier les sous-groupes paraboliques topologiques de $B(W)$ avec les sous-groupes paraboliques algébriques définis en utilisant cette structure de Garside. Ceci peut être utilisé pour montrer que deux structures de Garside définies sur le même groupe de tresses complexe et aussi agréables l'une que l'autre induisent la même collection de sous-groupes paraboliques, étant donné que les sous-groupes paraboliques topologiques ne dépendent pas du choix d'une structure de Garside. Avec de tels résultats, il est possible d'obtenir des résultats sur les sous-groupes paraboliques via des arguments de théorie de Garside. Par construction, cette approche fonctionne pour les groupes de tresses complexes qui peuvent être équipés d'une structure de groupe de Garside, en particulier, elle exclut $B(G_{31})$.

Pour les groupes de tresses complexes distincts de $B(G_{31})$, González-Meneses et Marin utilisent la méthode mentionnée ci-dessus dans [GM22] pour montrer que les sous-groupes paraboliques sont stables par intersection. Ils montrent d'abord que tout élément d'un groupe de Garside est contenu dans un plus petit sous-groupe parabolique (sa *clôture parabolique*), sous une condition appelée la *préservation du support* (notons que cet énoncé est celui de la conjecture (1) généralisé aux groupes de Garside). Godelle a prouvé dans [God07] que les sous-groupes paraboliques standards d'un groupe de Garside (G, M, Δ) sont stables par intersection. Étant donné un élément $x \in G$, il est alors possible de définir la *clôture parabolique standard* $\mathrm{SPC}(x)$ de x. Si x, y sont deux éléments de M qui sont conjugués dans G, disons par un élément $\alpha \in G$, la préservation du support nécessite par définition que α conjugue $\mathrm{SPC}(x)$ en $\mathrm{SPC}(y)$ et pas seulement en un sous-groupe parabolique arbitraire de G contenant y.

La propriété de préservation du support est difficile à vérifier en pratique. González-Meneses et Marin ont donc donné quelques réductions utiles pour démontrer cette propriété dans [GM22, Section 4.7], en utilisant la notion de *conjugant positif minimal.* Soit $x \in M$, un conjugant positif minimal de x est un élément $\alpha \in M$ qui conjugue x en un élément de M et tel qu'aucun diviseur strict à gauche de α dans M (à part l'élément neutre) ne conjugue également x en un élément de M. González-Meneses et Marin démontrent qu'il est suffisant de prouver que ces conjugants positifs minimaux préservent la clôture parabolique standard pour obtenir la préservation du support.

En utilisant l'existence de la clôture parabolique, González-Meneses et Marin montrent dans [GM22, Section 6.1] que les sous-groupes paraboliques standards dans un groupe de Garside *homogène* sont stables par intersection. Cela couvre en particulier le cas des groupes de tresses complexes irréductibles qui admettent une structure de groupe de Garside.

De plus, suivant [CGGW19], González-Meneses et Marin définissent un analogue algébrique du complexe de courbes en utilisant les sous-groupes paraboliques irréductibles comme les sommets d'un certain graphe. Toujours en utilisant des arguments issus de la théorie de Garside, ils donnent une caractérisation agréable de l'adjacence dans ce *graphe de courbes* pour tout les groupes de tresses complexes, sauf pour $B(G_{31})$.

Résultats principaux

On présente dans cette section les résultats principaux de la thèse.

Sous-groupes paraboliques et groupes de Garside

Avant d'étudier les groupes de tresses complexes et leurs sous-groupes paraboliques, nous avons besoin de définir les outils nécessaires de théorie de Garside. Nous avons mentionné que le livre de référence [DDGKM] ne donne pas explicitement de définition de groupoïde de Garside, mais il étudie plutôt des catégories munies de familles de Garside satisfaisant des conditions plus ou moins restrictives. Dans l'étude des groupes de tresses complexes, on peut se restreindre à une définition plus proche de celle donnée dans [Bes07]. Dans le langage de [DDGKM], ce que l'on appelle *groupoïde de Garside* est un triplet $(\mathcal{G}, \mathcal{C}, \Delta)$, où

- $\mathcal{G}$ est un (petit) groupoïde.
- $\mathcal{C}$ est une sous-catégorie de $\mathcal{G}$ qui engendre $\mathcal{G}$ et qui ne contient pas d'élément inversible non trivial.
- $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ est une *application de Garside* [DDGKM, Definition V.2.19], telle que l'ensemble $\mathcal{S}$ des diviseurs de Δ est fini.

Les groupes de la forme $\mathcal{G}(u, u)$, où $(\mathcal{G}, \mathcal{C}, \Delta)$ est un groupoïde et où $u \in \mathrm{Ob}(\mathcal{G})$, est appelé un *groupe de Garside faible*.

Dans le cas des groupes de Garside, les sous-groupes paraboliques (standards) ont été introduits par Godelle dans [God07]. Dans [GM22], González-Meneses et Marin donnent des arguments généraux permettant de prouver (sous des hypothèses convenables) que tout élément d'un groupe de Garside est contenu dans un plus petit sous-groupe parabolique et même que les sous-groupes paraboliques sont stables par intersections. Le point de départ de leur preuve est de considérer le plus petit sous-groupe parabolique standard contenant un élément donné. Un tel groupe existe toujours puisque les sous-groupes paraboliques standards d'un groupe de Garside sont toujours stables par intersection.

Une notion de sous-groupoïde parabolique (standard) d'un groupoïde de Garside a aussi été définie par Godelle dans [God10], mais pas étudiée en profondeur. Cette notion de sous-groupoïde parabolique standard induit une notion de sous-groupe parabolique (standard) d'un groupe de Garside faible. Cependant, contrairement au cas des groupes de Garside, les sous-groupoïdes paraboliques standards d'un groupoïde de Garside ne sont pas stables par intersection. Pour pouvoir adapter les arguments de [GM22], on choisit de se restreindre à des familles particulières de sous-groupoïdes paraboliques standards, que nous appelons les *bancs*. Un banc $\mathcal{T}$ pour un groupoïde de Garside $(\mathcal{G}, \mathcal{C}, \Delta)$ doit en particulier être stable par intersection. Fixant un banc $\mathcal{T}$, on peut définir des sous-groupes $\mathcal{T}$-paraboliques (standards) dans le groupoïde $\mathcal{G}$. Dans le cas particulier d'un groupe de Garside (G, M, Δ), l'ensemble de tous les sous-groupes paraboliques standards de G est un banc.

Fixant un banc $\mathcal{T}$ pour un groupoïde de Garside $(\mathcal{G}, \mathcal{C}, \Delta)$, on peut définir la clôture $\mathcal{T}$-parabolique standard d'un endomorphisme x dans $\mathcal{G}$ simplement en prenant l'intersection de tous les sous-groupes $\mathcal{T}$-paraboliques standards contenant x. Partant de là, on imite la définition de préservation du support donnée dans [GM22] (voir Définition 5.2.8). On obtient le résultat suivant :

Théorème (Existence de la clôture $\mathcal{T}$-parabolique). *(Théorème 5.2.17 et Corollaire 5.2.18) Soit $(\mathcal{G}, \mathcal{C}, \Delta)$ un groupoïde de Garside et soit $\mathcal{T}$ un banc pour $\mathcal{G}$ qui préserve le support. Tout endomorphisme x dans $\mathcal{G}$ admet une clôture $\mathcal{T}$-parabolique* $\mathrm{PC}(x)$. *De plus, si m est un entier non nul, alors on a* $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ *pour tout endomorphisme x dans $\mathcal{G}$.*

Gardons à l'esprit que les bancs sont une structure additionnelle sur un groupoïde de Garside. En étudiant les sous-groupes paraboliques de $B(G_{31})$ via son groupoïde de Springer associé, un point important sera de construire un banc rendant compte de la construction topologique des sous-groupes paraboliques de $B(G_{31})$.

Monoïde dual et tresses régulières

La préservation du support est une propriété importante et difficile à prouver en pratique. Dans le cas des groupoïdes de Springer (en particulier, celui associé à $B(G_{31})$), cette propriété provient de la préservation du support d'un autre banc, nommément le banc de tous les sous-groupes paraboliques standards du groupe dual associé à un groupe de réflexions complexe bien-engendré. Dans ce cas, la préservation du support est assuré par le théorème suivant (obtenu via la classification de Shephard-Todd), qui établit en fait un résultat plus fort sur les conjugants positifs minimaux dans les groupes duaux.

Théorème (Sous-groupes paraboliques dans les groupes duaux). *(Théorème 8.4.1 et Corollaire 8.4.2)*
Soit W un groupe de réflexions complexe bien-engendré et soit $(G(W), M(W), \Delta)$ le groupe dual de type W. Soit également $x \in M(W)$ et soit $\mathrm{SPC}(x) = G(W)_s$ *sa clôture parabolique standard pour un certain simple s. Si a est un atome de $M(W)$, alors on a soit :*

- *a divise $\overline{s}$ à gauche dans $M(W)$ et $\rho_a(x) = a$ est un conjugant positif minimal.*
- *a divise s à gauche dans $M(W)$ et $\rho_a(x) \in M(W)_s$.*
- *Ni l'un ni l'autre et $\rho_a(x)$ n'est pas un conjguant positif minimal.*

En particulier, le banc de tous les sous-groupes paraboliques standards de $G(W)$ préserve le support.

En utilisant les résultats de [GM22, Section 4.6 and Section 6.1], on obtient le corollaire suivant :

Corollaire. *(Théorème 8.4.5) Soit W un groupe de réflexions complexe bien-engendré. Les sous-groupes paraboliques de $B(W)$ sont stables par intersections.*

Ce résultat est déjà connu via les travaux de Cumplido, Gebhardt, González-Meneses, Wiest [CGGW19] et de González-Meneses, Marin [GM22]. Cependant, le théorème précédent n'est quant à lui pas une conséquence des résultats de [CGGW19] et de [GM22]. En effet, ces travaux étudient la préservation du support pour d'autres structures de Garside sur les groupes bien-engendrés (sauf pour $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$), or la préservation du support d'une structure de Garside sur un groupe donné n'entraîne pas la préservation du support d'une autre structure de Garside sur ce même groupe.

Dans [Bes15], Bessis parvient à étudier le groupe de tresses complexe $B(G_{31})$ en y associant un groupoïde de Garside, que nous appellerons le groupoïde de Springer. Le point de départ de cette construction est la théorie des éléments réguliers de Springer dans les groupes de réflexions complexes [Spr74]. Un *élément régulier* dans un groupe de réflexions complexe W pour une valeur propre ζ est un élément g de W qui admet des vecteurs propres pour la valeur propre ζ en dehors de tout hyperplan de réflexions de W. Le centralisateur W_g d'un tel élément agit comme un groupe de réflexions complexe sur l'espace propre de g associé à la valeur propre ζ. Quitte à prendre des puissances, on peut restreindre notre attention au cas où la valeur propre considérée est de la forme $\zeta_d := \exp(\frac{2i\pi}{d})$ pour $d \geqslant 1$. On appelle alors g un élément d-régulier et on dit que d est un nombre régulier pour W.

Un des thèmes récurrents dans les travaux de Bessis est la possibilité d'offrir un « relevé » de la théorie des éléments réguliers de Springer aux groupes de tresses complexes, répondant ainsi à des questions posées par Broué et Michel [Bes00] [BDM02]. Dans [Bes15, Section 12], Bessis montre qu'un tel relevé existe dans le cas des groupes de réflexions complexes irréductibles et bien-engendrés. Ce relevé repose sur le concept de *tresse régulière*.

Soit W un groupe de réflexions complexe et soit $\pi \in Z(P(W))$ l'élément mentionné plus haut. Pour $d \geqslant 1$, un élément $\rho \in B(W)$ est une tresse d-régulière si $\rho^d = \pi$ dans $B(W)$. Il est facile de montrer que tout élément ζ_d-régulier de W est l'image d'une tresse d-régulière par la projection $B(W) \twoheadrightarrow W$. Le théorème suivant prouve que les tresses régulières fournissent un bon analogue dans les groupes de tresses complexes des éléments réguliers dans les groupes de réflexions complexes.

Théorème (Théorie de Springer dans les groupes de tresses complexes). *(Théorème 9.1.7)*
Soit W un groupe de réflexions complexe irréductible et soit d un entier strictement positif.

(a) L'entier d est régulier pour W au sens de Springer si et seulement si $B(W)$ admet des tresses d-régulières.

(b) Si d est régulier, les tresses d-régulières dans $B(W)$ sont toutes conjuguées et une tresse d-régulière est envoyée sur un élément ζ_d-régulier dans W.

(c) Soit d un nombre régulier et soit ρ une tresse d-régulière dans $B(W)$. Soit g l'image de ρ dans W. Le groupe de tresses complexe $B(W_g)$ est isomorphe au centralisateur $C_{B(W)}(\rho)$. En d'autres termes, le centralisateur d'une tresse régulière est le groupe de tresses du centralisateur d'un élément régulier.

Insistons sur le fait que ce résultat a été obtenu par Bessis dans le cas où W est bien-engendré. Nous complétons la preuve en étudiant le cas des autres groupes de réflexions complexes irréductibles, en particulier G_{31}.

Dans le cas où W est bien-engendré, on peut considérer la structure duale $(G(W), M(W), \Delta)$ sur $B(W)$. L'élément $\pi \in B(W)$ apparaît alors comme une puissance de l'élément de Garside Δ. Une tresse régulière est alors un cas particulier d'élément périodique dans un groupe de Garside $(G(W), M(W), \Delta)$ (i.e. une racine d'une puissance de Δ). Bessis a introduit dans [Bes07] une construction générale de groupoïde de Garside équivalent au centralisateur d'un élément périodique dans un groupe de Garside. Appliquer cette construction à une tresse d-régulière induit alors un groupoïde de Garside qui est équivalent au centralisateur de cette tresse régulière dans $B(W)$. D'après le théorème ci-dessus, ce centralisateur est isomorphe au groupe de tresses du centralisateur d'un élément ζ_d-régulier de W. Nous appelons ce groupoïde le *groupoïde de*

Springer (associé à W et d).

Soit W un groupe de réflexions complexe bien-engendré et soit d un nombre régulier pour W. Soit également $(\mathcal{G}, \mathcal{C}, \Delta)$ le groupoïde de Springer associé. Par construction, $\mathcal{G}$ admet un foncteur $p : \mathcal{G} \to G(W)$ tel que, pour tout $u \in \mathrm{Ob}(\mathcal{G})$, p identifie $\mathcal{G}(u, u)$ avec le centralisateur dans $G(W)$ d'une certaine tresse régulière $\rho(u)$.

La construction de $\mathcal{G}$ à partir du groupe dual $(G(W), M(W), \Delta)$ permet de définir un banc $\mathcal{T}$ de sous-groupoïdes paraboliques standards de $\mathcal{G}$ en partant du banc de tous les sous-groupes paraboliques standards de $G(W)$. Le théorème sur les conjugants positifs minimaux dans $(G(W), M(W), \Delta)$ nous permet alors de montrer que ce banc préserve le support. L'objectif suivant est alors de montrer que les bancs ainsi construits pour les groupoïdes de Springer donnent effectivement les mêmes sous-groupes paraboliques que les sous-groupes paraboliques topologiques définis pour le groupe de tresses du centralisateur d'un élément régulier. Ceci nécessite de suivre la description topologique faite par Bessis du groupoïde de Springer et d'introduire la notion de groupe fondamental local dans ce contexte. On obtient finalement le théorème suivant :

Théorème. *(Section 9.2.4 et Théorème 9.2.42)*
Soit W un groupe de réflexions complexe bien-engendré et soit d un nombre régulier pour W. Soit également $(G(W), M(W), \Delta)$ le groupe dual de type W et soit $(\mathcal{G}, \mathcal{C}, \Delta)$ le groupoïde de Springer associé à W et d. Il y a un banc $\mathcal{T}$ sur $\mathcal{G}$ qui préserve le support et tel que, pour $u \in \mathrm{Ob}(\mathcal{G})$, on a

(a) Les sous-groupes $\mathcal{T}$-paraboliques de $\mathcal{G}(u, u) \simeq C_{G(W)}(\rho(u))$ sont exactement les intersections avec $C_{G(W)}(\rho(u))$ des sous-groupes paraboliques de $G(W)$ qui sont normalisés par $\rho(u)$.

(b) Deux sous-groupes paraboliques de $G(W)$ normalisés par $\rho(u)$ sont égaux si et seulement si leurs intersections avec $C_{G(W)}(\rho(u))$ sont égales.

(c) Soit g l'image de $\rho(u)$ par la projection $G(W) \simeq B(W) \to W$. L'isomorphisme $B(W_g) \simeq C_{G(W)}(\rho(u))$ identifie les sous-groupes paraboliques du premier avec les sous-groupes $\mathcal{T}$-paraboliques du second.

Ce théorème donne une description complète des sous-groupes paraboliques dans le centralisateur d'une tresse régulière dans un groupe de tresses bien-engendré. Le résultat suivant a été conjecturé par González-Meneses et Marin dans le cas où W est bien-engendré. Dans ce cas, il apparaît comme une conséquence du théorème ci-dessus et nous en donnons également la preuve pour les autres groupes de réflexions complexes irréductibles.

Théorème (Sous-groupes paraboliques d'un centralisateur régulier). *(Théorème 9.3.1)*
Soit W un groupe de réflexions complexe irréductible et soit d un nombre régulier pour W. Soit $\rho \in B(W)$ une tresse d-régulière et soit g son image dans W.

(a) L'isomorphisme $B(W_g) \simeq C_{B(W)}(\rho)$ identifie les sous-groupes paraboliques de $B(W_g)$ avec les intersections de $C_{B(W)}(\rho)$ et d'un sous-groupe parabolique de $B(W)$ normalisé par ρ.

(b) Deux sous-groupes paraboliques de $B(W)$ normalisés par ρ sont égaux si et seulement si leurs intersections avec $C_{B(W)}(\rho)$ sont égales.

Soit W un groupe de réflexions complexe et soit $g \in W$ un élément régulier. Si l'on sait déjà que les sous-groupes paraboliques de $B(W)$ sont stables par intersection, alors le théorème ci-dessus nous permet de déduire que les sous-groupes paraboliques de $B(W_g)$ sont également stables par intersection. En particulier, on a le corollaire suivant :

Corollaire. *Soit W un groupe de réflexions complexe bien-engendré et soit $g \in W$ un élément régulier. Les sous-groupes paraboliques de $B(W_g)$ sont stables par intersection.*

Notons que ce corollaire s'applique en particulier au cas de $B(G_{31})$. Combinant ce résultat avec le cas des groupes bien-engendrés, on obtient que les sous-groupes paraboliques d'un groupe de tresses complexe irréductible sont stables par intersection pour tous les cas sauf G_{12}, G_{13} et les groupes de la forme $G(de, e, n)$ pour $d, e, n > 2$ (notons que ces derniers peuvent être étudiés en tant que sous-groupes d'indice fini dans les groupes de la forme $G(r, 1, n)$, qui sont bien-engendrés).

Résultats sur le groupe de tresses complexes $B(G_{31})$

Le groupe de réflexions complexe G_{37} (qui est un groupe de Coxeter complexifié de type E_8) admet 4 comme nombre régulier et le centralisateur d'un élément 4-régulier dans G_{37} est un groupe de réflexions complexe de type G_{31}. Comme G_{37} est bien-engendré en tant que groupe de réflexions réel complexifié, on peut considérer le groupoïde de Springer $(\mathcal{B}_{31}, \mathcal{C}_{31}, \Delta)$ associé à G_{37} et au nombre régulier 4.

Comme mentionné dans la section précédente, pour $u \in \mathrm{Ob}(\mathcal{B}_{31})$, le foncteur naturel $\mathcal{B}_{31} \to B(G_{37})$ identifie $\mathcal{B}_{31}(u, u)$ au centralisateur dans $B(G_{37})$ d'une tresse 4-régulière $\rho(u)$; un tel centralisateur étant isomorphe au groupe de tresses complexe $B(G_{31})$.

Rappelons que le *centre* d'une catégorie $\mathcal{C}$ est défini comme l'ensemble des endomorphismes naturels du foncteur identité de $\mathcal{C}$. Dans un groupoïde de Garside $(\mathcal{G}, \mathcal{C}, \Delta)$, l'application de Garside Δ peut être vue comme une transformation naturelle du foncteur identité de $\mathcal{C}$ vers un certain automorphisme de $\mathcal{C}$. Cet automorphisme a un ordre fini, donc une certaine puissance de l'application de Garside Δ se trouve en fait dans le centre de $\mathcal{C}$, qui est inclus dans le centre de $\mathcal{G}$. Dans le cas du groupoïde de Springer $\mathcal{B}_{31}$, ceci décrit complètement le centre et on a le résultat suivant (le deuxième énoncé et le troisième apparaissent dans [DMM11, Theorem 1.4], mais avec une preuve différente) :

Théorème. *(Théorème 10.0.1) Les centres de $\mathcal{C}_{31}$ et de $\mathcal{B}_{31}$ sont monogènes et engendrés par Δ^{15}. De plus, si U est un sous-groupe d'indice fini de $B(G_{31})$, alors $Z(U) \subset Z(B(G_{31}))$. En particulier, le centre du groupe de tresses pures $P(G_{31})$ est monogène et on a une suite exacte courte*

$$1 \to P(G_{31}) \to B(G_{31}) \to G_{31} \to 1.$$

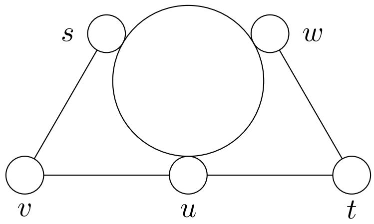
Comme nous l'avons dit plus haut, le deuxième énoncé ainsi que le troisième sont déjà prouvés dans [DMM11], mais notre preuve est une preuve de théorie de Garside et repose sur la conjugaison des réflexions tressées de $B(G_{31})$ vues dans le groupoïde de Springer $\mathcal{B}_{31}$.

Afin d'étudier des présentations de $B(G_{31})$, nous suivons l'idée de [Bes15, Remark 11.29] et nous cherchons à déduire une présentation de $B(G_{31})$ à partir d'une présentation de $\mathcal{B}_{31}$. Ceci est faisable en utilisant une version générale de la méthode de Reidemeister-Schreier que nous définissons. Partant d'un objet $u \in \mathrm{Ob}(\mathcal{B}_{31})$, on peut d'un côté « deviner une présentation » dont les générateurs sont les réflexions tressées dans $\mathcal{C}_{31}(u, u)$ (boucles atomiques) et d'un autre côté calculer une présentation de $\mathcal{B}_{31}(u, u)$ en utilisant la méthode de Reidemeister-Schreier pour les groupoïdes. Cette dernière présentation est très redondante en pratique et nous la simplifions en utilisant le retournement de mots. À chaque fois, nous obtenons que la présentation conjecturée est effectivement une présentation de $B(G_{31})$. On obtient alors plusieurs présentations de $B(G_{31})$, parmi lesquelles se trouve la présentation induite par le diagramme de BMR de G_{31} :

Théorème. *(Théorème 10.1.1 et Section 10.1.3)*
Le groupe $B(G_{31})$ admet la présentation suivante

$$B(G_{31}) = \left\langle s, t, u, v, w \;\middle|\; \begin{array}{l} st = ts,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw \end{array} \right\rangle,$$

dont les générateurs sont des réflexions tressées (distinguées). Cette présentation est celle induite par le diagramme de BMR de $B(G_{31})$



Ce théorème met un point final à la recherche de présentations des groupes de tresses complexes irréductibles commencée dans [BMR98].

Concernant les sous-groupes paraboliques de $B(G_{31})$, comme G_{37} est bien-engendré, on peut appliquer les résultats de la section précédente pour prouver

Théorème. *(Corollaire 9.3.2) Les sous-groupes paraboliques du groupe de tresses complexe $B(G_{31})$ sont stables par intersection.*

Ce théorème est [GM22, Theorem 1.1 and Theorem 1.2] dans le cas de $B(G_{31})$. Afin de compléter la preuve des résultats principaux [GM22], il reste à prouver [GM22, Theorem 1.3] dans le cas de $B(G_{31})$.

Soit $B_0 \subset B(G_{31})$ un sous-groupe parabolique irréductible. On sait que le centre de B_0 est monogène infini et on peut considérer son unique générateur z_{B_0} qui est positif en les générateurs s, t, u, v, w de $B(G_{31})$. Premièrement, on a que la conjugaison des sous-groupes paraboliques irréductibles de $B(G_{31})$ est complètement caractérisée par les éléments z_{B_0}.

Théorème. *(Section 10.2.2)*
Soit $B_0 \subset B(G_{31})$ un sous-groupe parabolique irréductible. Le plus petit sous-groupe parabolique de $B(G_{31})$ contenant z_{B_0} est B_0. Soient $B_1, B_2 \subset B(G_{31})$ deux sous-groupes paraboliques irréductibles. Un élément $b \in B(G_{31})$ conjugue B_1 en B_2 si et seulement si $z_{B_1}^b = z_{B_2}$.

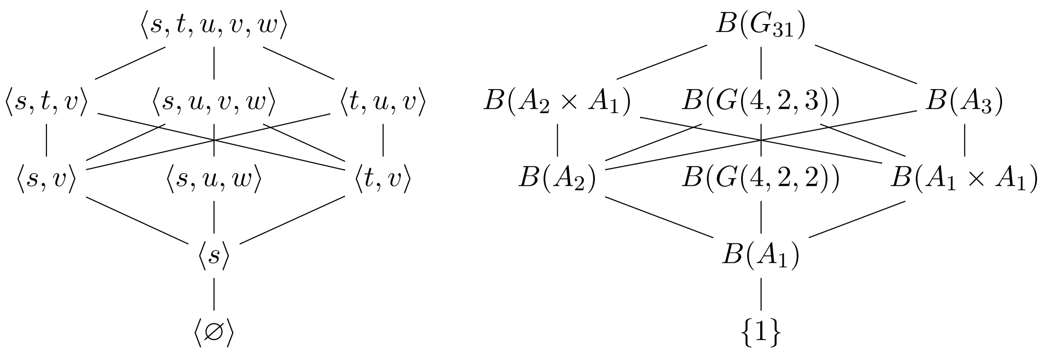
Suivant [GM22], le *graphe de courbes* de $B(G_{31})$ est défini comme le graphe dont les sommets sont les sous-groupes paraboliques irréductibles de $B(G_{31})$ et dans lequel deux sommets B_1, B_2 sont adjacents si et seulement si ils sont distincts et si l'on a soit $B_1 \subset B_2$, $B_2 \subset B_1$, soit $B_1 \cap B_2 = [B_1, B_2] = 1$. On a la caractérisation suivante de l'adjacence dans le graphe de courbes de $B(G_{31})$:

Théorème. *(Théorème 10.2.8)*
Deux sous-groupes paraboliques irréductibles $B_1, B_2 \subset B(G_{31})$ sont adjacents dans le graphe de courbe de $B(G_{31})$ si et seulement si z_{B_1} et z_{B_2} sont distincts et commutent.

Ceci termine la preuve de [GM22, Theorem 1.3] pour $B(G_{31})$, qui était le seul cas non étudié dans [GM22]. Nous avons ainsi obtenu que les résultats principaux de [GM22] sont vrais pour tous les groupes de tresses complexes irréductibles.

De plus, on donne une description des sous-groupes paraboliques de $B(G_{31})$ à conjugaison près. Considérons les générateurs s, t, u, v, w de la présentation de $B(G_{31})$ induite par son diagramme de BMR. Soient $\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}$ les images de s, t, u, v, w dans le groupe de réflexions G_{31}. Les éléments $\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}$ engendrent G_{31} et une présentation de G_{31} se déduit de son diagramme de BMR. Il est connu que tout sous-groupe parabolique de $B(G_{31})$ est déterminé à conjugaison près par un sous-ensemble S de $\{\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}\}$. De plus, le diagramme de BMR d'un tel sous-groupe parabolique est obtenu en prenant le sous-diagramme du diagramme de BMR de $B(G_{31})$ dont les nœuds sont les éléments de S. Le théorème suivant indique que la situation est la même au niveau du groupe de tresses $B(G_{31})$:

Théorème. *(Théorème 10.2.1) Le treillis des sous-groupes paraboliques de $B(G_{31})$ à conjugaison près est donné par*



Le treillis de droite, où A_n dénote le groupe de réflexions complexe $G(1,1,n+1)$, décrit le type d'isomorphie des sous-groupes paraboliques donnés à gauche. Pour chaque tel sous-groupe parabolique $\langle S\rangle \subset B(G_{31})$, une présentation est obtenue en prenant le sous-diagramme du diagramme de BMR de $B(G_{31})$ dont les nœuds sont les éléments de S.

Dans l'Annexe A, nous calculons l'homologie de $B(G_{31})$ partant de celle du groupoïde de Springer $\mathcal{B}_{31}$. Nous donnons les valeurs explicites de $H_*(B(G_{31}), M)$ pour les trois $B(G_{31})$-modules M suivants :

- $M = \mathbb{Z}$ avec action triviale de $B(G_{31})$.
- $M = \mathbb{Z}$ où les réflexions tressées de $B(G_{31})$ agissent par multiplication par -1. Ce module sera noté $\mathbb{Z}_\varepsilon$.
- $M = \mathbb{Q}[t, t^{-1}]$, où les réflexions tressées de $B(G_{31})$ agissent par multiplication par t.

Les résultats sont dans la table suivante (voir Table A.3, Table A.4, Table A.5). Pour $M = \mathbb{Q}[t, t^{-1}]$, $\mathbb{Q}$ dénote le quotient $\mathbb{Q}[t, t^{-1}]/(t-1)$ et un polynôme $P(t)$ représente le quotient $\mathbb{Q}[t, t^{-1}]/(P)$ (Φ_n est le n-ème polynôme cyclotomique).

$B(G_{31})$	H_0	H_1	H_2	H_3	H_4
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}_\varepsilon$	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_{20}$	0
$\mathbb{Q}[t, t^{-1}]$	$\mathbb{Q}$	0	Φ_6	$\frac{t^{10}-1}{t+1}\Phi_{15}$	0

Perspectives et conjectures

Les résultats que nous avons présentés plus haut offrent plusieurs perspectives naturelles à suivre.

Concernant la théorie de Garside, le concept de banc que nous définissons rend compte de l'intersection de sous-groupoïdes paraboliques standards. Dans [God10], Godelle introduit la notion de ruban entre deux sous-groupoïdes paraboliques standards dans un groupoïde de Garside. Ces rubans apparaissent comme des « conjugants élémentaires » de sous-groupoïdes paraboliques standards. Godelle montre ensuite que dans le cas des groupes de Garside (sous des hypothèses convenables), les rubans forment une catégorie de Garside, qu'il utilise ensuite pour étudier la conjugaison des sous-groupes paraboliques standards. Une amélioration de la définition de banc prenant les rubans en compte pourrait ouvrir la voie à une étude similaire pour les sous-groupes paraboliques dans les groupes de Garside faibles.

À présent, concernant les groupes de tresses complexes : dans [Shv96], Shvartsman étudie les groupes quotients de la forme $B(W)/Z(B(W))$. Il prouve que, dans le cas où W est un groupe de réflexions réel, ce groupe est réalisé comme le groupe fondamental d'une variété projective naturellement associée à W. Il conjecture ensuite que ceci est également valable pour les groupes de tresses complexes en général. La compréhension globale que nous avons à présent du centre $Z(B(W))$, ainsi que notre compréhension des tresses régulières, pourrait être suffisante pour démontrer cette conjecture. Une telle idée est déjà présente dans [BMR98, Proposition 2.23], mais la preuve admet un défaut, comme remarqué dans [DMM11] (la suite du bas n'est pas exacte et le carré inférieur droit n'est pas commutatif).

Les présentations que nous obtenons de $B(G_{31})$ via la méthode de Reidemeister-Schreier pour les groupoïdes sont assez grandes et les réductions de ces présentations sont effectuées via de lourds calculs directs. Même si une preuve totalement conceptuelle semble hors de portée (pour d'autres groupes exceptionnels comme G_{33} ou G_{34}, les preuves connues utilisent aussi des calculs directs avec le monoïde dual associé), il serait intéressant d'obtenir une preuve faisant appel à des calculs moins lourds. Par exemple, un bon choix de transversale de Schreier enracinée en $u \in \mathrm{Ob}(\mathcal{B}_{31})$ pourrait induire une présentation moins volumineuse de $\mathcal{B}_{31}(u, u)$, ou au moins une présentation qui pourrait être simplifiée plus facilement.

De plus, il pourrait être intéressant de vérifier si les autres présentations de $B(G_{31})$ que nous obtenons donnent également lieu à de bonnes descriptions des sous-groupes paraboliques à conjugaison près, dans l'esprit du Théorème 10.2.1.

Plus généralement, les travaux de González-Meneses et Marin dans [GM22], complétés par notre travail sur $B(G_{31})$, donnent ce qui semble être une bonne généralisation aux groupes de tresses complexes du complexe de courbes pour le groupe de tresses usuel. Il est alors naturel de chercher à montrer que cette généralisation satisfait les mêmes propriétés que le complexe de courbes usuel. Par exemple, nous avons mentionné que le complexe de courbes usuel est hyperbolique dans le sens de Gromov. Une telle propriété n'est pas encore connue dans le cas des groupes de tresses complexes (ni même dans le cas des groupes d'Artin sphériques). De plus, l'interprétation du groupe de tresses usuel comme groupe modulaire permet la définition de la classification de Nielsen-Thurston des tresses. La définition générale de sous-groupe parabolique pourrait permettre l'extension de la classification de Nielsen-Thurston aux groupes de tresses complexes arbitraires, au moins pour la notion de tresse réductible.

Dans le cas des sous-groupes paraboliques des groupes d'Artin sphériques, une question soulevée par Marin est de savoir si le plongement d'un sous-groupe parabolique peut rassembler des classes de conjugaison (stabilité de conjugaison). Une description complète de ce phénomène est donnée par Calvez, Cisneros et Cumplido [CCC20], en utilisant notamment la théorie de Garside. Il serait intéressant de voir si leur méthode peut être généralisée aux groupes de tresses complexes quelconques et donner une description complète de la stabilité de conjugaison dans ce cas.

Dans l'Annexe B, nous introduisons une classe particulière de groupes de Garside définis par générateurs et relations, que nous appelons *groupes circulaires*. Cette classe inclut en particulier les groupes de tresses complexes de rang 2. Par construction, un groupe de réflexions complexes de rang 2 peut être retrouvé à partir de son groupe de tresses (vu comme un groupe circulaire) en imposant des relations de torsion aux générateurs. La question inverse est alors : « Quand ajouter des relations de torsion à un groupe circulaire induit-il un groupe fini ? Et, quand c'est le cas, ce groupe est-il un groupe de réflexions complexe ? ». Les travaux d'Achar et Aubert [AA08] donnent en particulier une réponse positive à cette question dans le cas du groupe circulaire

$$G(3,3) := \langle a, b, c \mid abc = bca = cab \rangle.$$

Les travaux de Gobet [Gob24] donnent également une réponse positive dans de nombreux cas. Dans un travail commun en cours avec I. Haladjan, nous comptons décrire exactement quels sont les quotients finis des groupes circulaires obtenus par des relations de torsion sur les générateurs. Nous conjecturons qu'aucun nouveau quotient n'apparaît à part ceux décrits dans [AA08] et dans [Gob24].

Guide de lecture

Cette thèse est divisée en deux parties principales, complétées par deux annexes. La première partie est centrée autour de thèmes généraux en théorie de Garside, tandis que la seconde se concentre sur des applications de cette théorie dans l'étude des groupes de tresses complexes. Notre espoir est que ce découpage permettra aux lecteur.ice.s familier.e.s avec une partie ou l'autre de naviguer facilement entre ce qu'iels cherchent à savoir et ce qu'iels connaissent déjà. Nous avons inclus de nombreux exemples afin d'illustrer les diverses constructions et résultats que nous exposons.

Au début de chaque section, nous donnons une liste complète des objets considérés « par défaut » dans cette section, afin de ne pas avoir à les rappeler au début de chaque résultat. Cependant, dans le cas des théorèmes, nous donnerons systématiquement la liste complète des assertions sans faire référence implicitement au début de la section.

La première partie est dédiée aux catégories et aux groupoïdes de Garside. Cette première partie peut être considérée comme une référence auto-suffisante en théorie de Garside, dans un contexte bien plus restreint que celui de [DDGKM]. Notons toutefois que ce contexte est suffisant pour l'étude des groupes de tresses complexes et qu'il permet une exposition plus concise.

Nous commençons dans le premier chapitre par l'introduction de quelques constructions et définitions générales sur les catégories et les groupoïdes. En particulier c'est ici que nous donnons la définition de présentation de catégorie, qui est une notion centrale en théorie de Garside. Nous détaillons également une méthode générale permettant de calculer une présentation d'un groupe à partir d'une présentation d'un groupoïde équivalent. Nous appelons cette méthode la *méthode de Reidemeister-Schreier pour les groupoïdes* (elle généralise la méthode de Reidemeister-Schreier classique).

Dans le deuxième chapitre, nous présentons les notions de catégories et groupoïdes de Garside, en suivant principalement [DDGKM] que nous adaptons à notre contexte plus restreint.

Le Chapitre 3 présente les outils usuels pour étudier la conjugaison dans les groupoïdes de Garside (catégories de conjugaison, super summit-sets etc). En plus de ces définitions classiques (reprises de [DDGKM]), nous adaptons la notion de « swap », récemment définie pour les groupes de Garside par González-Meneses et Marin dans [GM22], au contexte des catégories. Nous étudions en particulier le comportement de l'opération de swap appliquée aux puissances d'un élément donné. Nous prouvons notamment que les puissances d'un élément récurrent pour le swap sont encore récurrentes.

Le Chapitre 4 peut être vu comme une boîte à outils pour construire de nouveaux groupoïdes de Garside à partir de groupoïdes de Garside déjà connus. La plupart des constructions que nous présentons ici apparaissent déjà dans la littérature (à l'exception de celle de la Section 4.2), mais souvent avec moins de détails qu'ici. Par exemple, le concept de graphe de conjugaison se trouve dans [FG03a] et l'idée de munir les catégories de conjugaison d'une structure de Garside se trouve dans [DDGKM], mais l'idée de munir un graphe de conjugaison d'une structure de Garside est nouvelle. De même, la construction du germe des éléments périodiques dans la Définition 4.5.7 est nouvelle, ainsi que les résultats que nous obtenons sur les classes de conjugaison des éléments périodiques.

Nous terminons la première partie en donnant dans le Chapitre 5 une étude approfondie des sous-groupoïdes paraboliques standards dans un groupoïde de Garside (la notion elle-même n'est pas nouvelle et a été introduite par Godelle). Nous construisons des sous-groupes paraboliques minimaux contenant un élément donné dans un groupoïde de Garside. La situation est assez différente de la situation plus classique des groupes de Garside, ce qui nous amène à introduire la notion de banc de sous-groupoïde parabolique standard comme un outil pour pouvoir gérer les intersections de sous-groupoïdes paraboliques standards dans un groupoïde de Garside général. Cette notion nous offre un cadre approprié pour adapter les résultats de González-Meneses et Marin sur les intersections de sous-groupes paraboliques arbitraires dans les groupes de Garside. Nous donnons ensuite différentes constructions de bancs adaptés aux différentes constructions de groupoïdes de Garside données dans le Chapitre 4. En particulier, la dernière section du Chapitre 5 peut aller de pair avec le chapitre 4 pour les lecteur.ice.s intéressé.e.s par l'étude des sous-groupoïdes paraboliques standards.

Nous entamons la seconde partie sur les groupes de réflexions complexes et leurs groupes de tresses dans le Chapitre 6, en rappelant quelques faits de base sur ces groupes. En particulier, nous rappelons la classification de Shephard-Todd des groupes de réflexions complexes irréductibles, avec d'un côté une série infinie de groupes de matrices monômiales et d'un autre côté la liste de 34 groupes exceptionnels (notés $G_4, \ldots, G_{37}$). Nous rappelons également la dichotomie entre groupes de réflexions complexes bien- et mal-engendrés, qui est importante dans la suite.

Un thème central de la seconde partie est celui des sous-groupes paraboliques des groupes de tresses complexes. Bien que nous les étudiions sous l'angle de la théorie de Garside, ces groupes sont en fait définis topologiquement comme des cas particuliers d'une notion générale de groupe fondamental local (voir [GM22]). Dans le Chapitre 7, nous proposons une légère extension du concept de groupe fondamental local défini dans [GM22] et nous prouvons que cette extension se comporte bien via certains revêtements particuliers, provenant d'actions de groupes. Nous rappelons également les premiers résultats de [GM22] sur les sous-groupes paraboliques des groupes de tresses complexes.

Dans le Chapitre 8, nous étudions le monoïde dual associé à un groupe de réflexions complexe bien-engendré. Ce monoïde de Garside défini par Bessis nous permet ensuite d'étudier le groupe de tresses d'un tel groupe de réflexions. Ces résultats ne sont pas nouveaux, mais nous étudions également les sous-groupes paraboliques des groupes de tresses complexes bien-engendrés en utilisant ces monoïdes. Nous montrons en particulier que les monoïdes duaux préservent le support dans le sens de [GM22], ce qui nous permet de décrire complètement les sous-groupes paraboliques du groupe de tresses associé et de fournir de nouvelles preuves des résultats de González-Meneses et Marin dans le cas des groupes bien-engendrés. Pour prouver ces différents résultats, nous utilisons notamment la correspondance entre les éléments simples du monoïde dual et certains treillis de partitions non-croisées dans le cas des groupes bien-engendrés de la série infinie.

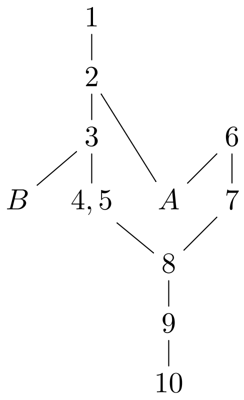
Le Chapitre 9 est consacré à l'étude des tresses régulières et de leurs centralisateurs dans les groupes de tresses complexes. Nous commençons par expliquer comment les tresses régulières fournissent de bons relèvements dans les groupes de tresses complexes des éléments réguliers dans les groupes de réflexions complexes. Dans le cas d'une tresse régulière ρ dans un groupe de tresses complexe bien-engendré $B(W)$, Bessis a montré que le centralisateur de ρ est équivalent (en tant que groupoïde) à un groupoïde de Garside défini à partir de la structure duale sur $B(W)$. Nous proposons une étude systématique de ces groupoïdes, que nous appelons *groupoïdes de Springer*. Nous montrons que ces groupoïdes fournissent un bon cadre de théorie de Garside pour étudier les centralisateurs réguliers dans les groupes de tresses complexes bien-engendrés. En particulier, ces groupoïdes sont dotés d'un banc de sous-groupoïdes paraboliques standards, adapté à l'étude des sous-groupes paraboliques (topologiques) des centralisateurs réguliers dans les groupes de tresses complexes bien-engendrés. Cela nous permet ensuite de donner une description complète (pour tous les groupes de tresses complexes) des sous-groupes paraboliques du centralisateur d'un élément régulier via les sous-groupes paraboliques du groupe ambiant.

Enfin, nous appliquons les résultats obtenus dans le Chapitre 9 au groupe de tresses complexe $B(G_{31})$ dans le Chapitre 10. Nous utilisons la méthode de Reidemeister-Schreier pour les groupoïdes afin de calculer des présentations de ce groupe en utilisant son groupoïde de Springer associé, comme dans [Gar23b]. Nous décrivons ensuite les sous-groupes paraboliques de $B(G_{31})$ à conjugaison près et nous montrons [GM22, Theorem 1.3] pour ce groupe. Ce dernier résultat est le dernier élément manquant pour compléter la preuve des résultats principaux de [GM22] pour tous les groupes de tresses complexes.

La première annexe consiste principalement en une généralisation d'un article de Dehornoy, Lafont [DL03] donnant une méthode efficace de calcul de l'homologie d'un groupe de Garside (d'un groupe Gaussien). Nous adaptons cette construction aux groupoïdes de Garside et nous montrons comment déduire des résultats sur l'homologie des groupes de Garside faibles. Nous appliquons ensuite cette construction pour calculer l'homologie des groupes de tresses complexes exceptionnels dans différents modules. En particulier, nous obtenons de nouveaux résultats sur le groupe de tresses complexe $B(G_{31})$.

La seconde annexe étudie une classe particulière de groupes de Garside, contenant notamment les groupes de tresses complexes de rang 2 et que nous appelons *groupes circulaires*. Nous prouvons en particulier que les racines sont uniques à conjugaison près dans ces groupes. Nous considérons également une généralisation des groupes circulaires, appelée *groupes de type hosoédral*. Ces groupes sont définis à partir des groupes circulaires via une procédure appelée le Δ-*produit*, que nous étudions en général. Nous décrivons également les sous-groupes paraboliques de ces groupes et prouvons qu'ils sont stables par intersection.

Pour faciliter la lecture, nous incluons ci-dessous un diagramme donnant les dépendances logiques entre les différents chapitres et annexes.



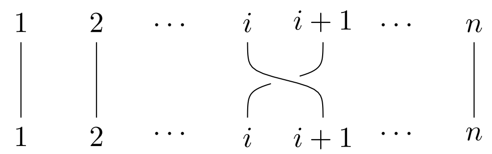
Introduction

Context and overview

A very brief history of the usual braid group

Mathematical braids were first formally described by Artin in [Art25], where they are defined as particular families of nonintersecting curves in the Euclidean space (geometric braids), up to some equivalence relation, obtained by deforming geometric braids. Of course "deforming geometric braids" may have several a priori different meanings. However, it was shown in [Art47] that most of the natural notions of deformation actually give the same set of equivalence classes (see also [Deh19, Chapter 1]).

Moreover, Artin showed in [Art25] that braids are naturally equipped with a composition law, making the set of braids on n-strands into a group B_n, of which he gave a presentation by generators and relations: The group B_n is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$ -the "Artin generators"- where σ_i is represented by the following geometric braid:



The defining relations are $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $|i-j| > 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \in [\![1, n-2]\!]$ (note that these relations can be easily seen to hold by drawing the associated diagram).

In the following years, some other fundamental properties of braid groups were described: The relation between braids, knots and links was investigated by Alexander [Ale28] (in which the *Alexander polynomial* of a knot is first defined) and Markov [Mar45] (see also [Bir74, Chapter 2]). The representation theory of braid groups was considered by Burau in [Bur35], in which he described an $n-1$-dimensional irreducible representation of the braid group on n-strands. This representation, now called the *Burau representation*, can be used to construct the Alexander polynomial, and showcases the importance of the representation theory of B_n, notably in knot theory. Later, during the 60's, Fox and Neuwirth realized in [FN62b] the group B_n as the fundamental group of the space of configurations of n points in the complex plane (see also [FN62a]), and they showed that the later space is a $K(\pi, 1)$-space (or Eilenberg-MacLane space) (this will be a recurrent theme, as seen in the next sections). The cohomology of the braid group was studied during the 70's, first by Arnold [Arn70], where he showed in particular theorems of finiteness, recurrence and stabilization of this cohomology. His work was continued by Fuks, who calculated the cohomology of the braid groups mod 2 in [Fuk70]. Independently, and with different methods, the homology of B_n was studied by Cohen [Coh73a], [Coh73b], [CLM76] (see the survey [Ver98] for more details).

Among the open problems given by Artin in [Art47] are the conjugacy problem for braid groups (is there an algorithm determining whether two braids are conjugate?), and the determination of the centralizer of a given braid. The first of these problems was solved by Garside in [Gar69], in which he introduced a particular submonoid B_n^+ of B_n, along with a distinguished element $\Delta \in B_n^+$. He then proved that the monoid B_n^+ is cancellative and admits greatest common divisors (gcds) and least common multiples (lcms), allowing to describe the whole group B_n using just B_n^+ and Δ: Every element of B_n can be uniquely written as $\Delta^m b$ with $m \in \mathbb{Z}$ maximal and $b \in B_n^+$. Using this description, Garside then defined a characteristic finite subset -the *summit-set*- of every conjugacy classes in B_n, thus giving a solution to the conjugacy problem.

The work of Garside was generalized in the 80's by constructing distinguished decompositions of elements of B_n involving the element Δ and the monoid B_n^+. While such decompositions do not appear explicitly in [Gar69], the fact that gcds exist in B_n^+, which appears in [Gar69], is a big step towards constructing them. The first explicit instances of such decompositions, or *normal forms*, were constructed independently by Adjan [Adj66], El-Rifai and Morton [EM94] and Thurston [Thu88]. The normal form later became an important tool in the study of braid groups. For instance, El-Rifai and Morton gave in [EM94] an improvement of Garside's solution to the conjugacy problem, by considering characteristic subsets of conjugacy classes strictly included in the summit-sets of Garside, the *super summit-sets*.

The second problem was solved theoretically by Makanin in [Mak71], in which he gives a way to compute generators of the centralizer of an arbitrary braid. Later, a more efficient algorithm was given by Franco and González-Meneses in [FG03a], again using generalizations of the work of Garside.

Yet another application of the normal form on B_n lies in its representation theory: The Burau representation we evoked earlier is known to be faithful for $n \leqslant 3$, but it was proven to be unfaithful for $n \geqslant 9$ by Moody [Moo91], then for $n \geqslant 6$ by Long and Paton [LP93], and for $n = 5$ by Bigelow [Big99]. The question of whether or not the Burau representation of B_4 is faithful is still an open problem. Here again, a recent result on this matter [GWY23] uses the normal form on B_4.

In the general case, unfaithfulness of the Burau representation of B_n leaves open the *linearity problem* for the braid group: Does B_n embed in a finite dimensional linear group over some field? In the beginning of the 2000's, Krammer constructed in [Kra00] a representation of the braid group B_n in $\mathrm{GL}_{\frac{n(n-1)}{2}}(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$, and he showed that this representation is linear in the case of the 4-strands braid group B_4. Later, Krammer [Kra02] and Bigelow [Big01] both proved that this representation of B_n is always faithful.

Another important description of the braid group is as a *mapping class group*. Recall that if S is a compact oriented surface (possibly with boundary), with a finite subset of marked points $p := \{p_1, \ldots, p_n\} \subset S$, then the mapping class group of S is defined as the set of isotopy classes of orientation preserving homeomorphisms of S which fix the boundary ∂S pointwise, and which leaves the set p globally invariant. Magnus proved in [Mag34] that the braid group B_n can be realized as the mapping class group of the closed unit disk with n punctures. This interpretation of the braid group is useful as it allows to extend the Nielsen-Thurston classification of homeomorphisms to braids: a braid is then either *periodic*, *reducible*, or *pseudo-Anosov*. This thrichotomy has been used to prove several results on braids. For instance, it was used by

González-Meneses in [Gon03] to show that two k-th roots of a same element of B_n are always conjugate. Also, González-Meneses and Wiest described in [GW04] the centralizer of a braid in terms of its Nielsen-Thurston type.

Moreover, seeing the braid group B_n as a mapping class group also induces a group action of B_n on any object in the punctured disk which is defined up to isotopy. As an example, B_n acts on the set of (isotopy classes of) non-denegerate simple closed curves drawn in the punctured disk. These curves naturally form the vertices of a simplicial complex, called the *curve complex*, and B_n acts by isometries on this complex. Since the curve complex was proven by Masur and Minsky to be δ-hyperbolic in [MM99], we obtain an action of B_n on a δ-hyperbolic simplicial complex, yielding results in geometric group theory.

Artin groups as a first generalization

Artin groups (or Artin-Tits groups) were first defined by Tits in [Tit66], by a presentation similar to the original presentation given by Artin for the n-strand braid group. This algebraic approach quickly found a topological counterpart in the work of Brieskorn [Bri71], where the author realizes the Artin group $A(W)$ attached to a Coxeter system (W, S) (with W finite) as the fundamental group of a topological space naturally attached to W (namely, the space of regular complex orbits). In the special case of where $W = \mathfrak{S}_n$ is the symmetric group, this last space is the space of configurations of n complex points. The result of Brieskorn is then a generalization of the result of Fox and Neuwirth [FN62b] in this case.

Starting from this, it is natural to consider Artin groups associated to finite Coxeter groups (i.e. *spherical Artin groups*) as a generalization of the usual braid group. This led to several research works aiming to generalize the known results on the usual braid group (for instance those we mentioned in the last section) to spherical Artin groups.

First, Brieskorn, Saito [BS72] and Deligne [Del72] solved independently the conjugacy problem and the word problem for spherical Artin groups. They also determined the center of these groups. Both of their approaches generalize the work of Garside [Gar69] by introducing a particular submonoid $A^+(W)$ of a spherical Artin group $A(W)$. Moreover, we said that the space of configurations of n complex points is a classifying space for the usual braid group on n-strands. Following this idea, Brieskorn conjectured in [Bri73] that the regular complex orbit space of a finite Coxeter group also provides a classifying space for the associated Artin group. He then proved this conjecture in several, but not all, irreducible cases. Quickly after that, this conjecture was proven uniformly by Deligne in [Del72], again using the submonoid $A^+(W)$.

Later, during the 90's, the (co)homology of spherical Artin groups was computed by Salvetti in [Sal94] using a geometric approach, and by Squier in [Squ94], using a more algebraic approach (again using generalizations of the work of Garside, Brieskorn-Saito and Deligne).

Concerning representation theoretic aspects, independent works of Cohen, Wales [CW02] and Digne [Dig03] proved that spherical Artin groups are linear. The representation they use is a generalization of the Krammer representation of the usual braid group. The proofs adapts arguments of Krammer, which themselves rely on the normal form for the braid group, using the monoid introduced by Garside.

We see that the influence of Garside's work on the usual braid group is omnipresent in the study of spherical Artin groups. This is mostly because the submonoid $A^+(W)$ of a spherical Artin group $A(W)$ behaves very similarly to the monoid B_n^+ introduced by Garside. This realization led in the end of the 90's to the development of the notion of "Garside monoid", which generalizes the good algebraic properties of the monoid B_n^+. We postpone this story to the next section.

Note that these various generalizations still leave some unanswered questions: for instance there is no general interpretation of spherical Artin groups as mapping class groups. In particular there is no geometrical way to define a generalized Nielsen-Thurston classification for elements of spherical Artin groups (the particular notion of periodic element can however be algebraically generalized as elements of a spherical Artin group which admit a central power). As a consequence, the arguments of González-Meneses [Gon03] proving that roots are unique up to conjugacy in the usual braid group cannot be readily generalized. Nevertheless, the result of González-Meneses was generalized by Lee and Lee [LL10] to Artin groups attached to Coxeter groups of type B, by embedding such an Artin group in the usual braid group.

Garside groups and Garside groupoids

As we mentioned in the last section, the work of Garside on the usual braid group was rapidly adapted to the more general case of spherical Artin groups. Later on this adaptation was formalized in a general algebraic theory, which is now called "Garside theory". The advent of this theory is described as a "natural but slowly emerging program" in the reference book [DDGKM], which was published 46 years after Garside's PhD thesis! It then seems reasonable to cut this rich story in several distinct acts, in order to put an emphasis on the many new ideas and frameworks which appeared over this time span.

The first act could be referred to as the proto-history of Garside structures, and contains most of the work we discussed in the last sections. That is, generalizations of the work of Garside [Gar69] to spherical Artin groups. This act ends in the 90's with the emergence of new structures resembling spherical Artin groups (and their associated positive monoids) while being distinct from them. For instance, Birman, Ko and Lee introduced in [BKL98] a new submonoid of the usual braid group, which gives rise to new solutions of the word and conjugacy problem, and which is not an Artin-Tits monoid. Around the same time, Dehornoy considered interactions between the usual braid group and selfdistributivity systems [Deh92], [Deh94]. In particular, he introduced a monoid M_{LD} in which a normal form reminiscent of the normal form on B_n exists. However, the monoid M_{LD} does not contain an element resembling the element $\Delta \in B_n$, which showcases a need of generalizing Garside's work.

The second act begins at the turn of the century with the foundational article by Dehornoy and Paris [DP99], which introduces the notions of Garside monoids and Garside groups, however not exactly in their current form: the authors consider the three distinct classes of Gaussian monoids, small Gaussian monoids, and Garside monoids (each class of monoids is bigger than the next).

Roughly speaking, a cancellative monoid M is *Gaussian* if the length of a decomposition of x in M as a product (with nontrivial entries) is bounded (the bound depending on x), and if the lcm of two elements of M always exists. A Gaussian monoid is *small* if if admits an element Δ whose left- and right-divisors coincide, are in finite number, and generate M. A small Gaussian monoid is *Garside* if the element Δ can be chosen to be the (left- and right-) lcm of the atoms of M. A Gaussian (resp. small Gaussian, Garside) monoid always embeds in its enveloping group $G(M)$, which can be described as a group of fractions. The group $G(M)$ is then called a Gaussian (resp. small Gaussian, Garside) group. More generally, we will denote a small Gaussian group by a triple (G, M, Δ), where (M, Δ) is a small Gaussian monoid, and where G is the enveloping group of M. By construction, the monoid $A^+(W)$ attached to a spherical Artin group $A(W)$ is a Garside monoid in the sense of [DP99], as well as the monoid introduced by Birman, Ko and Lee in [BKL98].

The terminology of (small) Gaussian groups was quickly abandoned in favor of Garside groups. In today's terminology, Garside groups are what Dehornoy and Paris called small Gaussian groups. This change of terminology is already apparent in [Deh02], in which Dehornoy credits the influence of works of Bessis, Charney, Digne and Michel among others. Nonetheless, several articles were written with the Gaussian groups terminology [Pic00], [Pic01a], [DL03], [Pic01b].

The algebraic formalization of Garside groups initiated broad research axes in the 2000's, aiming to describe both theoretical and algorithmic aspects of these groups: In [DL03], Dehornoy and Lafont give two algebraic methods to compute the homology of a Garside group. The second method is reminiscent of the work of Squier [Squ94] on the homology of Artin groups. A more topological approach is developed in [CMW04], expanding on earlier works by Bestvina [Bes99] on spherical Artin groups.

Conjugacy in Garside groups has been widely studied: First, the algorithm of El-Rifai, Morton [EM94] using the super summit-set was generalized to Garside groups in [Pic01b], and improved upon in [FG03b]. A particular subset of the super-summit set, called the *ultra summit-set* was introduced by Gebhardt in [Geb05]. This set was studied extensively by Birman, Gebhardt and González-Meneses in a series of articles [BGG07a],[BGG08], [BGG07b], in the hope of finding a polynomial-time algorithm for the conjugacy problem in braid groups. In [GG10b] and [GG10a], Gebhardt and González-Meneses introduce a new operation for solving the conjugacy problem in Garside groups. This new operation, called cyclic sliding, provides characteristic subsets of conjugacy classes which are even smaller than ultra summit-sets. These characteristic subsets are called *sets of sliding circuits*.

Among other works, let us cite [FG03a], in which Franco and González-Meneses give an algorithm for computing generating systems of centralizers in Garside groups. Also, the extraction of roots of elements in a Garside group was studied for instance by Sibert in [Sib02], and by Zheng in [Zhe06].

Note that the approach of Garside monoids and Garside groups is still not sufficient to study the monoid M_{LD} properly, as two elements of M_{LD} may not admit right-multiples at all (although those who do always have a right-lcm). However, Dehornoy was still able to study this monoid in [Deh00], and to connect it with the braid group B_∞ on an infinite number of strands.

The second act ends in the second part of the 2000's, with the realization that the mechanism of the normal form, as constructed in Garside monoids, can easily be generalized to categories instead of monoids. The third act is then marked by the advent of Garside categories and Garside groupoids. We can (more or less arbitrarily) fix the beginning of this third act with the publication by Krammer of [Kra08], which is the first work explicitly mentioning Garside structures on categories (it was first circulated as a preprint in 2005). Let us mention that earlier works also implicitly used related structures (see for instance [DL76], [Del72], [Deh94] or [God01]).

In less than two years after the first preprint by Krammer, other works of Digne, Michel [DM06] and Bessis [Bes07] also introduced categorical versions of Garside monoids and Garside groups. Both of these works were motivated by a situation in which categorical analogues of Garside monoids appeared spontaneously, and where Garside monoids were not sufficient:

respectively, the study of Deligne-Lusztig varieties, and the theory of periodic elements in Garside groups. Already in [DM06], the authors mention the project of writing a survey on the topic of Garside categories.

The third act ends (almost by definition) with the publication of the reference book [DDGKM] by Dehornoy, Digne, Godelle, Krammer and Michel, which is the outcome of the project we just mentioned. Over the seven years period in which this book was written, the axioms of Garside structures were loosened and stretched into the final approach present in [DDGKM]. This approach is centered around the concept of *greedy decompositions*, generalizing the normal form in the braid monoid B_n^+. A *Garside family* on a category is then defined as a subfamily of the category giving rise to such decomposition for every element of the category. This very broad approach can be applied in a wide variety of contexts (see [DDGKM, Part B]), but it gives much weaker results than the original theory of Garside groups. For instance the study of a groupoid starting from a subcategory is not reasonably possible starting from just a Garside family on this category. This leads the authors to consider particular stronger cases of Garside families, notably those arising from a *Garside map* (generalizing the Garside element Δ of a Garside monoid). Note however that no concept with the name "Garside category" is defined in [DDGKM].

Nowadays (in what we could call the fourth act), Garside groups are still studied for themselves and have not been supplanted by Garside groupoids, but rather completed by them. Recent works of Paolini, Salvetti [PS21], and Delucchi, Paolini, Salvetti [DPS22], use posets attached to particular non spherical Artin groups which are not lattices in general, and which as such do not give rise to Garside structures. Nonetheless, these posets satisfy a combinatorial property (lexicographic shellability), and they still provide ways of studying Eilenberg-MacLane spaces just like classical Garside structures. This again suggests a need for generalizations of Garside structures.

Moreover, it is said in [DDGKM] that the book provides the "ultimate generalization" in the direction of greedy decompositions (by definition, Garside families are exactly those subfamilies which give rise to greedy decompositions). However, the authors also point out that other directions of generalizations could also be considered. For instance they credit Crisp, McCammond and Krammer for the possibility of seeing a Garside group mainly as a group acting on a lattice with certain interval playing a distinguished role. In this spirit, Haettel and Huang recently proposed in [HH23] a characterization of (weak) Garside groups as groups acting on particular flag complexes.

Complex braid groups and the Broué-Malle-Rouquier program

Spherical Artin groups are known to share many properties with the usual braid group. Starting from this observation, it is natural to look for generalizations of spherical Artin groups, and to ask which properties these generalizations share with the usual braid group. In the last section we discussed the algebraic generalization provided by Garside groups (and Garside groupoids). In this section, we turn to a more topological/geometrical generalization, given by complex braid groups.

Although spherical Artin groups are defined starting from finite Coxeter groups, the work of Brieskorn [Bri71] considers finite subgroups of $\mathrm{GL}_n(\mathbb{R})$ which are generated by reflections (*real reflection groups*). These two concepts are equivalent in the following sense: First, if (W, S) is a Coxeter system with W finite, then W admits a faithful representation making it into a real reflection group (Tits representation). Conversely, any real finite reflection group $W \subset \mathrm{GL}_n(\mathbb{R})$

admits a Coxeter presentation. Under this correspondence, the regular orbit space evoked in an earlier section is simply the orbit space of W acting on $\mathbb{C}^n$ where the complexified hyperplanes attached to the reflections of W have been removed.

One possible direction for generalization is then to extend the base field from $\mathbb{R}$ to $\mathbb{C}$, and to work with *complex reflection groups* (all reflection groups will be assumed to be finite, unless specified otherwise). By construction, complex reflection groups are in particular a generalization of real reflection groups, but not every complex reflection group comes from a real reflection group (for instance, a complex reflection may have arbitrary order, while a real reflection is always involutive). One easily reduces the study of complex reflection groups to the study of *irreducible complex reflection groups*. That is, the complex reflection groups $W \subset \mathrm{GL}_n(\mathbb{C})$ such that $\mathbb{C}^n$ admits no nontrivial subspace which is globally W-invariant. The *rank* of an irreducible complex reflection group $W \subset \mathrm{GL}_n(\mathbb{C})$ is defined as the integer n.

Complex reflection groups were first considered systematically in the 50's. Shephard introduced a concept of *regular complex polytopes* [She52], and studied a particular class of complex reflection groups [She53]. Shortly after, the irreducible complex reflection groups were classified by Shephard and Todd in [ST54], using in particular existing literature on collineation groups in projective spaces. On the one hand, there is an "infinite series" of irreducible groups, denoted by $G(de, e, n)$ and depending on three integer parameters d, e, n. On the other hand, we have a finite sequence of 34 exceptional groups, labeled $G_4, \ldots, G_{37}$ in [ST54]. Note that several of these groups appear as symmetry groups of the regular complex polytopes introduced in [She52], these groups were named *Shephard groups* in [OS88b].

Using this classification result, it becomes possible to study complex reflection groups in a case-by-case fashion. A first instance of this is the proof by Shephard and Todd [ST54] that complex reflection groups are exactly the finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ whose ring of invariants in the polynomial algebra $\mathbb{C}[X_1, \ldots, X_n]$ is again a polynomial algebra (note that Chevalley later provided a case-free argument for this result in [Che55]).

It is known that *rational* reflection groups (i.e. Weyl groups) control several representation-theoretic aspects of algebraic groups (for instance). This can be understood by associating to a rational reflection group W a *Hecke algebra* $\mathcal{H}(W)$, defined as a deformation of the group algebra $\mathbb{Z}W$. In the more general case where W is a real reflection group, W admits a Coxeter presentation, which allows for a direct definition of $\mathcal{H}(W)$ by generators and relations. It is fairly easy to show that $\mathcal{H}(W)$ is free of rank $|W|$ over its ring of definition. These Hecke algebras associated to Coxeter groups are widely studied today (see for instance [GP00] or [Eli+20])

During the 90's, it was realized by Broué, Malle, Michel [BMM93] and by Broué, Malle [BM93] that complex reflection groups play a role in the representation theory of finite reductive groups. From there it seems natural to try and define a Hecke algebra for any complex reflection group, in order to understand this role further. The first papers on this topic by Ariki, Koike [AK94], Ariki [Ari95] and Broué, Malle [BM93] consider groups in the infinite series along with some exceptional groups. Their approach was by deforming a presentation of the reflection group W into a presentation of the Hecke algebra $\mathcal{H}(W)$. However, generalizing this to arbitrary complex reflection groups raises the problem of dependency on the chosen presentation. Indeed there is no good analogue of Coxeter presentation for complex reflection groups, and thus the choice of a presentation of W could impact the structure of the associated algebra $\mathcal{H}(W)$. This was one of the main motivations of Broué, Malle and Rouquier in [BMR98] for introducing *complex braid groups* attached to complex reflection groups.

To a complex reflection group $W \subset \mathrm{GL}_n(\mathbb{C})$, one can associate the set $\mathcal{A}$ of all the hyperplanes attached to reflections in W, as well as the complement X in $\mathbb{C}^n$ of the union of all the elements of $\mathcal{A}$. The *braid group* $B(W)$ (resp. the *pure braid group* $P(W)$) is then defined as the fundamental group of X/W (resp. of X). A classical theorem of Steinberg ensures that the action of W on X is free, so that we have a covering map $X \twoheadrightarrow X/W$. This covering map in turn induces a short exact sequence of groups

$$1 \to P(W) \to B(W) \to W \to 1.$$

In the case where W is a real reflection group, the braid group $B(W)$ was proven by Brieskorn [Bri71] to be isomorphic to the Artin group $A(W)$ attached to W: Complex braid groups provide yet another generalization of spherical Artin groups.

Broué, Malle and Rouquier define in [BMR98] a distinguished infinite set of generators for $B(W)$, that they call *generators of the monodromy.* Each such element $\sigma \in B(W)$ appears as a lift of a reflection of W under the projection map $B(W) \twoheadrightarrow W$. Later, these elements were called *braid reflections* in [Bro01], or *braided reflections* in [Mar09]. We will call them braided reflections from now on.

The first presentations of complex braid groups date back to Bannai in [Ban76], who computed the complex braid groups of irreducible complex reflection groups of rank 2. In [OS88b], Orlik and Solomon show that a Shephard group W is always attached to a unique Coxeter group W', such that the regular orbit spaces attached to W and to W' are homeomorphic. This implies in particular that $B(W) \simeq B(W') \simeq A(W')$ is a spherical Artin group.

Once the group $B(W)$ has been introduced, it is possible to define the Hecke algebra $\mathcal{H}(W)$ as a quotient of the group algebra of $B(W)$ over some ring of Laurent polynomials. This construction is independent of the choice of a presentation of W. This answers in particular the question of having a good (i.e. unambiguous) definition of the Hecke algebra attached to a complex reflection group. However, studying these algebras proved to be way harder than in the Coxeter case: the theorem that $\mathcal{H}(W)$ is free of rank $|W|$ over its ring of definition took more than twenty years to prove in every case [AK94], [Ari95], [BM93], [Mar14], [Mar12b], [MP17], [Cha18], [Mar19], [Tsu20].

Many important properties of (pure) complex braid groups are stated in [BMR98], generalizing properties of spherical Artin groups. However, most of them are only proven for groups belonging to the infinite series, or for groups of rank 2. This leaves out the family of groups $G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}$. Among those, G_{25}, G_{26} and G_{32} are Shephard groups. This often (but not always!) allows for arguments using their associated Coxeter group (and Artin groups). This highlights the family $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ as a particularly problematic one.

- (Center) If W is irreducible, then Schur's lemma implies that the center $Z(W)$ of W is cyclic of order say k. Broué, Malle and Rouquier introduce in [BMR98] two elements $\pi \in Z(P(W))$ and $\beta \in Z(B(W))$. They show that $\beta^k = \pi$, and that the image of β in W generates $Z(W)$. They conjecture that the center of $B(W)$ (resp. of $P(W)$) is cyclic and generated by β (resp. by π), and that we have a short exact sequence

$$1 \to Z(P(W)) \to Z(B(W)) \to Z(W) \to 1.$$

  They claim to prove this conjecture for every irreducible complex reflection group except $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ [BMR98, Theorem 2.24]. However, as pointed out in [DMM11], the argument for computing the center of $P(W)$ in the case of the Shephard groups G_{25}, G_{26}, G_{32} is flawed.

- (Presentation) If W is a real reflection group, then the Coxeter presentation of W can be summarized in a graph (the *Coxeter diagram*), which also induces a presentation of the Artin group $A(W)$. In [BMR98], Broué, Malle and Rouquier give presentations of irreducible complex reflection groups using diagrams mimicking Coxeter diagrams [BMR98, Table 1 to 5] (many of these presentations were already known). We call these diagrams *BMR-diagrams* from now on. The BMR-diagram of a Coxeter group W is equal to its Coxeter diagram. The BMR-diagram of a Shephard group coincides with the Coxeter diagram of its associated Coxeter group. In particular, the BMR-diagram of a Shephard group induces a presentation of $B(W)$. It is then possible to conjecture that all the BMR-diagrams provide presentations of the associated braid groups. More precisely, let W be an irreducible complex reflection group. The BMR-diagram D attached to W induces a group presentation, which defines a group $G(D)$. One can ask if there is an isomorphism $G(D) \simeq B(W)$ which sends the generators of $G(D)$ to braided reflections in $B(W)$. This is shown in [BMR98, Theorem 2.27] for every irreducible complex reflection group but the ones belonging to the family $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$. Broué, Malle, Rouquier conjecture that the diagram for G_{31} provides a presentation for the braid group $B(G_{31})$, they do not go as far as to conjecture that the same holds for the five other groups.
- (Monoid) Let W be an irreducible complex reflection group, and let D be its associated BMR-diagram. The group presentation induced by D is always a positive homogeneous group presentation. In other words, the relations are equalities between positive words of the same length in the generators. In particular, D can be used to define both a group G and a monoid M, with G being the enveloping group of M. In the case where G is known to be isomorphic to $B(W)$, Broué, Malle, Rouquier ask whether the natural morphism $\iota : M \to G \simeq B(W)$ is injective [BMR98, Question 2.28], and whether or not the group $B(W)$ can be described as $\{\pi^n \iota(b) \mid n \in \mathbb{Z}, b \in M\}$.

All these questions hint towards the existence of Garside group structures for complex braid groups (note that Garside groups were not defined at the time). Indeed, these questions could reasonably be answered in a Garside group: The study of the conjugacy problem in a Garside group often allows for the determination of its center. By construction, a Garside group comes equipped with a presentation, which can be manipulated using Tietze transformations. Moreover, a Garside group comes equipped with a distinguished submonoid for which the above description always holds (replacing π with the Garside element Δ). Thus, if the monoid M defined above is in fact Garside, with Garside element a root of π, then we can try to answer the conjectures of Broué, Malle, Rouquier.

However, the presentations induced by the BMR-diagrams for the groups $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ do not give rise to Garside structures in general. In particular the monoid defined for G_{31} does not embed in the group $B(G_{31})$ ([Pic00, Example 14]). Thus the above questions remained open for $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$ during the beginning of the 2000's.

Concerning presentations of complex braid groups, Bessis showed abstractly in [Bes01] that complex braid groups always admit presentations similar to the ones conjectured in [BMR98], in the sense that they are positive, homogeneous, and have braided reflections as generators. Later, Bessis and Michel [BM04] were able to directly obtain presentations for the groups $B(G_{24})$ and $B(G_{27})$ using computers. In particular, the presentation arising from the BMR-diagram of G_{27} does give a presentation of the associated braid group. Moreover, Bessis and Michel also proposed in [BM04] conjectural presentations for the complex braid groups $B(G_{29}), B(G_{31}), B(G_{33}), B(G_{34})$, backed by computational evidence.

Another motivation for searching for Garside structures on complex braid groups is that these structures naturally provide a way of computing Eilenberg-MacLane spaces. In the case of a real reflection group W (i.e. a finite Coxeter group), we mentioned that the space X (and thus X/W) is known to be an Eilenberg-MacLane space [Del72]. The conjecture that this also holds for all complex reflection groups appears perhaps first in the book by Orlik and Terao [OT92] on reflection arrangements. By the time of the publication of [BMR98], on top of the work already done in the real case, the result was known for groups in the infinite series thanks to Nakamura [Nak83], and for Shephard groups thanks to Orlik, Solomon [OS88a]. Since the case where W has rank two is immediate (see for instance [OT92, Proposition 5.6]), the conjecture was still open only for the groups $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$.

The lack of Garside structures for $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{31}), B(G_{33}), B(G_{34})$ was filled by the so-called *dual braid monoid*. This Garside monoid was first introduced by Bessis in [Bes03] as a new Garside structure on spherical Artin groups, generalizing the monoid defined by Birman, Ko and Lee [BKL98] for the usual braid group. Later, it was generalized by Bessis and Corran [BC06] to complex braid groups of the form $B(G(e,e,n))$ for $e, n > 1$. Then, in [Bes15], Bessis generalized the construction of the dual braid monoid to the class of *well-generated* complex reflection groups, which includes real reflection groups, and the groups $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$ (only G_{31} is missing compared to the list given above).

Let W be a well-generated complex reflection group. In modern terms, the dual braid monoid is defined as an *interval monoid* attached to W and to a (generalized) Coxeter element $c \in W$ (see for instance [GM22, Section 3.3] and [Nea18, Section 3.1.2]). The various possible choices for c induce isomorphic monoids. We then denote by $M(W)$ the dual braid monoid attached to W, and by $G(W)$ the enveloping group of $M(W)$. This definition is seemingly uniform, but showing that $(G(W), M(W), \Delta)$ is a Garside group structure for $B(W)$ relies on fundamental results on well-generated complex reflection groups which are proven case-by-case. The use of the Shephard-Todd classification is then limited, but crucial. We call $G(W)$ the *dual group* of type W, and $(G(W), M(W), \Delta)$ is the *dual structure* on $B(W)$. Note that the isomorphism $G(W) \simeq B(W)$ constructed by Bessis sends the generators of $M(W)$ to braided reflections in $B(W)$.

Using the dual structure, Bessis was able to show that the spaces X and X/W are Eilenberg-MacLane spaces for $W = G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$. He was also able to show that the centers of the associated braid groups are cyclic and generated by β, as conjectured by Broué, Malle and Rouquier. Moreover, the presentation obtained for the associated braid groups (using their Garside structures induced by the dual braid monoid) can be used to prove the conjectural presentations of [BM04].

A priori, this leaves the conjectures open for the case of $W = G_{31}$. However, this group embeds in the well-generated group G_{37} (complexified real reflection group of type E_8) as a centralizer of a regular element in the sense of Springer [Spr74]. Using this, Bessis attached to the complex braid group $B(G_{31})$ a Garside *groupoid*, which he then used in order to construct a classifying space of $B(G_{31})$, and finally showed that the associated space X/W is an Eilenberg-MacLane space. This groupoid is part of a general family of Garside groupoids attached by Bessis to centralizers of regular elements in well-generated complex reflection groups. We call these groupoids the *Springer groupoids*. Bessis was also able to show that the center of $B(G_{31})$ is cyclic and generated by β. Concerning a presentation for $B(G_{31})$, contrary to what is claimed in

[Bes15], it appears that one cannot readily apply [Bes15, Corollary 4.5] to compute a presentation of $B(G_{31})$, since the data given in [BM04] does not satisfy the assumptions of [Bes15, Corollary 4.5]. However, Bessis also mentions in [Bes15, Remark 11.29] that it should be possible to deduce a presentation of $B(G_{31})$ starting from its associated Garside groupoid. Following this remark, we will provide in this thesis a presentation of $B(G_{31})$.

Lastly, the computation of the center of the pure braid group $P(W)$ was completed by Digne, Marin and Michel in [DMM11]. In fact they showed a stronger result: If W is an irreducible complex reflection group, and if $U \subset B(W)$ is a finite index subgroup, then $Z(U) \subset Z(B(W))$. This can in particular be applied to $P(W) \subset B(W)$ to prove the conjecture of Broué, Malle, Rouquier. The proof given in [DMM11] relies on a limited case-by-case analysis: The authors provide a general Garside-theoretic argument, which they then apply to various Garside monoids suitable for studying every irreducible complex braid group but $B(G_{31})$. The case of $B(G_{31})$ is handled using the faithful linear representation of $B(G_{37}) \simeq A(E_8)$ defined in [CW02] and [Dig03].

Concerning the linearity of complex braid groups, Marin defined in [Mar12a] a representation of an arbitrary complex braid group which generalizes the Krammer representation of the usual braid group. This representation was already studied by Marin in particular cases of real reflection groups. Marin showed in particular in [Mar07] that this representation generalizes the earlier representation of Cohen, Wales in the case of Artin groups of type A, D, E. This representation is conjectured to be faithful in the general case. This would prove that complex braid groups are linear, a conjecture which is still open for the groups $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{33}), B(G_{34})$, as well as for the groups of the form $B(G(e, e, n))$ (see [Nea18, Chapter 5] for more details concerning this last family of groups).

Newer developments: Parabolic subgroups

In the above Sections, we purposely avoided any mention of the concept of parabolic subgroups, in order to talk about it here. We choose to dedicate a section to this topic because several main results of this thesis concern these particular subgroups and their behavior in complex braid groups.

Let (W, S) be a Coxeter system. A *standard parabolic subgroup* of W is a subgroup W_I generated by a subset I of S. The couple (W_I, I) is then a Coxeter system. A *parabolic subgroup* of W is the conjugate in W of a standard parabolic subgroup. Since the Artin group $A(W)$ is generated by a formal copy $\mathbf{S}$ of S, this situation transposes easily in $A(W)$: A standard parabolic subgroup of $A(W)$ is a subgroup of $A(W)$ generated by a subset $\mathbf{I}$ of $\mathbf{S}$, and a parabolic subgroup of $A(W)$ is the conjugate in $A(W)$ of a standard parabolic subgroup. A theorem of van der Lek [Lek83] ensures that, for $\mathbf{I} \subset \mathbf{S}$ (associated to $I \subset S$), the standard parabolic subgroup $\langle \mathbf{I} \rangle$ of $A(W)$ is isomorphic to $A(W_I)$.

One can state two important conjectures concerning parabolic subgroups of Artin groups:

(1) Any element of an Artin group is contained in a smallest (relative to inclusion) parabolic subgroup.

(2) The intersection of two parabolic subgroups of an Artin group is again a parabolic subgroup.

Of course, (2) implies (1), but (2) is a priori a stronger statement. Note that (2) becomes trivial when replacing "parabolic subgroups" with "standard parabolic subgroups". These two conjectures have been proven for spherical Artin groups by Cumplido, Gebhardt, González-Meneses, Wiest in [CGGW19]. They are also known for other particular families of Artin groups [CMV23], [Mor21], but the general case is still open.

In the first section, we mentioned that the usual braid group can be seen as the mapping class group of a punctured disk. In particular, it acts on isotopy classes of curves drawn in the punctured disk. These isotopy classes of curves form the vertices of the so-called *curve complex*, on which the usual braid group acts by isometries. We also mentioned that this situation is not immediately transposable to other spherical Artin groups. However, Cumplido, Gebhardt, González-Meneses, Wiest suggest in [CGGW19] that parabolic subgroups of spherical Artin groups provide a good algebraic analogue of isotopy classes of curves for the usual braid group. In particular, they also form the vertices of a simplicial complex on which the Artin group naturally acts.

The concept of parabolic subgroups in Artin groups was generalized to the context of Garside groups by Godelle in [God07]. Again, there is on the one hand a concept of standard parabolic subgroups, and on the other hand a concept of parabolic subgroups, which are the conjugates of the standard parabolic subgroups. Let (G, M, Δ) be a Garside group. For $s \in M$, we denote by M_s the submonoid of M generated by the divisors of s in M, and by G_s the subgroup of G generated by the divisors of s in M. A *standard parabolic subgroup* of a Garside group (G, M, Δ) is a subgroup of the form G_s, where s is a divisor of Δ in M satisfying particular assumptions. Note that (G_s, M_s, s) is then a Garside group. Note also that the parabolic subgroups of (G, M, Δ) depend on (M, Δ), and not only on G. In other words, the notion of parabolic subgroups on a Garside group G depends on its Garside structure.

Let $W \subset \mathrm{GL}_n(\mathbb{C})$ be a complex reflection group. A *parabolic subgroup* of W is defined as the pointwise stabilizer of some part of $\mathbb{C}^n$. A classical theorem of Steinberg then ensures that a parabolic subgroup $W_0 \subset W$ is generated by the reflections of W which it contains. In particular, $W_0 \subset \mathrm{GL}_n(\mathbb{C})$ is then again a complex reflection group. Note that there is no notion of standard parabolic subgroup for complex reflection groups.

Let $W_0 \subset W$ be a parabolic subgroup. Let also X_0 be the complement in $\mathbb{C}^n$ of the reflecting hyperplanes of W_0. We have $X \subset X_0$, whence a morphism $P(W) \to P(W_0)$. In [BMR98, Section 2.D], the authors show that the morphism $P(W) \to P(W_0)$ is a split surjection, and that there is a morphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & P(W_0) \arrow[r] \arrow[d, hook] & B(W_0) \arrow[r] \arrow[d, hook] & W_0 \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & P(W) \arrow[r] & B(W) \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

it is reasonable to define the image of the middle morphism as a parabolic subgroup of $B(W)$. The construction in [BMR98, Section 2.D] depends on some choices, but it is well-defined up to conjugacy. However, notice that this construction is primarily done in the space X. In particular, if two complex reflection groups W, W' are such that $X/W \simeq X'/W'$ (for instance, a Shephard group and its associated Coxeter group), then we have $B(W) \simeq B(W')$, but it is not clear that the collections of parabolic subgroups as defined above are the same.

To manage this problem, González-Meneses and Marin introduce in [GM22] a purely topological concept of a parabolic subgroup of a complex braid group. This concept relies on a general concept of local fundamental group defined for a topological pair. In the case of a complex reflection group $W \subset \mathrm{GL}_n(\mathbb{C})$, González-Meneses and Marin prove that local fundamental groups always exist for the topological pair $(X/W, \mathbb{C}^n/W)$. Their images in the ambient fundamental group $B(W)$ are by definition the parabolic subgroups of $B(W)$. Again, there is no concept of standard parabolic subgroup a priori.

When $B(W)$ is endowed with a Garside group structure, it is possible to try and identify the topological parabolic subgroups of $B(W)$ with the algebraic parabolic subgroups defined using this Garside structure. This can be used to show that different equally likable Garside group structures on the same complex braid group $B(W)$ give rise to the same collection of parabolic subgroups, since the topological parabolic subgroups do not depend on the choice of a Garside structure. With this at hand, one can prove results on parabolic subgroups using Garside-theoretic arguments. By construction, this approach works for the complex braid groups which can be endowed with a Garside group structure. In particular it excludes $B(G_{31})$.

For complex braid groups different from $B(G_{31})$, González-Meneses and Marin use this method in [GM22] to show that parabolic subgroups are stable under intersection. They first show that an arbitrary element of a Garside group is contained in a smallest parabolic subgroup (its *parabolic closure*) under an assumption that they call *support-preservingness* (note that this is conjecture (1) generalized to Garside groups). Godelle showed in [God07] that standard parabolic subgroups of a Garside group (G, M, Δ) are stable under intersection. Given an element $x \in G$, it is then possible to define the *standard parabolic closure* $\mathrm{SPC}(x)$ of x. If x, y are two elements of M which are conjugate in G, say by some α in G, support preservingness requires that α conjugates $\mathrm{SPC}(x)$ to $\mathrm{SPC}(y)$, and not only to an arbitrary parabolic subgroup of G containing y.

Support-preservingness is hard to check in practice. Thus, González-Meneses and Marin provide some reductions in [GM22, Section 4.7], using the concept of *minimal positive conjugators*. If $x \in M$, then a minimal positive conjugator of x is an element $\alpha \in M$ which conjugates x to an element of M, and such that no proper left-divisor of α in M (except 1) conjugates x to an element of M. González-Meneses and Marin prove that it is sufficient to show that the standard parabolic closure is preserved under minimal positive conjugators in order to have global support-preservingness.

Using the existence of parabolic closures, González-Meneses and Marin show in [GM22, Section 6.1] that parabolic subgroups of a *homogeneous* Garside group are stable under intersection. This covers in particular the case of irreducible complex reflection groups having a Garside group structure.

Moreover, following [CGGW19], González-Meneses and Marin define an analogue of the curve complex by using irreducible parabolic subgroups as the vertex set of a particular graph. Again by using (in particular) Garside-theoretic arguments, they give a convenient characterization of the adjacency in this *curve graph* for all complex braid groups but $B(G_{31})$.

Main results

In this section we present the main results of this thesis.

Parabolic subgroups of weak Garside groups

Before studying complex braid groups and their parabolic subgroups, we need to develop the needed Garside machinery. We mentioned that the reference book [DDGKM] does not give a definition of Garside groupoid per say, but rather it studies categories endowed with Garside families satisfying more or less restrictive assumptions. In the study of complex braid groups, we can restrict ourselves to a definition closer to what is given in [Bes07]. In the language of [DDGKM], what we call a *Garside groupoid* is a triple $(\mathcal{G}, \mathcal{C}, \Delta)$, where

- $\mathcal{G}$ is a (small) groupoid.
- $\mathcal{C}$ is a subcategory of $\mathcal{G}$ which generates $\mathcal{G}$ and which contains no nontrivial invertible elements.
- $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ is a *Garside map* [DDGKM, Definition V.2.19], such that the set $\mathcal{S}$ of divisors of Δ is finite.

Groups of the form $\mathcal{G}(u, u)$, where $(\mathcal{G}, \mathcal{C}, \Delta)$ is a Garside groupoid, and where $u \in \mathrm{Ob}(\mathcal{G})$, are called *weak Garside groups*.

In the case of Garside groups, (standard) parabolic subgroups have been defined by Godelle in [God07]. In [GM22], González-Meneses and Marin provide general arguments allowing to prove (under suitable assumptions) that arbitrary elements in Garside groups are contained in a smallest parabolic subgroup, and even that parabolic subgroups are stable under intersection. The starting point of their proof is to consider the smallest standard parabolic subgroup containing a given element. Such a group always exist since standard parabolic subgroups of Garside groups are always stable under intersection.

A concept of standard parabolic subgroupoid of a Garside groupoid has also been introduced by Godelle in [God10], but not studied in depth. This concept of standard parabolic subgroupoid induces in turn a concept of (standard) parabolic subgroup of a weak Garside group. However, contrary to the case of Garside groups, standard parabolic subgroupoids of a Garside groupoid are no longer stable under intersection. In order to mimick the arguments of [GM22], we choose to restrict ourselves to particular families of standard parabolic subgroupoids, that we call *shoals*. A shoal $\mathcal{T}$ for a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ should in particular be stable under intersection. Fixing a shoal $\mathcal{T}$, we can define $\mathcal{T}$-(standard) parabolic subgroups in the groupoid $\mathcal{G}$. In the special case of a Garside group (G, M, Δ), the set of all standard parabolic subgroups of G is a shoal.

Fixing a shoal $\mathcal{T}$ for a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, we can define $\mathcal{T}$-standard parabolic closures of an endomorphism x in $\mathcal{G}$ by simply intersecting all the $\mathcal{T}$-standard parabolic subgroups containing x. Starting from here, we mimick the definition of support-preservingness given in [GM22] (see Definition 5.2.8). We obtain the following result:

Theorem (Existence of $\mathcal{T}$-parabolic closure). *(Theorem 5.2.17 and Corollary 5.2.18) Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $\mathcal{T}$ be a support-preserving shoal for $\mathcal{G}$. Every endomorphism x in $\mathcal{G}$ admits a $\mathcal{T}$-parabolic closure* $\mathrm{PC}(x)$*. Moreover, if m is a nonzero integer, then we have* $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ *for all endomorphisms x in $\mathcal{G}$.*

Keep in mind that shoals are an additional structure on a Garside groupoid. When studying the parabolic subgroups of $B(G_{31})$ through its associated Springer groupoid, an important part will be to construct a shoal accounting for the topological construction of the parabolic subgroups of $B(G_{31})$.

Dual braid monoids and regular braids

Support-preservingness of a shoal is an important property which is hard to show in practice. In the case of Springer groupoids (in particular, the one attached to $B(G_{31})$), this property will come from the support-preservingness of another shoal, which is the shoal of all parabolic subgroups of the dual group attached to a well-generated complex reflection group. In this case, support-preservingness is ensured by the following Theorem (obtained in a case-by-case fashion), which establishes a stronger result about conjugating elements in dual groups.

Theorem (Parabolic subgroups in dual groups). *(Theorem 8.4.1 and Corollary 8.4.2) Let W be a well-generated complex reflection group, and let $(G(W), M(W), \Delta)$ be the dual group of type W. Let also $x \in M(W)$ have standard parabolic closure $\mathrm{SPC}(x) = G(W)_s$ for some simple element s. If a is an atom of $M(W)$, then we either have*

- *a is a left-divisor of $\overline{s}$ in $M(W)$, and $\rho_a(x) = a$ is a minimal positive conjugator.*
- *a is a left-divisor of s in $M(W)$, and $\rho_a(x) \in M(W)_s$.*
- *None of the above, and $\rho_a(x)$ is not a minimal positive conjugator.*

In particular, the shoal of all standard parabolic sugbroups of $G(W)$ is support-preserving.

Using the results of [GM22, Section 4.6 and Section 6.1], we deduce the following corollary:

Corollary. *(Theorem 8.4.5) Let W be a well-generated complex reflection group. Parabolic subgroups of $B(W)$ are stable under intersections.*

Note that this was already known by results of Cumplido, Gebhardt, González-Meneses, Wiest [CGGW19] and González-Meneses, Marin [GM22]. However, the above Theorem is not a consequence of the results in [CGGW19] and [GM22]. Indeed, they show support-preservingness of other Garside structures on well-generated groups (except for $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$).

In [Bes15], Bessis studies the complex braid group $B(G_{31})$ by endowing it with a Garside groupoid, which we call the Springer groupoid. The starting point of this construction is the theory of Springer regular elements in complex reflection groups [Spr74]. A *regular element* in a complex reflection group W for an eigenvalue ζ is an element g of W which admits a ζ-eigenvectors outside of the reflecting hyperplanes of W. The centralizer W_g of such an element acts as a complex reflection group on the ζ-eigenspace of g. Up to taking powers, we can restrict our attention to the case where the eigenvalue has the form $\zeta_d := \exp(\frac{2i\pi}{d})$ for $d \geqslant 1$. We then call g a d-regular element, and we say that d is a regular integer for W.

One of the themes in Bessis works was to give a “lift” of Springer’s theory of regular elements in complex braid groups, answering questions of Broué and Michel [Bes00] [BDM02]. In [Bes15, Section 12], Bessis provides such a lift in the case of well-generated irreducible complex reflection groups. This lift relies on the notion of *regular braids*.

Let W be a complex reflection group, and let $\pi \in Z(P(W))$ be the element mentioned earlier. For $d \geqslant 1$, an element $\rho \in B(W)$ is a d-regular braid if $\rho^d = \pi$ in $B(W)$. It is easy to show that every ζ_d-regular element in W is the image of a d-regular braid under the projection map $B(W) \twoheadrightarrow W$. The following theorem proves that regular braids are a good analogue in complex braid groups of regular elements in complex reflection groups.

Theorem (Springer theory in braid groups). *(Theorem 9.1.7)*
Let W be an irreducible complex braid group, and let d be a positive integer.

(a) The integer d is regular for W in the sense of Springer if and only if there are d-regular braids in $B(W)$.

(b) When d is regular, d-regular braids are all conjugate in $B(W)$, and a d-regular braid is mapped to a ζ_d-regular element in W.

(c) Let d be a regular number, and let ρ be a d-regular braid in $B(W)$. Let g be the image of ρ under the projection map $B(W) \to W$. The complex braid group $B(W_g)$ is isomorphic to the centralizer $C_{B(W)}(\rho)$. In other words, the centralizer of a regular braid is the braid group of the centralizer of a regular element.

Again, Bessis proved this result in the case where W is well-generated. We completed the proof by studying the other irreducible complex reflection groups, in particular G_{31}.

In the case where W is well-generated, we can consider the dual structure $(G(W), M(W), \Delta)$ on $B(W)$. The element $\pi \in B(W)$ then appears as a power of the Garside element Δ. A regular braid is then a particular case of periodic element in the Garside group $(G(W), M(W), \Delta)$ (i.e. a root of some power of Δ). Bessis introduced in [Bes07] a general construction of a Garside groupoid equivalent to the centralizer of a periodic element in a Garside group. Applying this construction to a d-regular braid then yields a Garside groupoid, which is equivalent to the centralizer of a regular braid in $B(W)$. By the above theorem, this centralizer is isomorphic to the braid group of the centralizer of a ζ_d-regular element in W. This Garside groupoid is then called the Springer groupoid (attached to W and to the integer d).

Let W be a well-generated complex reflection group, and d be a regular integer for W. Let also $(\mathcal{G}, \mathcal{C}, \Delta)$ be the associated Springer groupoid. By construction, $\mathcal{G}$ is endowed with a functor $p : \mathcal{G} \to G(W)$ such that, for all $u \in \mathrm{Ob}(\mathcal{G})$, p identifies $\mathcal{G}(u, u)$ with the centralizer in $G(W)$ of a regular braid $\rho(u)$.

The construction of $\mathcal{G}$ starting from the dual group $(G(W), M(W), \Delta)$ allows us to define a particular shoal $\mathcal{T}$ of standard parabolic subgroupoids of $\mathcal{G}$, starting from the shoal of all standard parabolic subgroups of $G(W)$. The theorem on the minimal positive conjugators in $(G(W), M(W), \Delta)$ then allows us to show that this shoal is support-preserving. The next task is to show that the shoal we constructed for Springer groupoids actually gives the same parabolic subgroups as the topological parabolic subgroup defined for the braid group of the centralizer of a regular element. This requires to follow the topological description made by Bessis of Springer groupoids, and to introduce the concept of local fundamental group in this context. We finally obtain the following theorem:

Theorem. *(Section 9.2.4 and Theorem 9.2.42)*
Let W be a well-generated complex reflection group, and let d be a regular integer for W. Let also $(G(W), M(W), \Delta)$ be the dual group of type W, and let $(\mathcal{G}, \mathcal{C}, \Delta)$ be the Springer groupoid associated to W and d. There is a support-preserving shoal $\mathcal{T}$ on $\mathcal{G}$ such that, for $u \in \mathrm{Ob}(\mathcal{G})$, we have

(a) The $\mathcal{T}$-parabolic subgroups of $\mathcal{G}(u, u) \simeq C_{G(W)}(\rho(u))$ are exactly the intersections with $C_{G(W)}(\rho(u))$ of the parabolic subgroups of $G(W)$ which are normalized by $\rho(u)$.

(b) Two parabolic subgroups of $G(W)$ which are normalized by $\rho(u)$ are equal if and only their intersections with $C_{G(W)}(\rho(u))$ are equal.

(c) Let g be the image of $\rho(u)$ under the projection map $G(W) \simeq B(W) \to W$. The isomorophism $B(W_g) \simeq C_{G(W)}(\rho(u))$ identifies the parabolic subgroups of the former with the

$\mathcal{T}$-parabolic subgroups of the latter.

This theorem gives a complete description of the parabolic subgroups of centralizers of regular braids in braid groups of well-generated complex reflection groups. The following result was conjectured by González-Meneses and Marin in the case where W is well-generated. In this case, it is a consequence of the above theorem, and we were able to also prove it for other irreducible complex reflection groups.

Theorem (Parabolic subgroups of regular centralizers). *(Theorem 9.3.1)*
Let W be an irreducible complex reflection group, and let d be a regular integer for W. Let $\rho \in B(W)$ be a d-regular braid, and let g be its image in W.

(a) The isomorphism $B(W_g) \simeq C_{B(W)}(\rho)$ identifies the parabolic subgroups of $B(W_g)$ with the intersections with $C_{B(W)}(\rho)$ of the parabolic subgroups of $B(W)$ which are normalized by ρ.

(b) Two parabolic subgroups of $B(W)$ which are normalized by ρ are equal if and only if their intersections with $C_{B(W)}(\rho)$ are equal.

Let W be a complex reflection group, and let $g \in W$ be a regular element. If we already know that the parabolic subgroups of $B(W)$ are stable under intersection, then the above theorem allows us to easily deduce that the parabolic subgroups of $B(W_g)$ are stable under intersection. In particular we have the following corollary:

Corollary. *Let W be a well-generated complex reflection group, and let $g \in W$ be a regular element. The parabolic subgroups of $B(W_g)$ are stable under intersection.*

Note that this corollary covers in particular the case of $B(G_{31})$. Combining this result with the well-generated case, we obtain that parabolic subgroups of irreducible complex braid groups are stable under intersection for all cases but G_{12}, G_{13} and groups of the form $G(de, e, n)$ for $d, e, n > 2$ (note that the later groups can be studied as finite index subgroups in groups of the form $G(r, 1, n)$, which are well-generated).

Results on the complex braid group $B(G_{31})$

The complex reflection group G_{37} (which is a complexified Coxeter group of type E_8) admits 4 as a regular integer. The centralizer of a 4-regular element in G_{37} is a complex reflection group of type G_{31}. Since G_{37} is well-generated as a complexified real reflection group, we can consider the Springer groupoid $(\mathcal{B}_{31}, \mathcal{C}_{31}, \Delta)$ attached to G_{37} and to the regular integer 4.

As we mentioned in the last section, for $u \in \mathrm{Ob}(\mathcal{B}_{31})$, the natural functor $\mathcal{B}_{31} \to B(G_{37})$ identifies $\mathcal{B}_{31}(u, u)$ with the centralizer in $B(G_{37})$ of a regular braid $\rho(u)$. This centralizer in turn is isomorphic to the complex braid group $B(G_{31})$.

Recall that the *center* of a category $\mathcal{C}$ is defined as the set of natural endomorphisms of the identify functor of $\mathcal{C}$. In a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, the Garside map Δ can be seen as a natural transformation from the identify functor of $\mathcal{C}$ to a particular automorphism of $\mathcal{C}$. This automorphism has finite order, thus some power of the Garside map Δ actually lies in the center of $\mathcal{C}$, which is included in the center of $\mathcal{G}$. In the case of the Springer groupoid $\mathcal{B}_{31}$, this completely describes the center, and we have the following result (the second and third statement appears in [DMM11, Theorem 1.4], but with a different proof):

Theorem. *(Theorem 10.0.1) The centers of $\mathcal{C}_{31}$ and $\mathcal{B}_{31}$ are cyclic and generated by Δ^{15}. Moreover, if U is a finite index subgroup of $B(G_{31})$, then $Z(U) \subset Z(B(G_{31}))$. In particular, the* *center of the pure braid group $P(G_{31})$ is cyclic and we have a short exact sequence*

$$1 \to P(G_{31}) \to B(G_{31}) \to G_{31} \to 1.$$

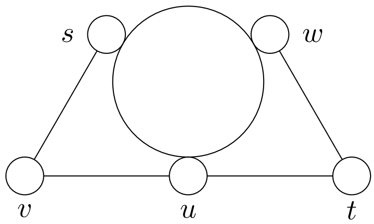
As we said above, the second and third statements were already proven in [DMM11], but our proof is Garside-theoretic, and relies on the conjugacy of braided reflections of $B(G_{31})$ seen in the Springer groupoid $\mathcal{B}_{31}$.

In order to study presentations of $B(G_{31})$, we follow the idea of [Bes15, Remark 11.29], and we aim to deduce a presentation of $B(G_{31})$ starting from a presentation of $\mathcal{B}_{31}$. This is possible using a generalized version of the Reidemeister-Schreier method that we introduce. Starting from an object $u \in \mathrm{Ob}(\mathcal{B}_{31})$, we can on the one hand "guess a presentation" where the generators are the braided reflections in $\mathcal{C}_{31}(u,u)$ (atomic loops), and on the other hand compute a presentation of $\mathcal{B}_{31}(u,u)$ using the Reidemeister-Schreier method for groupoids. This last presentation is very redundant in practice, and we simplify it using word-reversing. Each time, we obtain that the guessed presentation was indeed a presentation of $B(G_{31})$. We then obtain several presentations of $B(G_{31})$, among which is the presentation induced by the BMR-diagram of $B(G_{31})$:

Theorem. *(Theorem 10.1.1 and Section 10.1.3)*
The group $B(G_{31})$ admits the following presentation

$$B(G_{31}) = \left\langle s,t,u,v,w \;\middle|\; \begin{array}{l} st = ts,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw \end{array} \right\rangle,$$

where the generators are (distinguished) braided reflections. This presentation is the one induced by the BMR-diagram of $B(G_{31})$



This theorem finally closes the search for presentations of irreducible complex braid groups initiated in [BMR98].

Concerning parabolic subgroups of $B(G_{31})$, since G_{37} is well-generated, we can apply the results in the above section to obtain

Theorem. *(Corollary 9.3.2) The parabolic subgroups of the complex braid group $B(G_{31})$ are stable under intersection.*

This extends [GM22, Theorem 1.1 and Theorem 1.2] to the case of $B(G_{31})$. Since $B(G_{31})$ was the only case not covered in [GM22], this completes the proof of [GM22, Theorem 1.1 and Theorem 1.2] for all irreducible complex reflection groups. In order to complete the proof of all the main results of [GM22], it remains to show [GM22, Theorem 1.3] in the case of $B(G_{31})$, which is again the only case not covered in [GM22].

Let $B_0 \subset B(G_{31})$ be an irreducible parabolic subgroup (i.e. the associated parabolic subgroup of G_{31} is irreducible). We know that the center of B_0 is infinite cyclic, and we can consider the unique generator z_{B_0} which is positive in the generators s, t, u, v, w of $B(G_{31})$. First, we have that conjugacy of irreducible parabolic subgroups of $B(G_{31})$ are completely characterized by the elements z_{B_0}.

Theorem. *(Section 10.2.2)*
Let $B_0 \subset B(G_{31})$ be an irreducible parabolic subgroup. The smallest parabolic subgroup of $B(G_{31})$ containing z_{B_0} is B_0. Let $B_1, B_2 \subset B(G_{31})$ be two irreducible parabolic subgroups. An element $b \in B(G_{31})$ conjugates B_1 to B_2 if and only if $z_{B_1}^b = z_{B_2}$.

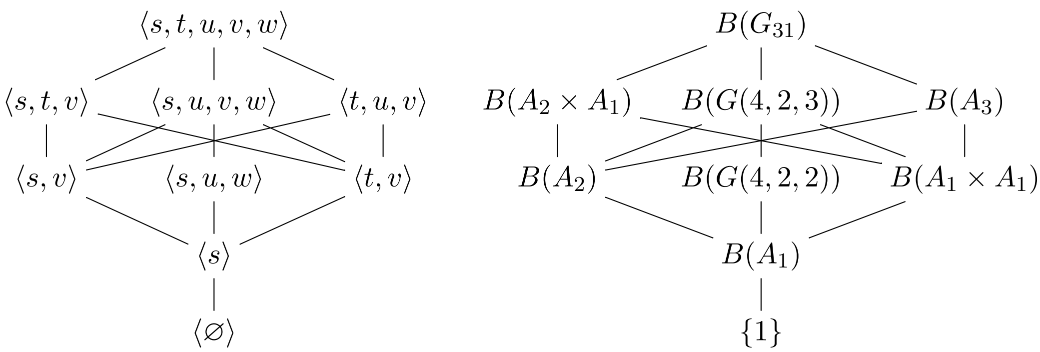
Following [GM22], the *curve graph* of $B(G_{31})$ is defined as the graph whose vertices are the irreducible parabolic subgroups of $B(G_{31})$, and where two vertices B_1, B_2 are adjacent if and only if they are distinct, and if we either have $B_1 \subset B_2, B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = 1$. We have the following characterization of adjacency in the curve graph of $B(G_{31})$:

Theorem. *(Theorem 10.2.8)*
Two irreducible parabolic subgroups $B_1, B_2 \subset B(G_{31})$ are adjacent in the curve graph of $B(G_{31})$ if and only if z_{B_1} and z_{B_2} are distinct and commute.

This extends [GM22, Theorem 1.3] to the case of $B(G_{31})$. Since $B(G_{31})$ was the only case not covered in [GM22], this completes the proof of [GM22, Theorem 1.3] for all irreducible complex reflection groups. We then have that the main results of [GM22] hold for every irreducible complex braid group.

Moreover, we give a description of the parabolic subgroups of $B(G_{31})$ up to conjugacy. Consider the generators s, t, u, v, w of the presentation of $B(G_{31})$ obtained using its BMR-diagram. Let us denote by $\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}$ the images of s, t, u, v, w in the reflection group G_{31}. The elements $\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}$ generate G_{31}, and a presentation is obtained from the BMR-diagram of G_{31}. It is known that every parabolic subgroup of G_{31} is, up to conjugacy, generated by some subset S of $\{\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}\}$. Moreover, the BMR-diagram of such a parabolic subgroup is obtained by taking the subdiagram of the BMR-diagram of $B(G_{31})$ whose nodes are the elements of S. The following theorem states that the situation is the same at the level of the braid group $B(G_{31})$:

Theorem. *(Theorem 10.2.1) The lattice of parabolic subgroups of $B(G_{31})$ up to conjugacy is given by*



The lattice on the right, where A_n denotes the complex reflection group $G(1,1,n+1)$, describes the isomorphism type of the parabolic subgroups given on the left. For each such parabolic subgroup $\langle S \rangle \subset B(G_{31})$, a presentation is obtained by taking the subdiagram of the BMR-diagram of $B(G_{31})$ whose nodes are the elements of S.

In Appendix A, we are able to compute the homology of $B(G_{31})$ starting from that of the Springer groupoid $\mathcal{B}_{31}$. We make explicit computations of $H_*(B(G_{31}), M)$ for the following three

$B(G_{31})$-modules M:

- $M = \mathbb{Z}$ with trivial action of $B(G_{31})$.
- $M = \mathbb{Z}$ where the braided reflections of $B(G_{31})$ act by multiplication by -1. We denote this module by $\mathbb{Z}_\varepsilon$.
- $M = \mathbb{Q}[t, t^{-1}]$, where the braided reflections of $B(G_{31})$ act by multiplication by t.

The results are in the following table (see Table A.3, Table A.4, Table A.5). For $M = \mathbb{Q}[t, t^{-1}]$, $\mathbb{Q}$ denotes the quotient $\mathbb{Q}[t, t^{-1}]/(t-1)$, and a polynomial $P(t)$ stands for the quotient $\mathbb{Q}[t, t^{-1}]/(P)$ (Φ_n denotes the n-th cyclotomic polynomial).

$B(G_{31})$	H_0	H_1	H_2	H_3	H_4
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}_\varepsilon$	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_{20}$	0
$\mathbb{Q}[t, t^{-1}]$	$\mathbb{Q}$	0	Φ_6	$\frac{t^{10}-1}{t+1}\Phi_{15}$	0

Perspectives and conjectures

The results we presented above offer several interesting perspectives to follow.

Concerning Garside theory, the concept of shoal we introduce takes into account the intersection of standard parabolic subgroupoids. In [God10], Godelle introduced a concept of ribbon between two standard parabolic subgroupoids in a Garside groupoid. These ribbons appear as “elementary conjugators” of standard parabolic subgroupoids. Godelle then showed that, in the case of a Garside groups (under suitable assumptions), ribbons form a Garside groupoid, which he then uses to study the conjugacy of standard parabolic subgroups. Upgrading the definition of shoal in order to take ribbons into account could pave the way to a similar study of parabolic subgroups in weak Garside groups.

Now on complex braid groups. In [Shv96], Shvartsman studies quotient groups of the form $B(W)/Z(B(W))$. He proves that, in the case where W is a real reflection group, this group appears as the fundamental group of a projective variety naturally associated with W. He then conjectures that this also holds for general complex braid groups. The general understanding we now have of the center $Z(B(W))$, along with our understanding of regular braids, could be sufficient to prove this conjecture. This idea is already present in [BMR98, Proposition 2.23], but the proof is flawed, as pointed out in [DMM11] (the bottom sequence is not exact, and the bottom-right square is not commutative).

The presentations of $B(G_{31})$ we obtain through the Reidemeister-Schreier method for groupoids are quite big, and the reduction of these presentations is done using a lot of direct computations. Even though finding a completely conceptual proof of a presentation of $B(G_{31})$ seems unreasonable (for other exceptional braid groups, like G_{33} or G_{34}, the known proofs are also direct computations using the dual braid monoid), it would be interesting to have a proof relying on less tedious computations. For instance a good choice of Schreier transversal rooted in $u \in \mathrm{Ob}(\mathcal{B}_{31})$ could provide a much less voluminous presentation of $\mathcal{B}_{31}(u, u)$, or at least a presentation that could be simplified more easily.

Moreover, it could be interesting to see whether or not the other presentations of $B(G_{31})$ we obtain also allow for a convenient description of parabolic subgroups up to conjugacy, in the spirit of Theorem 10.2.1.

More generally, the work of González-Meneses and Marin in [GM22], completed by our work on $B(G_{31})$, provides what seems to be a good generalization of the usual curve complex for arbitrary complex braid groups. It is then natural to try and show that this generalization satisfies the same properties as the original curve complex. For instance, we mentioned that the usual curve complex is hyperbolic in the sense of Gromov, and we do not know yet if this property also holds in the case of complex braid groups (the case of spherical Artin groups is not yet known either). Moreover, the interpretation of the usual braid group as a mapping class group allows one to define the Nielsen-Thurston classification of braids. The general definition of parabolic subgroups may allow for an extension of the Nielsen-Thurston classification to arbitrary complex braid groups, or at least of the notion of reducible braid.

In the case of parabolic subgroups of spherical Artin groups, a question raised by Marin was to know whether or not the embedding of parabolic subgroups merges together conjugacy classes (the conjugacy stability problem). A complete description of this phenomenon was given by Calvez, Cisneros and Cumplido [CCC20], notably using Garside theory. It would be interesting to see if their methods can be generalized to arbitrary complex braid groups, and to give a complete description of conjugacy stability in this case.

In Appendix B, we introduce a particular class of Garside groups defined by generators and relations, that we call *circular groups*. This class contains in particular the complex braid groups of rank 2. By construction, a complex reflection group of rank 2 can be recovered from its braid group (seen as a circular group) by adding torsion relations to the generators. The inverse question is to ask when does adding torsion relations to the generators of a circular group produce a finite group? And, when this is the case, is it always a complex reflection group? The work of Achar and Aubert [AA08] gives in particular a positive answer to this question in the case of the circular group

$$G(3,3) := \langle a, b, c \mid abc = bca = cab \rangle.$$

The work of Gobet [Gob24] also gives a positive answer is many cases. In a joint work in progress with I. Haladjian, we plan on describing exactly which quotients of circular groups by torsion relations on the generators are finite. We conjecture that no new finite quotients arise apart from the ones described in [AA08] and [Gob24].

Reader's guide

This thesis is split in two main parts (plus appendix). The first part is centered around general themes in Garside theory, while the second focuses on applications of Garside theory in the study of complex braid groups. Our hope is that this splitting will allow readers familiar with one part or the other to navigate easily between what they want to know and what they already know. We included many examples in order to illustrate the various constructions and results we discuss.

At the beginning of each section, we give a complete list of the objects we consider "by default" in the section, so as not to recall them at the beginning of each result. However, in the case of theorems, we still give the complete list of assumptions without implicitly referring to the beginning of the section.

As we said, the first part is devoted to Garside categories and Garside groupoids. It may be considered as a short general self-contained reference of Garside theory, in a context much more narrow than that of [DDGKM]. Note however that this context is sufficient for applications concerning complex braid groups, and it allows for a much shorter exposition.

We start in the first chapter by introducing some general definitions and constructions about categories and groupoids. In particular this is where we introduce the notion of categorical presentation, which is central in Garside theory. We also detail a general method suitable for computing presentations of groups, given a presentation of an equivalent groupoid. We call this method the *Reidemeister-Schreier method for groupoids* (it generalizes the classical Reidemeister-Schreier method). In the second chapter, we present the notions of Garside category and Garside groupoid, following mostly [DDGKM], adapted to our narrower context.

Chapter 3 presents the usual tools for studying conjugacy in Garside groupoids (conjugacy category, super summit-sets...). Along these classical definitions (taken from [DDGKM]), we adapt the notion of swap, recently defined for Garside groups by González-Meneses and Marin in [GM22], to the categorical context. We study in particular the behavior of the swap when applied to powers, allowing us to prove that powers of recurrent elements for swaps are again recurrent.

Chapter 4 can be seen as a toolbox for constructing new Garside groupoids starting from known ones. Most of the constructions we present there already appear in the literature (except the one in Section 4.2), but often with less details than here. For instance, the concept of conjugacy graphs originates from [FG03a], and the idea of endowing conjugacy categories with Garside structures appears in [DDGKM], but the idea of endowing conjugacy graphs themselves with a Garside structure is new. Likewise, the construction of the germ of periodic elements in Definition 4.5.7 is new, as well as the results we obtain on conjugacy classes of periodic elements.

We close the first part by giving in Chapter 5 an in-depth study of standard parabolic subgroupoids in Garside groupoids (the concept itself is not new and was introduced by Godelle). We try and construct smallest parabolic subgroups in Garside groupoids containing a given element. The situation is quite different than the more classical situation of Garside groups, which leads us to introduce the concept of shoal of standard parabolic subgroupoids, as a way to manage intersections of standard parabolic subgroupoids in a general Garside groupoid. This concept provides us with a framework suitable for adapting the results of González-Meneses and Marin on the intersection of arbitrary parabolic subgroups in a Garside group. We then provide various constructions of shoals adapted to the various constructions of Garside groupoids detailed in Chapter 4. In particular, the last section of Chapter 5 can go hand in hand with Chapter 4 for the reader interested in studying standard parabolic subgroupoids.

We begin the second part on complex reflection groups and their braid groups by recalling well-known facts about them in Chapter 6. In particular, we recall the Shephard-Todd classification of irreducible complex reflection groups, between the infinite series of monomial matrix groups on the one hand, and the 34 exceptional cases (labeled $G_4, \ldots, G_{37}$) on the other hand. We also recall the dichotomy between well-generated and badly-generated complex reflection groups, which is important in the sequel.

A crucial theme in the second part is that of parabolic subgroups of complex braid groups. Although we study them through a Garside-theoretic lens, they are defined topologically as a particular case of a general concept of local fundamental group (see [GM22]). In Chapter 7, we provide a slight extension of the concept of local fundamental group defined in [GM22], and we prove that this extension behaves well with respect to certain covering maps. We also recall the first results of [GM22] on parabolic subgroups of complex braid groups.

In Chapter 8, we study the dual braid monoid associated to a well-generated complex reflection group. This Garside monoid introduced by Bessis then allows us to study the braid groups of such a reflection group. While this is not new, we also study parabolic subgroups of well-generated complex braid groups using this monoid. We show in particular that dual braid monoids are support-preserving in the sense of [GM22], allowing us to completely describe parabolic subgroups of the associated braid groups, and giving new proofs of results of González-Meneses and Marin regarding parabolic subgroups in this case. In order to prove this result in general, we use the correspondence between simple elements in dual braid monoids and noncrossing partition lattices, in the case of well-generated groups in the infinite series.

Chapter 9 is devoted to the study of regular braids and their centralizers in complex braid groups. We start by explaining how this theory provides a lift in complex braid groups of Springer theory of regular elements. In the case of a regular braid ρ inside of a well-generated complex braid group $B(W)$, the centralizer of ρ was shown by Bessis to be equivalent (as a groupoid) to a Garside groupoid attached to the dual structure on $B(W)$. We provide a systematic study of such groupoids, that we call *Springer groupoids*. We show how they provide a good Garside-theoretic framework for studying regular centralizers in braid groups of well-generated complex reflection groups. In particular, these groupoids come equipped with a shoal of standard parabolic subgroupoids, which is well-suited for studying the (topological) parabolic subgroups of regular centralizers in well-generated groups. This allows us to give a complete description (for all complex braid groups) of the parabolic subgroups of the centralizer of a regular braid in terms of parabolic subgroups of the ambient group.

Lastly, we apply the results obtained in Chapter 9 to the complex braid group $B(G_{31})$ in Chapter 10. We use the Reidemeister-Schreier method for groupoids to compute particular presentations of this groups using its associated Springer groupoid, as in [Gar23b]. Then we describe its parabolic subgroups up to conjugacy, and we show that [GM22, Theorem 1.3] holds also for this group. This was the last result needed to show that the main theorems of [GM22] hold for every complex braid group.

The first appendix mainly consists in a generalization of a work of Dehornoy, Lafont [DL03] providing an efficient way to compute the homology of a Garside group (Gaussian group). We adapt this construction to Garside groupoids, and we show how to deduce results on the homology of weak Garside groups. We then apply this construction to compute the homology of exceptional complex braid groups with various coefficients. In particular, we obtain new results for the complex braid group $B(G_{31})$.

The second appendix studies a particular class of Garside groups, which contains in particular complex braid groups of rank 2, and which we call circular groups. We mainly prove that roots are unique in these groups, up to conjugacy. We also consider a generalization of circular groups, called hosohedral-type groups. These groups are defined using circular groups, and a procedure called the Δ-product, which we study in generality. We are also able to describe (Garside-theoretic) parabolic subgroups of these groups, and to show that they are stable under intersection.

For reader convenience, we include a diagram showing the interdependencies among the chapters and appendices:

$$
\begin{tikzcd}
 & 1 \arrow[d, no head] & & \\
 & 2 \arrow[d, no head] \arrow[ddr, no head] & & \\
 & 3 \arrow[dl, no head] \arrow[d, no head] & & 6 \arrow[dl, no head] \arrow[d, no head] \\
B & 4,5 \arrow[dr, no head] & A & 7 \arrow[dl, no head] \\
 & & 8 \arrow[d, no head] & \\
 & & 9 \arrow[d, no head] & \\
 & & 10 &
\end{tikzcd}
$$

Notations and conventions

i) All categories considered here are assumed to be small.

ii) Let $\mathcal{C}$ be a category. We consider $\mathcal{C}$ as the set of its morphisms. That is, the statement $f \in \mathcal{C}$ expresses that f is a morphism between two objects of $\mathcal{C}$.

iii) Let $\mathcal{C}$ be a category, and consider a diagram in $\mathcal{C}$ of the following form:

$$u \xrightarrow{f} v \xrightarrow{g} w.$$

The composition of this diagram will be denoted by fg. Alternatively, we will call fg the composition or the product of f and g.

iv) We compose permutations like maps. For instance the product $(1\ 2)(2\ 3)$ in the group $\mathfrak{S}_3$ is given by $(1\ 2\ 3)$.

v) Let E be a topological space, and let $\gamma_1, \gamma_2 : [0,1] \to E$ be two paths in E, such that $\gamma_1(1) = \gamma_2(0)$. The concatenation $\gamma_1 * \gamma_2$ will denote the path

$$t \mapsto \begin{cases} \gamma_1(2t) & \text{if } t \leqslant 1/2, \\ \gamma_2(2t-1) & \text{if } t \geqslant 1/2. \end{cases}$$

That is "γ_1 and then γ_2".

Part I

Garside theory

Chapter 1

Generalities about categories and groupoids

In this first chapter, we introduce some general definitions and constructions about categories and groupoids which will be useful throughout this thesis when handling Garside categories. We detail in particular the notion of categorical presentation, which is central in Garside theory. The first three sections in this chapter are standard and contain no new material. We give in Section 1.4 a general groupoid construction which will later be used to construct important examples of Garside groupoids.

In Section 1.5, we detail a general method suitable for computing presentations of groups, given a presentation of an equivalent groupoid. This method generalizes the classical Reidemeister-Schreier method.

Contents

The first part of this thesis is organized around the notion of Garside groupoid. Before getting to this notion, we introduce in this first chapter the general categorical framework we will use. As in [DDGKM], we use categories mostly as a generalization of monoids in which the product needs not be defined everywhere. In particular, we will not use more advanced category theory, only the language of categories.

In the first section, we consider divisibility relations in a category, along with the main combinatorial assumptions we can make on a category regarding these relations (cancellativity, Noetherianity). In the second section, we give the definition of a categorical presentation, a concept which will be central for studying Garside categories (which can be though of as "categories equipped with some nice presentations"). Categorical presentation are also useful for defining (abstractly) the enveloping groupoid of a category. In the third section, we give a classical condition under which the enveloping groupoid of a category can be conveniently described as a groupoid of left-fractions.

In the last section, we propose a general combinatorial method for deducing group presentations from groupoid presentations. This method is reminiscent of the classical Reidemeister-Schreier method, when we apply it to the so-called groupoid of cosets, associated to a finite index subgroup of a presented group. As the construction of groupoid of cosets will be important later on, we detail it in the fourth section.

1.1 Divisibility in a category

One of the key points in the definition of a Garside category is the lattice condition on the divisibility partial order. We introduce in this short section the main terminology we will use in the next chapters. The definitions are taken from [DDGKM].

Let $\mathcal{C}$ be a category, and let $u, v \in \mathrm{Ob}(\mathcal{C})$. The set of morphisms in $\mathcal{C}$ from u to v (resp. with source u, with target v) will be denoted by $\mathcal{C}(u, v)$ (resp. $\mathcal{C}(u, -)$, $\mathcal{C}(-, v)$).

Definition 1.1.1 (Divisibility). [DDGKM, Definition II.2.1 and Definition II.2.5]
Let $\mathcal{C}$ be a category, and let $u \in \mathrm{Ob}(\mathcal{C})$.

- For $f, g \in \mathcal{C}(u, -)$, we say that f *left-divides* g, or, equivalently, that g is a *right-multiple* of f, written $f \preccurlyeq g$, if there exists h in $\mathcal{C}$ satisfying $fh = g$.
- For $f, g \in \mathcal{C}(-, u)$, we say that f *right-divides* g, or, equivalently, that g is a *left-multiple* of f, written $g \succcurlyeq f$, if there exists h in $\mathcal{C}$ satisfying $g = hf$.

Note in particular that $f \preccurlyeq g$ is not equivalent to $g \succcurlyeq f$. These two notions admit a "strict" version. For instance we say that f *strictly* left-divides g, written $f \prec g$ if $f \preccurlyeq g$ and $f \neq g$. Likewise, we denote strict right-divisibility by $\succ$. Another strict version is introduced in [DDGKM, Definition II.2.24] taking into account the possible existence of non invertible morphisms. Since we will work under the assumption that no nontrivial invertible morphism exist, we will not need this level of generality.

Note that left-divisibility is invariant under left-multiplication. In other words, $f \preccurlyeq g$ implies $hf \preccurlyeq hg$ for all $h \in \mathcal{C}$ whose target is the common source of f and g [DDGKM, Lemma II.2.4]. Symmetrically, right-divisibility is invariant under right-multiplication.

Let $\mathcal{G}$ be a groupoid, and let $u \in \mathrm{Ob}(\mathcal{G})$. For all $f, g \in \mathcal{G}(u, -)$, we have $f(f^{-1}g) = g$ and thus $f \preccurlyeq g$. Likewise, we have $g \succcurlyeq f$ for all $f, g \in \mathcal{G}(-, u)$. The relations $\preccurlyeq$ and $\succcurlyeq$ are then not very informative in a groupoid. More generally, let $\mathcal{C}$ be a category, $f \in \mathcal{C}(u, v)$ and let $\varphi \in \mathcal{C}(v, -)$ be an isomorphism. We have $f \prec f\varphi \prec f$ and $\preccurlyeq$ is not a partial order on $\mathcal{C}(u, -)$. We will thus often make the assumption that there are no nontrivial isomorphisms in $\mathcal{C}$, that is, the set $\mathcal{C}^{\times}$ of isomorphisms in $\mathcal{C}$ is reduced to $\{1_u\}_{u \in \mathrm{Ob}(\mathcal{C})}$. Another assumption we need to make in order for divisibility to be a partial order is that of cancellativity.

Definition 1.1.2 (Cancellativity). [DDGKM, Definition II.1.14] A category $\mathcal{C}$ is called *left-cancellative* (resp. *right-cancellative*) if every relation $fg = fg'$ (resp. $gf = g'f$) with $f, g, g' \in \mathcal{C}$ implies $g = g'$. A category which is both left- and right-cancellative is called *cancellative*.

In more classical category theory language, saying that a category $\mathcal{C}$ is left-cancellative (resp. right-cancellative) amounts to say that every morphism in $\mathcal{C}$ is an epimorphism (resp. a monomorphism). Since an isomorphism is both an epimorphism and a monomorphism, we see that a groupoid is always cancellative. More generally, a subcategory of a groupoid is always cancellative.

Example 1.1.3. The monoid $M = \langle a, b, c \mid ac = bc\rangle^+$ is not cancellative (for instance $a \neq b$ while $ac = bc$). It is not a submonoid of a group, since every morphism of monoids $\varphi : M \to G$ where G is a group must be so that $\varphi(a) = \varphi(b)$, whereas $a \neq b$ in M.

It is an easy exercise to show that, if $\mathcal{C}$ is a left- (resp. right-)cancellative category with no nontrivial isomorphisms, then for all $u \in \mathrm{Ob}(\mathcal{C})$, the relation $\preccurlyeq$ (resp. $\succcurlyeq$) is a partial order on $\mathcal{C}(u, -)$ (resp. on $(\mathcal{C}(-, u))$.

Definition 1.1.4 (Atom). [DDGKM, Definition II.2.52] Let $\mathcal{C}$ be a category. An element $a \in \mathcal{C}$ is an *atom* if it admits no strict left-divisor (or equivalently, no strict right-divisor).

Note that this definition is less general than [DDGKM, Definition II.2.52]. But it is sufficient for our purpose as we will only consider cancellative categories with no nontrivial isomorphisms in the sequel (in this context, the two definitions coincide).

Let $\mathcal{C}$ be a cancellative category with no nontrivial isomorphisms, and let $u \in \mathrm{Ob}(\mathcal{C})$. By definition, the atoms in $\mathcal{C}(u, -)$ are the $\preccurlyeq$-minimal elements in $\mathcal{C}(u, -) \setminus \{1_u\}$. A number of results in Garside category are proven using inductions on the relations $\preccurlyeq$ and $\succcurlyeq$, in which case the base case will be that of atoms. In order for such inductions to be possible, we need the additional assumption that divisibility is a well-founded relation. A classical condition ensuring this is that of (left- and right-)Noetherianity.

Definition 1.1.5 (Noetherianity). [DDGKM, Definition II.2.26] A category $\mathcal{C}$ is called left-Noetherian (resp. right-Noetherian) if there are no infinite strictly descending chain for $\preccurlyeq$ (resp. for $\succcurlyeq$). A category which is both left- and right-Noetherian will be called *Noetherian*.

Remark 1.1.6. The actual statement in [DDGKM, Definition II.2.26] on left- and right-Noetherianity is slightly broader than the one given here, as it must take into account the possible existence of invertible morphisms. Our definition prevents the existence of invertible morphisms altogether. In the case of a category with no nontrivial morphisms (i.e. the case we will consider throughout this thesis), the two definitions coincide.

Moreover, the definition of Noetherianity given in [DDGKM, Definition II.2.26] is stronger than the one given here, but the two definitions coincide in the case of a cancellative category.

Example 1.1.7. [DDGKM, Section I.3.2] The monoid $K^+ := \langle a, b \mid a = bab\rangle^+$ is neither left-Noetherian nor right-Noetherian. Indeed for $n \geqslant 0$, we have

$$b^{n+1}a \preccurlyeq b^{n+1}ab = b^n a \text{ and } ab^n = bab^{n+1} \succcurlyeq ab^{n+1}.$$

Thus, we have infinite strictly decreasing sequences for both $\preccurlyeq$ and $\succcurlyeq$

The Noetherianity assumption has for first consequence that atoms form a generating set. Moreover, the converse, stating that a generating set of a Noetherian category always contains its atoms, is also true [DDGKM, Corollary II.2.59]. When studying Garside categories, we will be able to use a condition which is stronger than Noetherianity.

Definition 1.1.8 (Strongly Noetherian). [DDGKM, Definition II.2.51] A category $\mathcal{C}$ is called *strongly Noetherian* if, for every $g \in \mathcal{C}$, the length of a decomposition of g as a product of elements of $\mathcal{C}$ without trivial entries admits an upper bound.

Before giving in Example 1.1.10 a monoid which is Noetherian but not strongly Noetherian, we give a characterization of strongly Noetherian categories. This characterization proves in particular that strong Noetherianity implies Noetherianity.

Lemma 1.1.9. *[DDGKM, Corollary II.2.49] A category $\mathcal{C}$ is strongly Noetherian if and only if there exists $\lambda : \mathcal{C} \to \mathbb{N}$ satisfying $\lambda(fg) \geqslant \lambda(f) + \lambda(g)$ and f nontrivial implies $\lambda(f) \geqslant 1$.*

Example 1.1.10. Consider the monoid $M = \langle a, b \mid ab = b\rangle^+$. It is not strongly Noetherian, as we would have $\lambda(b) \geqslant \lambda(a) + \lambda(b)$ thus $\lambda(a) = 0$, which contradicts $a \neq 1$. However, one can show that it is left-Noetherian.

Any element in M can be written in as a product of the form $b^p a^q$ with $p, q \in \mathbb{N}$. Furthermore, such a decomposition is unique. Indeed, if $b^p a^q = b^{p'} a^{q'}$, then $p = p'$ as the relations do not change the number of occurrences of b in a word, and one sees that we also have $q = q'$: the only possible use of a relation would be to either to add or to delete an occurrence of a on the left of an occurrence of b, such transformations cannot change the exponent of a on the right. The product is given by

$$b^p a^q b^{p'} a^{q'} = \begin{cases} b^{p+p'} a^{q'} & \text{if } p' \geqslant 1, \\ b^p a^{q+q'} & \text{if } p' = 0. \end{cases}$$

In particular, the strict left-divisors of $b^p a^q$ are exactly the $b^m a^n$ with either $m < p$ and $n \in \mathbb{N}$, or $m = p$ and $n < q$. Let $(x_n)_{n \in \mathbb{N}}$ be a strictly-$\preccurlyeq$-decreasing sequence in M, and let $x_i = b^{p_i} a^{q_i}$. For $i \in \mathbb{N}$, we either have $p_{i+1} = p_i$ and $q_{i+1} < q_i$, or $p_{i+1} < p_i$ and $q_{i+1} \in \mathbb{N}$. Since p_0 is finite, there can only be a finite number of integers i such that $p_{i+1} < p_i$. Let i_0 be such that $p_{i+1} = p_i$ for $i \geqslant i_0$. The sequence $(q_i)_{i \geqslant i_0}$ is then a strictly decreasing sequence of positive integers, which is impossible.

A particular case of strongly Noetherian categories is given by homogeneous categories.

Definition 1.1.11 (Homogeneous category). Let $\mathcal{C}$ be a category. A *length functor* is a functor $\ell : \mathcal{C} \to (\mathbb{N}, +)$ such that $\mathcal{C}$ is generated by morphisms of positive length. A *homogeneous category* is a pair $(\mathcal{C}, \ell)$, where $\mathcal{C}$ is a category, and ℓ is a length functor on $\mathcal{C}$.

Lemma 1.1.12. *Let $(\mathcal{C}, \ell)$ be a homogeneous category. The category $\mathcal{C}$ is strongly Noetherian and contains no nontrivial isomorphisms.*

Proof. Since ℓ is a functor, we have $\ell(1_u) = 0$ for all $u \in \mathrm{Ob}(\mathcal{C})$. If f is an invertible morphism in $\mathcal{C}$, we have $\ell(f) + \ell(f^{-1}) = 0$. Since ℓ is valued in $\mathbb{N}$, we deduce that $\ell(f) = \ell(f^{-1}) = 0$. Since $\mathcal{C}$ is generated by morphisms of positive length, this implies that both f and f^{-1} are trivial. Furthermore, ℓ respects the assumptions of Lemma 1.1.9, thus $\mathcal{C}$ is strongly Noetherian. □

Homogeneity is a very strong condition, which is often superfluous when compared to Noetherianity. However, most Garside structures involving braid groups will be homogeneous, making homogeneity a convenient condition in practice.

1.2 Presentation of categories and groupoids

As we said earlier, the concept of categorical presentation will be used many times throughout this thesis. We recall basic facts about this concept here, mostly in order to set the terminology and notation for the sequel. We also detail the particular case of presentations arising from germs, which will later be used to construct Garside categories and Garside groupoids.

1.2.1 Oriented graphs, relations

This section follows the exposition of [DDGKM, Section II.1], one could also consult [Bor94, Chapter 5].

Definition 1.2.1 (Oriented graph). [DDGKM, Definition II.1.4 and Definition II.1.32]
An *oriented graph* (or precategory) is a pair of sets $(\mathcal{O}, \mathcal{S})$ endowed with two maps $s, t : \mathcal{S} \to \mathcal{O}$. The elements of $\mathcal{O}$ are called *objects* and those of $\mathcal{S}$ are called *elements* (or morphisms, or arrows). The maps s and t are called *source* and *target*, respectively.

A *morphism* between two oriented graphs $(\mathcal{O}, \mathcal{S})$ and $(\mathcal{O}', \mathcal{S}')$ is given by two maps $\phi_0 : \mathcal{O} \to \mathcal{O}'$, $\phi_1 : \mathcal{S} \to \mathcal{S}'$ which preserve the source and target:

$$\forall f \in \mathcal{S}, s(\phi_1(f)) = \phi_0(s(f)) \text{ and } t(\phi_1(f)) = \phi_0(t(f)).$$

Note that we make no assumption regarding the existence of loops or multiple arrows with the same source and target. In practice, we often amalgamate an oriented graph $(\mathcal{O}, \mathcal{S})$ with its set of arrows $\mathcal{S}$. Like for categories, the set of objects will then be denoted by $\mathrm{Ob}(\mathcal{S})$, and, for $u, v \in \mathrm{Ob}(\mathcal{S})$, we will denote by $\mathcal{S}(u, -)$ (resp. $\mathcal{S}(-, v)$, $\mathcal{S}(u, v)$) the set of arrows in $\mathcal{S}$ with source u (resp. with target v, with source u and target v).

Definition 1.2.2 (Path). [DDGKM, Definition II.1.28]
Let $\mathcal{S}$ be an oriented graph. For $u, v \in \mathrm{Ob}(\mathcal{S})$, a *path* of length $p \geqslant 1$ in $\mathcal{S}$ from u to v is a finite sequence $(g_1, \ldots, g_p)$ of elements of $\mathcal{S}$ such that

$$s(g_1) = u, \ t(g_p) = v \text{ and } t(g_i) = s(g_{i+1}), \forall i \in [\![1, p-1]\!],$$

where $[\![n, m]\!] = \{n, n+1, \ldots, m\}$ for $n, m \in \mathbb{Z}$. For $u \in \mathrm{Ob}(\mathcal{C})$, one also defines an *empty path* from u to itself, denoted by 1_u, and of length 0 by definition.

Remark 1.2.3. An oriented graph is sometimes also called a quiver. Given a field k, the set of paths in an oriented graph $\mathcal{S}$ can be used to construct the *path algebra* of $\mathcal{S}$ over k. Such objects form the basis of the representation theory of quivers (see for instance [ASS06]).

Definition 1.2.4 (Free category). [DDGKM, Definition II.1.28 and Proposition 1.33]
Let $\mathcal{S}$ be an oriented graph. The *free category* on $\mathcal{S}$, denoted by $\mathcal{S}^*$, is defined as follows

- $\mathrm{Ob}(\mathcal{S}^*) := \mathrm{Ob}(\mathcal{S})$.
- For $u, v \in \mathrm{Ob}(\mathcal{S}^*)$, $\mathcal{S}^*(u, v)$ is the set of paths from u to v in $\mathcal{S}$.
- Composition is given by concatenation of paths.
- The identity of some object u is the empty path 1_u.

This category is free in the following sense: Let $\mathcal{S}$ be an oriented graph, and let $\mathcal{C}$ be a category. Any morphism of oriented graphs $\phi : \mathcal{S} \to \mathcal{C}$ induces a unique functor $\mathcal{S}^* \to \mathcal{C}$, sending a path $(g_1, \ldots, g_p)$ to the composition $\phi_1(g_1) \cdots \phi_1(g_p)$ in $\mathcal{C}$. In practice, a path $(f_1, \ldots, f_p)$ will often be denoted as a formal composition $f_1 \cdots f_p$ so that we have

$$\phi(f_1 \cdots f_p) = \phi_1(f_1) \cdots \phi_1(f_p).$$

An easy consequence of this "adjointness property" is that free categories are homogeneous.

Lemma 1.2.5. *Let $\mathcal{S}$ be an oriented graph. There is a well-defined functor $\ell : \mathcal{S}^* \to (\mathbb{N}, +)$, sending a path to its length. The category $\mathcal{S}^*$ endowed with this functor is a homogeneous category.*

This convenient definition of free category allows for defining relations and presentations of categories. Recall that a *congruence* on a category $\mathcal{C}$ is an equivalence relation $\equiv$ on $\mathcal{C}$ which is compatible both with composition and with the source and target. That is, the conjunction of $f \equiv f'$ and $g \equiv g'$ implies $fg \equiv f'g'$ (if fg and $f'g'$ are defined). If $\equiv$ is a congruence on a category $\mathcal{C}$, one can form the quotient category $\mathcal{C}/\equiv$. It has the same objects as $\mathcal{C}$, and its morphisms are $\equiv$-equivalence classes of morphisms in $\mathcal{C}$.

Definition 1.2.6. [DDGKM, Definition II.1.36 and Lemma II.1.37] Let $\mathcal{C}$ be a category. A *relation* on $\mathcal{C}$ is a pair (g, h) of morphisms in $\mathcal{C}$ sharing the same source and the same target. If $\mathcal{R}$ is a family of relations on $\mathcal{C}$, there exists a smallest congruence $\equiv_{\mathcal{R}}$ on $\mathcal{C}$ which includes $\mathcal{R}$.

The congruence $\equiv_{\mathcal{R}}$ is the reflexive-transitive closure of

$$\{(fgh, fg'h) \in \mathcal{C} \times \mathcal{C} \mid (g, g') \in \mathcal{R} \text{ or } (g', g) \in \mathcal{R}\}.$$

For readability purposes, it is convenient to write a relation (f, g) as an equality $f = g$ instead of a couple of paths. We use this convention from now on.

Definition 1.2.7 (Category presentation). [DDGKM, Definition II.1.38]
A *category presentation* is a pair $(\mathcal{S}, \mathcal{R})$, where $\mathcal{S}$ is an oriented graph, and $\mathcal{R}$ is a set of relations on $\mathcal{S}^*$. We call $\mathcal{S}$ the *generators* and $\mathcal{R}$ the *relations*. If $(\mathcal{S}, \mathcal{R})$ is a category presentation, the quotient category $\mathcal{S}/\equiv_{\mathcal{R}}$ is denoted by $\langle \mathcal{S} \mid \mathcal{R} \rangle^+$.

Remark 1.2.8. If we consider a graph $\mathcal{S}$ with one object, we recover the classical notion of monoid presentation.

Let $(\mathcal{S}, \mathcal{R})$ be a categorical presentation, and let $\mathcal{C}$ be a category. Any morphisms of oriented graphs $\phi : \mathcal{S} \to \mathcal{C}$ such that

$$\forall f_1 \cdots f_p = g_1 \cdots g_q \in \mathcal{R},\ \phi_1(f_1) \cdots \phi_1(f_p) = \phi_1(g_1) \cdots \phi_1(g_p) \in \mathcal{C}$$

induces a unique functor from the presented category $\langle \mathcal{S} \mid \mathcal{R} \rangle^+$ to $\mathcal{C}$. A first application of this property is the following easy lemma:

Lemma 1.2.9. *Let $\mathcal{C} := \langle \mathcal{S} \mid \mathcal{R} \rangle^+$ be a presented category. If $\mathcal{R}$ consists of relations between paths of the same length in $\mathcal{S}^*$, then the map sending $f \in \mathcal{C}$ to the length of any representative of f in $\mathcal{S}^*$ gives a length functor on $\mathcal{C}$, making it into a homogeneous category. We then say that $(\mathcal{S}, \mathcal{R})$ is a* homogeneous presentation.

Example 1.2.10. Consider the following directed graph $\mathcal{S}$:

$$\begin{tikzcd} u \arrow[r, bend left=60, "a"] \arrow[r, bend right=30, "b"'] & v \arrow[l, bend right=30, "\alpha"'] \arrow[l, bend left=60, "\beta"] \end{tikzcd}$$

We endow $\mathcal{S}$ with the relations $\mathcal{R} := \{a\alpha a = b\beta b, \alpha a\alpha = \beta b\beta\}$. The presentation $(\mathcal{S}\ ,\ \mathcal{R})$ is homogeneous, and the category $\mathcal{C}$ presented by $\mathcal{S}$ and $\mathcal{R}$ is homogeneous.

The notion of categorical presentation is also useful for defining presentation of groupoids. Let $\mathcal{C} = \langle \mathcal{S} \mid \mathcal{R} \rangle^+$ be a presented category. We consider $\overline{\mathcal{S}}$ a formal copy of $\mathcal{S}$, with source and target reversed. We consider the set $I(\mathcal{S})$ of relations on $\mathcal{S} \sqcup \overline{\mathcal{S}}$ defined by

$$I(\mathcal{S}) := \{x\overline{x} = 1_{s(x)} \mid x \in \mathcal{S}\} \sqcup \{\overline{x}x = 1_{t(x)} \mid x \in \mathcal{S}\}.$$

Lemma 1.2.11. *[DDGKM, Definition II.3.3 and Proposition II.3.5]*
The category $\mathcal{G}(\mathcal{C}) := \langle \mathcal{S} \sqcup \overline{\mathcal{S}} \mid \mathcal{R} \cup I(\mathcal{S}) \rangle^+$ is a groupoid. The inclusion map $\mathcal{S} \hookrightarrow \mathcal{S} \cup \overline{\mathcal{S}}$ induces a functor $\iota : \mathcal{C} \to \mathcal{G}(\mathcal{C})$. Every functor $\phi : \mathcal{C} \to \mathcal{D}$ which sends $\mathcal{C}$ to invertible morphisms induces a unique functor $\widetilde{\phi} : \mathcal{G}(\mathcal{C}) \to \mathcal{D}$ such that $\widetilde{\phi} \circ \iota = \phi$.

The last statement implies that the groupoid $\mathcal{G}(\mathcal{C})$ is unique up to (canonical) isomorphism of groupoids. In particular it depends only on the category $\mathcal{C}$ and not on its presentation. By convention, if $\mathcal{C} = \langle \mathcal{S} \mid \mathcal{R} \rangle^+$ is a presented category, the presentation of the enveloping groupoid of $\mathcal{C}$ will be denoted by $\langle \mathcal{S} \mid \mathcal{R} \rangle$. If $\mathcal{C} = \mathcal{S}^*$ is the free category on a graph $\mathcal{S}$, then $\mathcal{G}(\mathcal{C})$ will also be denoted by $\mathcal{F}(\mathcal{S})$ and be called the *free groupoid* on the oriented graph $\mathcal{S}$. As in the classical case, a morphism in $\mathcal{F}(\mathcal{S})$ can be represented uniquely by a *reduced path* of the form

$$s_1^{\varepsilon_1} \cdots s_r^{\varepsilon_r},$$

where $\varepsilon_i \in \{\pm 1\}$ for $i \in [\![1, r]\!]$ (reduced means that there are no subpaths of the form ss^{-1} or $s^{-1}s$).

Corollary 1.2.12 (Enveloping groupoid). *[DDGKM, Proposition II.3.5]*
Let $\mathcal{C}$ be a category. There is a groupoid $\mathcal{G}(\mathcal{C})$, endowed with a functor $\iota : \mathcal{C} \to \mathcal{G}(\mathcal{C})$ such that, for every functor $\phi : \mathcal{C} \to \mathcal{D}$ which sends $\mathcal{C}$ to invertible morphisms in $\mathcal{D}$, there is a unique functor $\widetilde{\phi} : \mathcal{G}(\mathcal{C}) \to \mathcal{D}$ such that $\widetilde{\phi} \circ \iota = F$. The groupoid $\mathcal{G}(\mathcal{C})$ is called the enveloping groupoid *of $\mathcal{C}$.*

1.2.2 The particular case of germs

Definition 1.2.13 (Germ). [DDGKM, Definition VI.1.3] A *germ* is a couple $(\mathcal{S}, \cdot)$, where $\mathcal{S}$ is an oriented graph, and $\cdot$ denotes a partial product from $\mathcal{S}^2$ to $\mathcal{S}$ such that

- If $s \cdot t$ is defined, then the target of s is the source of t, the source of $s \cdot t$ is the source of s, and the target of $s \cdot t$ is the target of t.
- For all $u \in \mathrm{Ob}(\mathcal{S})$, there is an element $1_u \in \mathcal{S}(u, u)$ such that, for all $s \in \mathcal{S}(u, v)$, the products $s \cdot 1_v$ and $1_u \cdot s$ are both defined and equal to s.
- The product $(r \cdot s) \cdot t$ is defined if and only if the product $r \cdot (s \cdot t)$ is defined, in which case they are equal.

Let $(\mathcal{S}, \cdot)$ be a germ. The category $\mathcal{C}((\mathcal{S}, \cdot))$ generated by the germ $(\mathcal{S}, \cdot)$ is defined by the categorical presentation $\mathcal{C}((\mathcal{S}, \cdot)) := \langle \mathcal{S} \mid \mathcal{R}_{\cdot} \rangle^+$ where $\mathcal{R}_{\cdot}$ is the family of all relations $st = s \cdot t$ with $s, t \in \mathcal{S}$ and $s \cdot t$ defined.

If the context is clear, we will often denote a germ $(\mathcal{S}, \cdot)$ using only the letter $\mathcal{S}$. In particular, we will write $\mathcal{C}(\mathcal{S})$ instead of $\mathcal{C}((\mathcal{S}, \cdot))$ if the map $\cdot$ is clear. Note that, if $(\mathcal{S}, \cdot)$ is a germ and if u is an object of $\mathcal{S}$, then $1_u \in \mathcal{S}$ is the identity of the object u in $\mathcal{C}(\mathcal{S})$.

There is a more general definition of germ in [DDGKM] which relaxes the associativity condition. The germs we consider here are the associative germs in the sense of [DDGKM].

Example 1.2.14. Let $\mathcal{C}$ be a category. It can be seen as a germ where $\cdot$ is simply given by the composition in $\mathcal{C}$. In fact, a germ $(\mathcal{S}, \cdot)$ is a category if and only if $s \cdot t$ is defined whenever the target of s is the source of t. In this case, we have $\mathcal{C}(\mathcal{S}) = \mathcal{C}$.

A germ naturally comes equipped with an analogue of the divisibility relations of Definition 1.1.1.

Definition 1.2.15 ($\mathcal{S}$-divisibility). [DDGKM, Definition VI.1.18]
Let $(\mathcal{S}, \cdot)$ be a germ, and let $u \in \mathrm{Ob}(\mathcal{S})$.

- For $s, t \in \mathcal{S}(u, -)$, we say that s *left-$\mathcal{S}$-divides* t, or, equivalently, that t is a *right-$\mathcal{S}$-multiple* of s, written $s \preccurlyeq_{\mathcal{S}} t$, if there exists r in $\mathcal{S}$ satisfying $s \cdot r = t$.
- For $s, t \in \mathcal{S}(-, u)$, we say that s *right-$\mathcal{S}$-divides* t, or, equivalently, that t is a *left-$\mathcal{S}$-multiple* of s, written $t \succcurlyeq_{\mathcal{S}} s$, if there exists r in $\mathcal{S}$ satisfying $t = r \cdot s$.

Again, these two notions admit "strict" versions, denoted by $\prec_{\mathcal{S}}$ and $\succ_{\mathcal{S}}$, respectively.

As in the case of divisibility in a category, a cancellativity condition, along with the absence of nontrivial invertible elements can be used to prove that the $\mathcal{S}$-divisibility relations are partial orders.

Definition 1.2.16 (Cancellativity). [DDGKM, Definition VI.1.17] A germ $(\mathcal{S}, \cdot)$ is called *left-cancellative* (resp. *right-cancellative*) if, for $s, t, t' \in \mathcal{S}$, having $s \cdot t = s \cdot t'$ (resp. $t \cdot s = t' \cdot s$) implies that $t = t'$. It is called *cancellative* if it is both left- and right-cancellative.

A priori, this condition is weaker than asking for the category $\mathcal{C}(\mathcal{S})$ to be cancellative. However, cancellativity of the germ $\mathcal{S}$, combined with other assumptions, will be sufficient to prove that $\mathcal{C}(\mathcal{S})$ is cancellative (see Section 2.2).

Let $(\mathcal{S}, \cdot)$ be a germ. There is a natural notion of invertible element of $\mathcal{S}$. We say that $\varphi \in \mathcal{S}(u, v)$ is invertible if there exists some $\varphi' \in \mathcal{S}(v, u)$ such that $\varphi \cdot \varphi' = 1_u$ and $\varphi' \cdot \varphi = 1_v$. We denote by $\mathcal{S}^\times$ the family of invertible elements in $\mathcal{S}$. The notion of cancellativity of a germ is sufficient to prove that the relations $\preccurlyeq_{\mathcal{S}}$ and $\succcurlyeq_{\mathcal{S}}$ are partial orders.

Lemma 1.2.17. *Let $(\mathcal{S}, \cdot)$ be a germ such that $\mathcal{S}^\times = \{1_u\}_{u \in \mathrm{Ob}(\mathcal{S})}$ and let $u \in \mathrm{Ob}(\mathcal{C})$.*

(a) If $\mathcal{S}$ is left-cancellative, then $\preccurlyeq_{\mathcal{S}}$ is a partial order on $\mathcal{S}(u, -)$.

(b) If $\mathcal{S}$ is right-cancellative, then $\succcurlyeq_{\mathcal{S}}$ is a partial order on $\mathcal{S}(-, u)$.

The presentations arising from germs will play an important role later on, as we will able to infer properties of the category $\mathcal{C}(\mathcal{S})$ using only combinatorial assumptions on the germ $\mathcal{S}$ (see Section 2.2). First we have that a germ $(\mathcal{S}, \cdot)$ always embeds in the category $\mathcal{C}(\mathcal{S})$.

Proposition 1.2.18. *[DDGKM, Proposition VI.1.14] If $(\mathcal{S}, \cdot)$ is a germ, the natural map $\mathcal{S} \to \mathcal{C}(\mathcal{S})$ is injective, and composition in $\mathcal{C}(\mathcal{S})$ extends the partial product $\cdot$. Furthermore, for $f, g \in \mathcal{C}(\mathcal{S})$, the assertion that fg lies in the image of $\mathcal{S}$ implies that both f and g also lie in the image of $\mathcal{S}$.*

Using this proposition, we will now always identify a germ $\mathcal{S}$ with its image in its associated category $\mathcal{C}(\mathcal{S})$. Moreover, we can identify the invertible elements of $\mathcal{C}(\mathcal{S})$ starting from the invertible elements of $\mathcal{S}$.

Lemma 1.2.19. *[DDGKM, Lemma VI.1.19] Let $(\mathcal{S}, \cdot)$ be a germ. The set $\mathcal{C}(\mathcal{S})^\times$ is equal to the image of $\mathcal{S}^\times$ in $\mathcal{C}$.*

Using Proposition 1.2.18, one can show that the restriction in $\mathcal{C}(\mathcal{S})$ of the divisibility relations to the image of $\mathcal{S}$ agrees with $\mathcal{S}$-divisibility.

Lemma 1.2.20. *[DDGKM, Lemma VI.1.19] Let $(\mathcal{S}, \cdot)$ be a germ. Let also $f \in \mathcal{C}(\mathcal{S})$ and $t \in \mathcal{S}$.*

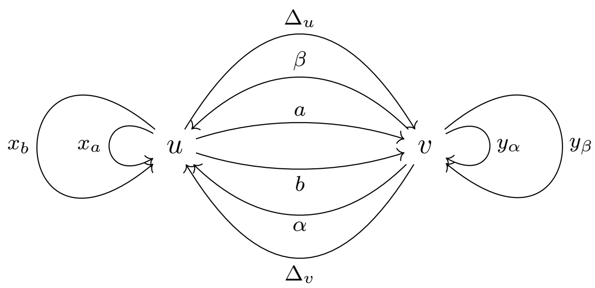
(a) We have $f \preccurlyeq t$ in $\mathcal{C}(\mathcal{S})$ if and only if $f \in \mathcal{S}$ and $f \preccurlyeq_{\mathcal{S}} t$.

(b) We have $t \succcurlyeq f$ in $\mathcal{C}(\mathcal{S})$ if and only if $f \in \mathcal{S}$ and $t \succcurlyeq_{\mathcal{S}} f$.

This lemma allows us to characterize the atoms of the category $\mathcal{C}(\mathcal{S})$ using only the germ $\mathcal{S}$ (see the discussions after [DDGKM, Definition VI.1.20]).

Lemma 1.2.21. *Let $(\mathcal{S}, \cdot)$ be a germ. The atoms of the category $\mathcal{C}(\mathcal{S})$ are the elements of $\mathcal{S}$ which have no proper left-$\mathcal{S}$-divisor.*

Example 1.2.22. Consider the following oriented graph $\mathcal{S}$ with two vertices u and v.

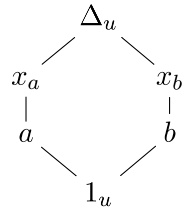


We can endow $\mathcal{S}$ with a partial product $\cdot$, as given in the tables below. The first (resp. second) table gives the partial product (should it be defined) of an element of $\mathcal{S}(-,u)$ (resp. $\mathcal{S}(-,v)$) with an element of $\mathcal{S}(u,-)$ (resp. $\mathcal{S}(v,-)$).

u	1_u	a	b	x_a	x_b	Δ_u
1_u	1_u	a	b	x_a	x_b	Δ_u
α	α	y_α		Δ_v		
β	β		y_β		Δ_v	
x_a	x_a	Δ_u				
x_b	x_b		Δ_u			
Δ_v	Δ_v					

v	1_v	α	β	y_α	y_β	Δ_v
1_v	1_v	α	β	y_α	y_β	Δ_v
a	a	x_a		Δ_u		
b	b		x_b		Δ_u	
y_α	y_α	Δ_v				
y_β	y_β		Δ_v			
Δ_u	Δ_u					

One readily checks that $(\mathcal{S}, \cdot)$ is a cancellative germ. The Hasse diagram of the poset $(\mathcal{S}(u,-), \preccurlyeq_{\mathcal{S}})$ is given by



In particular, $(\mathcal{S}(u,-), \preccurlyeq_{\mathcal{S}})$ is a lattice. By definition, the category $\mathcal{C}(\mathcal{S})$ is given by the presentation

$$\begin{aligned}
C(\mathcal{S}) &= \left\langle a, b, \alpha, \beta, x_a, x_b, y_\alpha, y_\beta, \Delta_u, \Delta_v \;\middle|\; \begin{cases} \alpha a = y_a,\ a\alpha = x_a,\ \beta b = y_\beta,\ b\beta = x_b \\ \Delta_u = x_a a = a y_\alpha = x_b b = b y_\beta \\ \Delta_v = y_\alpha \alpha = \alpha x_a = y_\beta \beta = \beta x_b \end{cases} \right\rangle^+ \\
&= \left\langle a, b, \alpha, \beta, x_a, x_b, y_\alpha, y_\beta \;\middle|\; \begin{cases} \alpha a = y_a,\ a\alpha = x_a,\ \beta b = y_\beta,\ b\beta = x_b \\ x_a a = a y_\alpha = x_b b = b y_\beta \\ y_\alpha \alpha = \alpha x_a = y_\beta \beta = \beta x_b \end{cases} \right\rangle^+ \\
&= \left\langle a, b, \alpha, \beta \;\middle|\; \begin{cases} a\alpha a = a\alpha a = b\beta b = b\beta b \\ \alpha a\alpha = \alpha a\alpha = \beta b\beta = \beta b\beta \end{cases} \right\rangle^+ \\
&= \langle a, b, \alpha, \beta \mid a\alpha a = b\beta b,\ \alpha a\alpha = \beta b\beta \rangle^+ .
\end{aligned}$$

The atoms of $\mathcal{C}(\mathcal{S})$ are given by a, α, b, β, and we see that the category $\mathcal{C}(\mathcal{S})$ is isomorphic to the category $\mathcal{C}$ of Example 1.2.10.

1.3 Groupoids of fractions

We saw in Corollary 1.2.12 that every category admits an enveloping groupoid. However, this formal groupoid may actually have very little to see with the starting category. For instance, it is hard to state whether or not a given category embeds in its enveloping groupoid, or even if the enveloping groupoid is nontrivial.

Example 1.3.1. Let p be a prime number. Let M_p denote the monoid

$$\langle a, z \mid za = az = z^2 = z, a^{p-1} = 1 \rangle^+.$$

This monoid contains p elements $z, 1, a, a^2, \ldots a^{p-2}$. It is actually isomorphic to the monoid $(\mathbb{F}_p, \times)$ by sending z to 0 and a to any generator of $(\mathbb{F}_p)^*$. The enveloping group of M_p is

$$\begin{aligned}
G_p &= \langle a, z \mid za = az = z = z^2, a^{p-1} = 1 \rangle \\
&= \langle a, z \mid a = z, z^2 = z, a^{p-1} = 1 \rangle \\
&= \langle z \mid z = 1 \rangle = \{1\}.
\end{aligned}$$

More generally, if R is a ring with unit, then $(R, \times)$ is a monoid whose enveloping group is trivial.

Like in the case of localization in a ring (with unit), we consider conditions on a category $\mathcal{C}$ which induce a convenient description of the enveloping groupoid of $\mathcal{C}$ as a groupoid of fractions. The material in this section is mostly taken from [DDGKM, Section II.3.2 and Appendix 1] and is given here for the sake of self-containment.

Definition 1.3.2 (Left-Ore category). [DDGKM, Definition 3.10] A category $\mathcal{C}$ is said to be a *left-Ore* category if it is cancellative and any two elements with the same target admit a common left-multiple. Likewise, a *right-Ore* category is a cancellative category where any two elements with the same source admit a common right-multiple. An *Ore* category is a category which is both a left- and right-Ore category.

For the remainder of this section, we fix a left-Ore category $\mathcal{C}$. For $u, v \in \mathrm{Ob}(\mathcal{C})$, we define

$$F(u, v) := \bigsqcup_{x \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(w, u) \times \mathcal{C}(w, v).$$

We morally think of $(a, b) \in F(u, v)$ as a formal left-fraction of the form $a^{-1}b$. We endow $F(u, v)$ with a binary relation $\bowtie$ given by

$$(a, b) \bowtie (a', b') \Leftrightarrow \exists h, h' \in \mathcal{C} \mid (ha = h'a',\ hb = h'b').$$

The situation giving $(a, b) \bowtie (a', b')$ can be summarized in a commutative diagram of the form

$$\begin{tikzcd}
& w \arrow[r, "a"] \arrow[ddr, "b" near start] & u \\
{} \arrow[ur, dashed, "\exists h"] \arrow[dr, dashed, "\exists h'"'] & & \\
& w' \arrow[uur, "a'" near end] \arrow[r, "b'"'] & v
\end{tikzcd}$$

The following lemma is an easy exercise using the definition of $\bowtie$.

Lemma 1.3.3. *[DDGKM, Appendix Claim 3] For all $u, v \in \mathrm{Ob}(\mathcal{C})$, $\bowtie$ is an equivalence relation on $F(u, v)$. The equivalence class of $(a, b) \in F(u, v)$ will be denoted by $\widetilde{a}b$.*

Using the relation $\bowtie$, we define an oriented graph $\mathcal{G}$. We set $\mathrm{Ob}(\mathcal{G}) := \mathrm{Ob}(\mathcal{C})$, and, for $u, v \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u, v) := F(u, v)/ \bowtie$. We claim that we can endow $\mathcal{G}$ with a composition law making it into the enveloping groupoid of $\mathcal{C}$.

Let $u, v, w \in \mathrm{Ob}(\mathcal{G})$, let $\widetilde{a}b \in \mathcal{G}(u, v), \widetilde{c}d \in \mathcal{G}(v, w)$. We are going to build a commutative diagram:

$$\begin{tikzcd}
& & {} \arrow[dl, dashed, "\exists x"'] \arrow[dr, dashed, "\exists y"] & & \\
& {} \arrow[dl, "a"'] \arrow[dr, "b"] & & {} \arrow[dl, "c"'] \arrow[dr, "d"] & \\
u & & v & & w
\end{tikzcd}$$

We have $b, c \in \mathcal{C}(-, v)$ and we can consider a common left-multiple $xb = yc$ of b and c. We define

$$\widetilde{a}b \circ \widetilde{c}d = \widetilde{(xa)}yd \in \mathcal{G}(u, w).$$

A priori, this definition seems to rely on several choices: the choice of a common multiple of b and c, and the choice of representatives of $\widetilde{a}b$ and $\widetilde{c}d$. The following proposition ensures that $\circ$ is well-defined and makes $\mathcal{G}$ into a groupoid.

Proposition 1.3.4 (Groupoid of fractions). *[DDGKM, Appendix Claim 4 to 6] The graph $\mathcal{G}$ endowed with the composition map $\circ$ is a groupoid. The identity of an object $u \in \mathrm{Ob}(\mathcal{G})$ is given by $\widetilde{1_u}1_u \in \mathcal{G}(u, u)$.*

Theorem 1.3.5 (Groupoid of fractions is enveloping groupoid). *[DDGKM, Appendix Claim 6] Let $\mathcal{C}$ be an Ore category. The map $\iota : \mathcal{C} \to \mathcal{G}$ sending $f \in \mathcal{C}(u, -)$ to $\widetilde{1_u}f$ is a faithful functor, making $\mathcal{G}$ into the enveloping groupoid of $\mathcal{C}$.*

Notation 1.3.6. Let $\mathcal{C}$ be a left-Ore category. Since the natural functor from $\mathcal{C}$ to its enveloping groupoid $\mathcal{G}$ is faithful, we can see $\mathcal{C}$ as a subcategory of $\mathcal{G}$. Composition in $\mathcal{G}$ will then be denoted as a product, dropping the symbol $\circ$. Moreover, we saw in the above proof that $\widetilde{a}b = a^{-1}b$. The elements of $\mathcal{G}$ will thus be denoted as fractions from now on. In particular, we will write a^{-1} for $\widetilde{a}1_u$, where u is the source of a in $\mathcal{C}$.

Remark 1.3.7. In the case where $\mathcal{C}$ is a right-Ore category, $\mathcal{C}^{\mathrm{op}}$ is a left-Ore category. In this case, Theorem 1.3.5 gives that $\mathcal{C}$ also embeds in its enveloping groupoid, and that the latter can be conveniently described a a groupoid of right-fractions, of the form xy^{-1}.

Example 1.3.8. Let $\mathbb{N}^*$ denote the set of positive integers. The product of natural numbers endows $\mathbb{N}^*$ with the structure of a left-Ore monoid. One easily sees that the enveloping groupoid of $\mathbb{N}^*$ is the group $(\mathbb{Q}_+^*, \times)$ of positive rational numbers, and that the elements of $\mathbb{Q}_+^*$ are all described as fractions of elements of $\mathbb{N}^*$.

1.4 Groupoids of cosets

Let $\mathcal{G}$ be a connected groupoid (i.e. for all $x, y \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(x, y)$ is nonempty), let $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u, u)$ be a subgroup. For $v \in \mathrm{Ob}(\mathcal{G})$, the set $\mathcal{G}(u, v)$ is endowed with a binary relation $\equiv_H$ given by

$$\forall g, g' \in \mathcal{G}(u, v),\ g \equiv_H g' \Leftrightarrow gg'^{-1} \in H.$$

One immediately checks that $\equiv_H$ is an equivalence relation.

Definition 1.4.1 (Right-coset). The equivalence class of $g \in \mathcal{G}(u, v)$ is denoted by Hg and called a *right-coset* of H in $\mathcal{G}(u, v)$. We denote by $H\backslash\mathcal{G}(u, v)$ the set of equivalence classes in $\mathcal{G}(u, v)$ for the relation $\equiv_R$. More generally, we consider the set of *right-cosets* of H in $\mathcal{G}$, defined by

$$H\backslash\mathcal{G} := \bigsqcup_{v \in \mathrm{Ob}(\mathcal{G})} H\backslash\mathcal{G}(u, v)$$

Let $g \in \mathcal{G}(u, v)$. One readily checks that $Hg = \{hg \mid h \in H\}$. For readability purposes, we will also denote by $[g]$ the coset Hg if the context is clear. Note that H acts on $\mathcal{G}(u, v)$ by multiplication on the left, and that the right-coset $[g]$ of $g \in \mathcal{G}(u, v)$ is simply the H-orbit of g.

Remark 1.4.2. Assume that $\mathcal{G} = G$ is a group (i.e. a groupoid with one object). The equivalence class Hg of $g \in \mathcal{G}(u, u) = G$ is the right-coset Hg in the usual sense.

The set $H\backslash\mathcal{G}(u, v)$ is endowed with a right-action of the group $\mathcal{G}(v, v)$, given by

$$(Hg).\sigma = H(g\sigma) \in H\backslash\mathcal{G}(u, v).$$

Lemma 1.4.3. *Let $t \in \mathcal{G}(u, v)$, and let $H_t := t^{-1}Ht \subset \mathcal{G}(v, v)$. The map $Hg \mapsto H_t(t^{-1}g)$ induces an isomorphism of right $\mathcal{G}(v, v)$-sets between $H\backslash\mathcal{G}(u, v)$ and $H_t\backslash\mathcal{G}(v, v)$.*

Proof. First, since $\mathcal{G}$ is a groupoid, precomposition by t^{-1} induces a bijection between $\mathcal{G}(u, v)$ and $\mathcal{G}(v, v)$. This bijection in turn induces a bijection φ between the sets $H\backslash\mathcal{G}(u, v)$ and $H_t\backslash\mathcal{G}(v, v)$. Indeed, for $g, g' \in \mathcal{G}(u, v)$, we have

$$g \equiv_H g' \Leftrightarrow gg'^{-1} \in H \Leftrightarrow t^{-1}gg'^{-1}t = t^{-1}g(t^{-1}g')^{-1} \in H_t.$$

Lastly, we show that the bijection φ is compatible with the action of $\mathcal{G}(v, v)$ on the right. Let $g \in \mathcal{G}(u, v)$ and let $\sigma \in \mathcal{G}(v, v)$. We have

$$\varphi(Hg).\sigma = (H_t t^{-1}g).\sigma = H_t t^{-1}g\sigma = \varphi(Hg\sigma) = \varphi((Hg).\sigma),$$

as we wanted to show. □

Corollary 1.4.4. *The set $H\backslash\mathcal{G}$ is in bijection with $\mathrm{Ob}(\mathcal{G}) \times H\backslash\mathcal{G}(u, u)$. In particular, if both $\mathrm{Ob}(\mathcal{G})$ and $[\mathcal{G}(u, u) : H]$ are finite, then $|H\backslash\mathcal{G}| = |\mathrm{Ob}(\mathcal{G})|[\mathcal{G}(u, u) : H]$ (where $|-|$ denotes the cardinality of a set).*

Proof. Since $\mathcal{G}$ is connected, we can consider, for each $v \in \mathrm{Ob}(\mathcal{G})$, a morphism $t_v : u \to v$ in $\mathcal{G}$. For $v \in \mathrm{Ob}(\mathcal{G})$, we know that conjugation by t_v induces an isomorphism between $\mathcal{G}(u,u)$ and $\mathcal{G}(v,v)$, which sends H to $H_v := H_{t_v}$. Conjugation by t_v then induces a bijection between $H\backslash\mathcal{G}(u,u)$ and $H_v\backslash\mathcal{G}(v,v)$. Using Lemma 1.4.3, we then get

$$H\backslash\mathcal{G} \simeq \bigsqcup_{v\in\mathrm{Ob}(\mathcal{G})} H\backslash\mathcal{G}(u,u) \simeq \mathrm{Ob}(\mathcal{G}) \times H\backslash\mathcal{G}(u,u).$$

□

Definition 1.4.5 (Groupoid of cosets). Let $\mathcal{G}$ be a connected groupoid, and let $H \subset \mathcal{G}(u,u)$ be a subgroup for some $u \in \mathrm{Ob}(\mathcal{G})$. The *groupoid of cosets* $\mathcal{G}_H$ is defined as follows:

- $\mathrm{Ob}(\mathcal{G}_H)$ is the set $H\backslash\mathcal{G}$ of right-cosets of H in $\mathcal{G}$.
- For $g \in \mathcal{G}(u,v)$, $g' \in \mathcal{G}(u,v')$. We set

  $$\mathcal{G}_H([g],[g']) = \{f \in \mathcal{G}(v,v') \mid [gf] = [g']\}.$$

  We denote by $f_{[g]}$ the morphism from $[g]$ to $[gf]$ represented by f.
- Composition is given the same way as in G

  $$\forall g \in \mathcal{G}(u,v), f \in \mathcal{G}(v,v'), f' \in \mathcal{G}(v',v''), \quad f_{[g]}f'_{[gf]} = (ff')_{[g]}.$$

Lemma 1.4.6. *Let $\mathcal{G}$ be a connected groupoid, let $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u,u)$ be a subgroup.*

(a) The groupoid $\mathcal{G}_H$ is connected and equivalent to H as a category.

(b) For $g \in \mathcal{G}(u,v)$, we set $\pi([g]) := v$. For $f_{[g]} \in \mathcal{G}_H([g],-)$, we set $\pi(f_{[g]}) := f$. The map π induces a functor $\pi : \mathcal{G}_H \to \mathcal{G}$.

(c) For $g \in \mathcal{G}(u,v)$, the map $\mathcal{G}_H([g],-) \to \mathcal{G}(v,-)$ induced by the functor π is a bijection. The inverse bijection sends f to $f_{[g]}$.

(d) For $g \in \mathcal{G}(u,v)$, the map $\mathcal{G}_H(-,[g]) \to \mathcal{G}(-,v)$ induced by the functor π is a bijection. The inverse bijection sends f to $f_{[gf^{-1}]}$.

Proof. (1) Let $[g] \in \mathrm{Ob}(\mathcal{G}_H)$, the morphism $g_{[1_u]}$ is a morphism in $\mathcal{G}_H$ from $[1_u] = H$ to $[g]$, thus $\mathcal{G}_H$ is connected. Moreover, for $g \in \mathcal{G}(u,v)$ and $g' \in \mathcal{G}(u,v')$, we have

$$\begin{aligned}\mathcal{G}_H([g],[g']) &= \{f \in \mathcal{G}(v,v') \mid [gf] = [g']\}\\ &= \{f \in \mathcal{G}(v,v') \mid gfg^{-1} \in H\}\\ &= g^{-1}Hg' \subset \mathcal{G}(v,v').\end{aligned}$$

In particular, $\mathcal{G}_H([1_u],[1_u]) = H \subset \mathcal{G}(u,u)$ and $\mathcal{G}_H$ is equivalent to H. The points (2), (3) and (4) are immediate consequences of the definition of $\mathcal{G}_H$. □

The construction of the groupoid of cosets is remarkably natural, it can even be interpreted topologically given the right context (see Section 7.2.1). For now, we show that a presentation of the groupoid $\mathcal{G}$ naturally induces a presentation of an associated groupoid of cosets.

Let $\mathcal{G} := \langle \mathcal{S} \mid \mathcal{R}\rangle$ be a connected groupoid (i.e. with $\mathcal{S}$ a connected graph), and let $H \subset \mathcal{G}(u,u)$ be a subgroup for some $u \in \mathrm{Ob}(\mathcal{G})$. Let also $\pi : \mathcal{G}_H \to \mathcal{G}$ be the functor of Lemma 1.4.6, and let $\widetilde{\mathcal{S}} := \pi^{-1}(\mathcal{S})$ be the subgraph of $\mathcal{G}_H$ made of the morphisms in $\mathcal{G}_H$ whose image under π lie in $\mathcal{S}$. The restriction of π to $\widetilde{\mathcal{S}}$ induces a functor $\widetilde{\pi} : \mathcal{F}(\widetilde{\mathcal{S}}) \to \mathcal{F}(\mathcal{S})$ between the associated free groupoids. The inclusion $\widetilde{\mathcal{S}} \to \mathcal{G}_H$ (resp. $\mathcal{S} \to \mathcal{G}$) induces a functor $\varphi : \mathcal{F}(\widetilde{\mathcal{S}}) \to \mathcal{G}_H$ (resp. $\phi : \mathcal{F}(\mathcal{S}) \to \mathcal{G}$). We have the following commutative square:

$$\begin{tikzcd}
\mathcal{F}(\widetilde{\mathcal{S}}) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\varphi"'] & \mathcal{F}(\mathcal{S}) \arrow[d, "\phi"] \\
\mathcal{G}_H \arrow[r, "\pi"'] & \mathcal{G}
\end{tikzcd}$$

Let $v, v' \in \mathrm{Ob}(\mathcal{S})$ and let $m := s_1^{\varepsilon_1} \cdots s_r^{\varepsilon_r}$ be a path from v to v' in $\mathcal{F}(\mathcal{S})$, and let $g \in \mathcal{G}(u, v)$. There is a unique path in $\mathcal{F}(\widetilde{\mathcal{S}})$ which starts at $[g]$ and whose image under $\widetilde{\pi}$ is m. This path is given by

$$m_{[g]} := (s_{1[g]})^{\varepsilon_1} (s_{2[gs_1^{\varepsilon_1}]})^{\varepsilon_2} \cdots (s_{r[gs_1^{\varepsilon_1} \cdots s_{r-1}^{\varepsilon_{r-1}}]})^{\varepsilon_r}.$$

Note that $\varphi(m_{[g]}) = \phi(m)_{[g]}$. In particular we see that $\widetilde{\mathcal{S}}$ generates $\mathcal{G}_H$ since $\mathcal{S}$ generates $\mathcal{G}$.

Proposition 1.4.7 (Presentation of the groupoid of cosets). *Let $\mathcal{G} := \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a connected groupoid, and let $H \subset \mathcal{G}(u, u)$ be a subgroup for some $u \in \mathrm{Ob}(\mathcal{G})$. With the above notation, the groupoid of cosets $\mathcal{G}_H$ admits the presentation $\langle \widetilde{\mathcal{S}} \mid \widetilde{\mathcal{R}} \rangle$, where*

$$\widetilde{\mathcal{R}} := \{ r_{[g]} = r'_{[g]} \mid (r, r') \in \mathcal{R} \text{ and } [g] \in H \setminus \mathcal{G} \}.$$

Proof. Since $\mathcal{G}$ is a groupoid, every relation $(r, r') \in \mathcal{R}$ is equivalent to $rr'^{-1} = 1_v$ (where v is the source of r in $\mathcal{G}$). We can then assume that $\mathcal{R}$ contains only relations of the form $(r, 1_v)$ for some $r \in \mathcal{F}(\mathcal{S})(v, -)$. The associated elements of $\widetilde{\mathcal{R}}$ have the form $(r_{[g]}, 1_{[g]})$ for some $g \in \mathcal{G}(u, v)$.

Let $\mathcal{G}' := \langle \widetilde{\mathcal{S}} \mid \widetilde{\mathcal{R}} \rangle$. Let $r_{[g]} = 1_{[g]}$ be a relation in $\widetilde{\mathcal{R}}$. We see r as a path in $\mathcal{F}(\widetilde{\mathcal{S}})$. By definition, $\phi(\widetilde{\pi}(r)) = \pi(\varphi(r))$ is trivial in $\mathcal{G}$. By Lemma 1.4.6 (3), we obtain that $\varphi(r)$ is trivial. Thus the identity map $\widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}}$ induces a well-defined functor $\mathcal{G}' \to \mathcal{G}_H$.

Since the graph $\widetilde{\mathcal{S}}$ generates $\mathcal{G}_H$, the functor $\mathcal{G}' \to \mathcal{G}_H$ is full. Now consider two paths in $\mathcal{F}(\widetilde{\mathcal{S}})$

$$m := s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_r^{\varepsilon_r} \quad \text{and} \quad m' := t_1^{\eta_1} t_2^{\eta_2} \cdots t_q^{\eta_q}.$$

By Lemma 1.4.6 (3), saying that these two paths induce the same morphism in $\mathcal{G}_H$ amounts to say that they share the same source and that $\phi(\widetilde{\pi}(m)) = \phi(\widetilde{\pi}(m'))$.

Suppose that m and m' induce the same element in $\mathcal{G}_H$. We have $\widetilde{\pi}(m) = \widetilde{\pi}(m')$ in $\mathcal{G}$. Thus there is a finite sequence of paths $m_1, \ldots, m_p$ in $\mathcal{F}(\mathcal{S})$ such that

- $m_1 = \widetilde{\pi}(m), m_p = \widetilde{\pi}(m')$
- For $i \in [\![1, p-1]\!]$, m_i is equivalent to m_{i+1} by the use of one relation in $\mathcal{R}$.

Let $[g]$ be the common source of m and m' in $\mathcal{F}(\widetilde{\mathcal{S}})$. We consider the paths $m_{i[g]}$ in $\mathcal{F}(\widetilde{\mathcal{S}})$. We have $m_{1[g]} = m$, $m_{p[g]} = m'$. For $i \in [\![1, p-1]\!]$, the path $m_{i[g]}$ is equivalent to $m_{i+1[g]}$ by the use of one relation in $\widetilde{\mathcal{R}}$ by definition.

Thus, m and m' induce the same element in $\mathcal{G}'$, and the functor $\mathcal{G}' \to \mathcal{G}_H$ is faithful. Since it is bijective on objects by definition, it is an isomorphism of groupoids. $\square$

Example 1.4.8. Let $\mathcal{S}$ be the oriented graph given by

$$\begin{tikzcd} u \arrow[r, bend left, "a"] & v \arrow[l, bend left, "b"] \end{tikzcd} ,$$

and let $\mathcal{G} := \mathcal{F}(\mathcal{S})$ be the free groupoid on this graph. We have

$$\begin{cases} \mathcal{G}(u,u) = \langle ab \rangle \simeq \mathbb{Z}, \\ \mathcal{G}(u,v) = \{(ab)^k a \mid k \in \mathbb{Z}\}, \\ \mathcal{G}(v,v) = \langle ba \rangle \simeq \mathbb{Z}, \\ \mathcal{G}(v,u) = \{(ba)^k b \mid k \in \mathbb{Z}\}. \end{cases}$$

We consider the subgroup $H := \langle (ab)^3 \rangle$ of $\mathcal{G}(u,u)$. In $\mathcal{G}(u,u)$, we have three right-cosets $H, Hab, H(ab)^2$. For $g := (ab)^k a, g' := (ab)^{k'} a \in \mathcal{G}(u,v)$, we have

$$g \equiv_H g' \Leftrightarrow gg'^{-1} = (ab)^k a a^{-1} (ab)^{-k'} = (ab)^{k-k'} \in H \Leftrightarrow 3|k-k'.$$

Thus, we also have three right-cosets $Ha, Haba, H(ab)^2 a$.

If we consider $t_v := a$, we have $H_v := t_v^{-1} H t_v = a^{-1} \langle (ab)^3 \rangle a = \langle (ba)^3 \rangle$, and Lemma 1.4.3 gives the following bijection between $H \backslash \mathcal{G}(u,v)$ and $H_v \backslash \mathcal{G}(v,v)$

$$\begin{cases} Ha \mapsto H_v(a^{-1}a) = H_v, \\ Haba \mapsto H_v ba, \\ H(ab)^2 a \mapsto H_v (ba)^2. \end{cases}$$

By Proposition 1.4.7 (presentation of groupoid of cosets), the groupoid of cosets $\mathcal{G}_H$ is then given as the free groupoid on the following graph:

$$\begin{tikzcd} H \arrow[r, "a_{[1]}"] & Ha \arrow[r, "b_{[a]}"] & Hab \arrow[d, "a_{[ba]}"] \\ H(ab)^2 a \arrow[u, "b_{[(ab)^2 a]}"] & H(ab)^2 \arrow[l, "a_{[(ab)^2]}"] & Haba \arrow[l, "b_{[aba]}"] \end{tikzcd}$$

We see in particular that it is equivalent to $H \simeq \mathbb{Z}$.

Example 1.4.9. Consider the group G given by the presentation $G = \langle s, t \mid s^3 = t^3 \rangle$, and consider the subgroup H of G normally generated by st^{-1} and s^2. We have

$$G/H = \langle s, t \mid s^3 = t^3, s = t, s^2 = 1 \rangle = \langle s \mid s^2 = 1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

We then have two right-cosets $H = [1]$ and $Hs = [s]$ of H in G. By Proposition 1.4.7 (presentation of groupoid of cosets), the groupoid G_H is generated by the following oriented graph:

$$\begin{tikzcd} {[1]} \arrow[r, bend left=10, "s_{[1]}"] \arrow[r, bend right=10, "t_{[1]}"'] & {[s]} \arrow[l, bend right=50, "s_{[s]}"'] \arrow[l, bend left=50, "t_{[s]}"] \end{tikzcd}$$

endowed with the relations $s_{[1]} s_{[s]} s_{[1]} = t_{[1]} t_{[s]} t_{[1]}$ and $s_{[s]} s_{[1]} s_{[s]} = t_{[s]} t_{[1]} t_{[s]}$. We see that G_H is isomorphic to the enveloping groupoid of the category $\mathcal{C}$ of Examples 1.2.10 and 1.2.22.

1.5 The Reidemeister-Schreier method for groupoids

In this section, we present a general method for computing a presentation of a group G starting from a presentation of a groupoid to which G is equivalent. This method provides a generalization of the "classical" Reidemeister-Schreier method, which is used to compute a presentation of a finite index subgroup of a presented group. This section is taken from my third preprint [Gar23b].

1.5.1 Schreier transversal and presentation

Let $\mathcal{G} = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a presented connected groupoid, and let $\mathcal{F}(\mathcal{S})$ be the free groupoid on the graph $\mathcal{S}$. Let also φ be the quotient map $\varphi : \mathcal{F}(\mathcal{S}) \to \mathcal{G}$.

Definition 1.5.1 (Schreier transversal). Let u be an object of $\mathcal{F}(\mathcal{S})$. A *Schreier transversal of $\mathcal{F}(\mathcal{S})$ rooted in u* is a family of reduced paths $T := \{t_v\}_{v \in \mathrm{Ob}(\mathcal{F}(\mathcal{S}))}$ satisfying:

- For all object v of $\mathcal{S}$, the path t_v has source u and target v.
- The family $\{t_v\}$ is stable under prefix: if t_v is written as the concatenation of two paths $m_1 m_2$, then $m_1 \in T$. In particular we have $t_u = 1_u$.

Remark 1.5.2. Since $\mathcal{G}$ is connected, it is also the case of $\mathcal{S}$ and $\mathcal{F}(\mathcal{S})$. In particular, a Schreier transversal of $\mathcal{F}(\mathcal{S})$ rooted in u exists for all object u of $\mathcal{F}(\mathcal{S})$.

Let u_0 be an object of $\mathcal{F}(\mathcal{S})$, and let $T := \{t_v\}_{v \in \mathrm{Ob}(\mathcal{G})}$ be a Schreier transversal in $\mathcal{F}(\mathcal{S})$ rooted in u_0. For $s \in \mathcal{S}(u,v)$, we define $\gamma(s) := t_u s t_v^{-1} \in \mathcal{F}(\mathcal{S})(u_0,u_0)$. Let S_1 be the set of all elements $\gamma(s) \neq 1_{u_0}$ for $s \in \mathcal{S}$.

Lemma 1.5.3 (Schreier's Lemma). *The group $\mathcal{G}(u_0,u_0)$ is generated by the $\varphi(\gamma(s))$ for $\gamma(s) \in S_1$.*

Proof. Let $g \in \mathcal{G}(u_0,u_0)$. Since $\mathcal{G}$ is generated by $\mathcal{S}$, we can write

$$g = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k},$$

with $s_i \in \mathcal{S}$ for $i \in [\![1,k]\!]$ and $\varepsilon_i \in \{\pm 1\}$ for $i \in [\![1,k]\!]$. We denote by u_i the target of $s_i^{\varepsilon_i}$ for $i \in [\![1,k-1]\!]$. In $\mathcal{F}(\mathcal{S})$ we have

$$\begin{aligned} s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} &= s_1^{\varepsilon_1} t_{u_1}^{-1} t_{u_1} \cdots t_{u_{k-1}}^{-1} t_{u_{k-1}} s_k^{\varepsilon_k} \\ &= t_{u_0} s_1^{\varepsilon_1} t_{u_1}^{-1} t_{u_1} \cdots t_{u_{k-1}}^{-1} t_{u_{k-1}} s_k^{\varepsilon_k} t_{u_0}^{-1} \\ &= \gamma(s_1)^{\varepsilon_1} \cdots \gamma(s_k)^{\varepsilon_k}. \end{aligned}$$

Thus, we have $g = \varphi(\gamma(s_1))^{\varepsilon_1} \cdots \varphi(\gamma(s_k))^{\varepsilon_k}$ in $\mathcal{G}(u_0,u_0)$. □

Proposition 1.5.4 (Reidemeister-Schreier method for groupoids). *Let S^* be a set of elements $\gamma(s)^*$ in one-to-one correspondence with those of S_1. Let also R^* be the set of all relations*

$$\gamma(s_1)^{*\varepsilon_1} \cdots \gamma(s_k)^{*\varepsilon_k} = \gamma(t_1)^{*\varepsilon'_1} \cdots \gamma(t_{k'})^{*\varepsilon'_{k'}},$$

where $s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = t_1^{\varepsilon'_1} \cdots t_{k'}^{\varepsilon'_{k'}}$ is in $\mathcal{R}$. The map $S^* \to \mathcal{G}(u_0,u_0)$ sending $\gamma(s)^*$ to $\varphi(\gamma(s))$ induces an isomorphism of groups between $G^* := \langle S^* \mid R^* \rangle$ and $\mathcal{G}(u_0,u_0)$.

Proof. Since $\mathcal{G}$ is a groupoid, every relation defining $\mathcal{G}$ can be rewritten into a relation of the form

$$s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = 1_u,$$

where $s_i \in \mathcal{S}$ and $\varepsilon_i \in \{\pm 1\}$ for $i \in [\![1,k]\!]$, and where u is the source of $s_1^{\varepsilon_1}$. We assume that every relation in $\mathcal{R}$ has this form.

First, we prove that the map $\gamma(s)^* \mapsto \varphi(\gamma(s))$ is compatible with the set of relations R^*. Let $\gamma^*(s_1)^{\varepsilon_1} \cdots \gamma^*(s_k)^{\varepsilon_k} = 1$ be in R^*. We have an equality $s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} = 1_u$ in $\mathcal{G}$, and we have

$$\varphi(\gamma(s_1))^{\varepsilon_1} \cdots \varphi(\gamma((s_k))^{\varepsilon_k} = \varphi(m_u) s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} \varphi(m_u)^{-1} = 1_{u_0}.$$

Let $\pi : G^* \to \mathcal{G}(u_0,u_0)$ be the morphism induced by $\gamma(s)^* \mapsto \varphi(\gamma(s))$. We know that π is surjective by Lemma 1.5.3.

Conversely, the map $\mathcal{S} \to S^*$ sending s to $\gamma(s)^*$ induces a functor $\phi : \mathcal{F}(\mathcal{S}) \to G^*$. Let v be an object of $\mathcal{F}(\mathcal{S})$. We show that ϕ sends t_v to 1 by induction on the length of t_v as an $\mathcal{S}$-path. First if $t_v = s_1^{\varepsilon_1}$ has length 1, we have $\phi(s_1^{\varepsilon_1}) = t_{u_0} t_v t_v^{-1} = 1$. Now if $t_v = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}$ is a decomposition of t_v on $\mathcal{S}$, we denote by v' the source of $s_k^{\varepsilon_k}$. By definition of a Schreier transversal we have $t_{v'} = s_1^{\varepsilon_1} \cdots s_{k-1}^{\varepsilon_{k-1}}$ and $\phi(t_{v'}) = 1$ by induction hypothesis. As we also have $\phi(s_k^{\varepsilon_k}) = t_{v'} s_k^{\varepsilon_k} t_v^{-1} = t_v t_v^{-1} = 1$, we get $\phi(t_v) = 1$.

By definition of the set R^*, ϕ induces a functor $\overline{\phi} : \mathcal{G} \to G^*$. We call ι the restriction of this functor to $\mathcal{G}(u_0,u_0)$. Let $\gamma(s)^*$ be a generator of G^*, with $s \in \mathcal{S}(u,v)$. We have

$$\begin{aligned}
\iota(\pi(\gamma(s)^*)) &= \iota(\varphi(\gamma(s))) \\
&= \iota(\varphi(t_u) s \varphi(t_v)^{-1}) \\
&= \overline{\phi}(\varphi(t_u)) \overline{\phi}(s) \overline{\phi}(\varphi(t_v))^{-1} \\
&= \phi(t_u) \gamma(s)^* \phi(t_v)^{-1} = \gamma(s)^*.
\end{aligned}$$

So $\iota \circ \pi$ induces the identity on the generators of G^*. We get $\iota \circ \pi = 1_{G^*}$ and π is injective. ☐

Corollary 1.5.5. *If $\mathcal{G}$ is a finitely presented groupoid, then for every object u of $\mathcal{G}$, the group $\mathcal{G}(u,u)$ is finitely presented.*

Example 1.5.6. Consider the category $\mathcal{C}$ of Example 1.2.10, and let $\mathcal{G}$ be the enveloping groupoid of $\mathcal{C}$. By construction, $\mathcal{G}$ is generated as a groupoid by the graph

$$\begin{tikzcd}
u \arrow[r, bend left=70, "a"] \arrow[r, bend right=25, "b"'] & v \arrow[l, bend right=25, "\alpha"'] \arrow[l, bend left=70, "\beta"]
\end{tikzcd}$$

endowed with the relations $\mathcal{R} := \{a\alpha a = b\beta b, \alpha a \alpha = \beta b \beta\}$. The family $T = \{1_u, a\}$ is a Schreier transversal of $\mathcal{F}(\mathcal{S})$ rooted in u. We have

$$\gamma(a) = 1_u, \gamma(\alpha) = a\alpha, \gamma(b) = ba^{-1}, \gamma(\beta) = a\beta.$$

Let us write $z := \gamma(\alpha)$, $x := \gamma(b)$, and $y := \gamma(\beta)$. By the Reidemeister-Schreier method, we obtain

$$\mathcal{G}(u,u) = \langle x,y,z \mid z^2 = yxy,\ z = xyx \rangle = \langle x,y \mid (xyx)^2 = yxy \rangle.$$

Combining this with Example 1.4.9, we obtain that the normal closure of st^{-1} and s^2 in $G := \langle s, t \mid s^3 = t^3 \rangle$ is isomorphic to $\mathcal{G}(u,u)$ by $x \mapsto ts^{-1}, y \mapsto st$.

Moreover, the group $\mathcal{G}(u,u)$ is known to be isomorphic to $G' = \langle g, h \mid g^3 = h^3 \rangle \simeq G$ by $g \mapsto xyx$ and $h \mapsto yx$ (see [Pic00, Monoïdes de type $(2,1)$]). When composed with the above isomorphism, we get that H and G' are isomorphic with $g \mapsto s^2$ and $h \mapsto st^2s^{-1}$.

1.5.2 The particular case of a subgroup of a presented group

The classical Reidemeister-Schreier method is used to compute a presentation of a subgroup H of a presented group G. We prove in this section that the Reidemeister-Schreier method for groupoids provides a generalization of this classical setting.

Let $G = \langle X \mid R \rangle$ be a presented group, and let $H \subset G$ be a subgroup. Let $F(X)$ be the free group on the set X, and let $\phi : F(X) \to G$ be the associated projection. We consider the coset groupoid G_H, and the natural functor $\pi : G_H \to G$. By Proposition 1.4.7 (presentation of groupoid of cosets), we have a presentation of G_H given by $\langle \mathcal{X} \mid \mathcal{R} \rangle$ where $\mathcal{X} := \pi^{-1}(X)$.

Let $\widetilde{H} = \phi^{-1}(H) \subset F(X)$ be the preimage of H under ϕ. Recall from [LS01, Proposition I.3.8] that a Schreier transversal (in the classical sense) for $\widetilde{H}$ in $F(X)$ is a set of words T such that

- The map $t \mapsto Ht$ is a bijection between T and $H \setminus G$.
- The set T is stable under prefix (in particular, the empty word lies in T).

Let T be a Schreier transversal in the classical sense for $\widetilde{H}$ and $F(X)$. The set of paths $\{t_H\}_{t \in T}$ is a Schreier transversal rooted in H in $\mathcal{F}(\mathcal{X})$ in the sense of Definition 1.5.1. Conversely, if $\{m_{[g]}\}_{Hg \in \mathrm{Ob}(\mathcal{F}(\mathcal{X}))}$ is a Schreier transversal rooted in H in the sense of Definition 1.5.1, then the set of words $\{\widetilde{\pi}(m_{[g]})\}_{[g] \in H \setminus G}$ is a Schreier transversal for $\widetilde{H}$ and $F(X)$ in the classical sense.

Let T be a Schreier transversal for $\widetilde{H}$ in $F(X)$, and let $\{m_{[g]}\}_{[g] \in \mathrm{Ob}(\mathcal{F}(\mathcal{X}))}$ be the associated Schreier transversal of $\mathcal{F}(\mathcal{X})$ rooted in H. For $g \in G$, let $\overline{g}$ denote the element of T such that $Hg = H\overline{g}$. The elements of $\mathcal{X}$ are given by

$$\mathcal{X} = \{x_{[g]} \mid x \in X, [g] \in \mathrm{Ob}(G_H)\}.$$

The set of generators of $G_H(H,H) = H$ we obtain by our method is

$$\begin{aligned} &\{\gamma(x_{[g]}) = m_{[g]} x m_{[gx]}^{-1} \mid x \in X, [g] \in \mathrm{Ob}(G_H)\} \\ =&\{\gamma(x_{[t]}) = m_{[t]} x m_{[tx]}^{-1} \mid x \in X, t \in T\} \\ =&\{tx(\overline{tx})^{-1} \mid x \in X, t \in T\}, \end{aligned}$$

which is the same set as given in [LS01, Proposition 4.1]. Following [LS01, Proposition 4.1], we denote $\gamma(t,x) := tx(\overline{tx})^{-1}$. This element is equal to what we denoted earlier by $\gamma(x_{[t]})$.

Now for the relators. Let $r = 1$ be a relation of G. It induces the following family of relations on G_H:

$$\{r_{[g]} = 1_{[g]} \mid [g] \in \mathrm{Ob}(G_H)\} = \{r_{[t]} = 1_{[t]} \mid t \in T\}.$$

Each relation $r_{[t]} = 1_{[t]}$ induces a relation on $G_H(H,H)$, given by

$$m_{[t]} r_{[t]} m_{[t]}^{-1} = 1.$$

If $r = y_1y_2y_3\cdots y_k$ is expressed as a word in $X \cup X^{-1}$, we have

$$\begin{aligned} m_{[t]}r_{[t]}m_{[t]}^{-1} &= \gamma(y_{1[t]})\gamma(y_{2[ty_1]})\cdots\gamma(y_{k[ty_1\cdots y_{k-1}]}) \\ &= \gamma(t, y_1)\gamma(\overline{ty_1}, y_2)\cdots\gamma(\overline{ty_1\cdots y_{k-1}}, y_k) \\ &= \gamma(1,t)\gamma(t, y_1)\gamma(\overline{ty_1}, y_2)\cdots\gamma(\overline{ty_1\cdots y_{k-1}}, y_k)\gamma(\overline{ty_1\cdots y_k}, t^{-1}), \end{aligned}$$

since both $\gamma(1,t)$ and $\gamma(\overline{ty_1\cdots y_k}, t^{-1})$ are trivial. Thus the relations we obtain with our method are the same as those given in [LS01, Proposition 4.1], and Proposition 1.5.4 applied to a groupoid of cosets gives a new proof of [LS01, Proposition 4.1].

Remark 1.5.7. Let X be a transitive G-set, and let $x \in X$. There is a natural isomorphism between the category of cosets for the stabilizer of x in G and the graph of the action of G on X. In particular, our method gives a way to compute a presentation of the stabilizer starting from the graph of the action.

Chapter 2

Garside categories and Garside groupoids

In this introductory chapter, we present the notion of Garside category, which we will use to study complex braid groups in the second part. This chapter is extracted mostly from [DDGKM] and contains no new result. We include it in order to set the notation we will use throughout this work.

Contents

The concepts of Garside categories and Garside groupoids originate from works of Krammer [Kra08], Digne and Michel [DM06], and Bessis [Bes07]. They aim to generalize the prior notions of Garside groups and monoids, introduced by Dehornoy and Paris in [DP99]. More recently, these notions were unified and generalized into the concept of category endowed with a Garside family. This is the point of view adopted in [DDGKM].

In our study of complex braid groups, we will not need the generality provided by [DDGKM]. Thus we can restrict our attention to a concept of Garside category closer to the definition given in [Bes07]. This chapter serves as a summary of some of the main results of [DDGKM] adapted to our setting.

2.1 Garside maps

As we said, the general objects of study in [DDGKM] are categories endowed with so-called Garside families. That is, distinguished families of morphisms giving rise to interesting decompositions of elements of the ambient category. We restrict ourselves to the situation of a finite Garside family, arising as the set of divisors of a particular family of elements: a Garside map. We give in this section the definition of a Garside map, and how it induces a distinguished decompositions of elements of the ambient category.

2.1.1 First definitions

Throughout this section, we consider a category $\mathcal{C}$ with no nontrivial invertible morphisms.

Definition 2.1.1 (Lcm and gcd). Let $u \in \mathrm{Ob}(\mathcal{C})$. Assume that $\mathcal{C}$ is left-cancellative, and let $f, g \in \mathcal{C}(u, -)$.

- The join of f, g in $(\mathcal{C}(u, -), \preccurlyeq)$ (should it exist) is denoted by $f \vee g$ and is called the *right-lcm* of f and g.
- The meet of f, g in $(\mathcal{C}(u, -), \preccurlyeq)$ (should it exist) is denoted by $f \wedge g$ and is called the *left-gcd* of f and g.

Assume now that $\mathcal{C}$ is right-cancellative, and let $f, g \in \mathcal{C}(-, u)$.

- The join of f, g in $(\mathcal{C}(-, u), \succcurlyeq)$ (should it exist) is denoted by $f \vee_L g$ and is called the *left-lcm* of f and g.
- The meet of f, g in $(\mathcal{C}(-, u), \succcurlyeq)$ (should it exist) is denoted by $f \wedge_R g$ and is called the *right-gcd* of f and g.

The left-cancellativity (resp. right-cancellativity) condition ensures that $(\mathcal{C}(u, -), \preccurlyeq)$ (resp. $(\mathcal{C}(-, u), \succcurlyeq)$) is a poset, so that we can define joins and meets.

Remark 2.1.2. It is possible to formulate definitions for lcms and gcds without requiring $\mathcal{C}^\times$ to be trivial. For instance, a right-lcm of $f, g \in \mathcal{C}(u, -)$ would be a morphism $m \in \mathcal{C}(u, -)$ such that, for all $h \in \mathcal{C}(u, -)$, $f, g \preccurlyeq h$ implies $m \preccurlyeq h$. This situation can be summarized in the following diagram

$$\begin{tikzcd}
{} \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, bend left, "m"] \arrow[ddrr, "h"] & {} \arrow[d] \arrow[ddr, bend left] & \\
{} \arrow[r] \arrow[drr, bend right] & {} \arrow[dr, dashed, "\exists!"] & \\
& & {}
\end{tikzcd}$$

One may recognize a pushout diagram in $\mathcal{C}$ (left-cancellativity ensures that the dashed map is unique). In this general setting, gcds and lcms are defined by universal properties and are only unique up to (canonical) isomorphisms.

Since left-divisibility is known to be invariant under left-multiplication [DDGKM, Lemma II.2.4], we easily obtain that left-gcds and right-lcms (should they exist) are also invariant under left-multiplication.

Lemma 2.1.3 (Invariance of $\wedge$ and $\vee$ under left-multiplication). *Let $u, v \in \mathrm{Ob}(\mathcal{C})$, and assume that $\mathcal{C}$ is left-cancellative. Let also $f \in \mathcal{C}(u, v)$ and $g, h \in \mathcal{C}(v, -)$ be such that $fg \wedge fh$ (resp. $fg \vee fh$) exists. We have that $g \wedge h$ (resp. $g \vee h$) exists and*

$$f(g \wedge h) = fg \wedge fh \text{ (resp. } f(g \vee h) = fg \vee fh).$$

Dually, one can show that, since right-divisibility is invariant under right-multiplication, right-gcds and left-lcms (should they exist) are invariant under right-multiplication.

Lemma 2.1.4 (Invariance of $\wedge_R$ and $\vee_L$ under right-multiplication). *Let $u, v \in \mathrm{Ob}(\mathcal{C})$, and assume that $\mathcal{C}$ is right-cancellative. Let also $f \in \mathcal{C}(u, v)$ and $g, h \in \mathcal{C}(-, u)$ be such that $gf \wedge_R hf$ (resp. $gf \vee_L hf$) exists. We have that $g \wedge_R h$ (resp. $g \vee_L h$) exists and*

$$(g \wedge_R h)f = gf \wedge_R hf \text{ (resp. } (g \vee_L h)f = gf \vee_L hf).$$

Let $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ be a family of morphisms in $\mathcal{C}$ indexed by the objects of $\mathcal{C}$. We denote

$$\mathrm{Div}(\Delta) := \bigcup_{u \in \mathrm{Ob}(\mathcal{C})} \mathrm{Div}(\Delta(u)) \text{ and } \mathrm{Div}_R(\Delta) := \bigcup_{u \in \mathrm{Ob}(\mathcal{C})} \mathrm{Div}_R(\Delta(u)),$$

where $\mathrm{Div}(\Delta(u))$ (resp. $\mathrm{Div}_R(\Delta(u))$) denotes the set of morphisms in $\mathcal{C}$ which left-divide (resp. right-divide) $\Delta(u)$.

Definition 2.1.5 (Garside category). [DDGKM, Definition V.2.19]
Let $\mathcal{C}$ be a left-cancellative category with no nontrivial isomorphism. A *Garside map* in $\mathcal{C}$ is a map $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ satisfying the following assumptions:

- For $u \in \mathrm{Ob}(\mathcal{C})$, the source of $\Delta(u)$ is u. The target of $\Delta(u)$ is denoted by $\phi(u)$.
- The family $\mathcal{S} := \mathrm{Div}(\Delta)$ is finite and generates $\mathcal{C}$. We call its elements the *simple morphisms*.
- The families $\mathrm{Div}(\Delta)$ and $\mathrm{Div}_R(\Delta)$ are equal. We say that Δ is a *balanced map*.
- For $u \in \mathrm{Ob}(\mathcal{C})$ and $g \in \mathcal{C}(u, -)$, the elements g and $\Delta(u)$ admit a left-gcd.

A couple $(\mathcal{C}, \Delta)$, where $\mathcal{C}$ is left-cancellative with no nontrivial invertible morphisms, and Δ is a Garside map in $\mathcal{C}$, is called a *Garside category*.

In [DDGKM, Definition V.2.19], there is no assumption regarding the finiteness of the set $\mathcal{S}$. We take this condition as it is necessary for most effective applications (see for instance Chapter 3). On the other hand, the authors of *loc. cit.* require an additional assumption that a Garside map should be *target injective*, that is, if $u \neq v$ are objects of $\mathcal{C}$, then the targets of $\Delta(u)$ and $\Delta(v)$ should be different. This more general situation where Δ is target-injective and where $\mathcal{S}$ is not finite will be called a *quasi-Garside* category. Although we will not investigate this kind of structure in depths, we will encounter natural examples later on.

Let $(\mathcal{C}, \Delta)$ be a Garside category. By construction, identity morphisms in $\mathcal{C}$ are simple morphisms. In particular, the finiteness of $\mathcal{S}$ implies the finiteness of $\mathrm{Ob}(\mathcal{C})$. Furthermore, since Δ is balanced, identity morphisms are also right-divisors of Δ, and the map $u \mapsto \phi(u)$ is surjective. By finiteness of $\mathrm{Ob}(\mathcal{C})$ we obtain that it is bijective (i.e. Δ is always target injective when $\mathcal{S}$ is finite). Moreover, the map $u \to \phi(u)$ on $\mathrm{Ob}(\mathcal{C})$ actually extends to a finite order automorphism of the category $\mathcal{C}$.

Proposition 2.1.6 (Garside automorphism). *[DDGKM, Proposition V.1.28 and V.2.17] Let $(\mathcal{C}, \Delta)$ be a Garside category. For all $f \in \mathcal{C}(u, v)$, there is a unique $\phi(f) \in \mathcal{C}(\phi(u), \phi(v))$ such that*

$$\Delta(u)\phi(f) = f\Delta(v).$$

The map $f \mapsto \phi(f)$ provides a finite order automorphism of $\mathcal{C}$, and Δ provides a natural transformation from $1_{\mathcal{C}}$ to ϕ. We call ϕ the Garside automorphism *of $(\mathcal{C}, \Delta)$.*

Remark 2.1.7. Let $(\mathcal{C}, \Delta)$ be a Garside category. By definition of a functor, the Garside automorphism ϕ preserves divisibility. Furthermore, as ϕ is an automorphism, two morphisms f and g admits a left-gcd (resp. left-lcm, right-gcd, right-lcm) if and only if $\phi(f)$ and $\phi(g)$ do, and this left-gcd (resp. left-lcm, right-gcd, right-lcm) is preserved by ϕ.

Example 2.1.8. Consider M a monoid, seen as a category with one object $\bullet$. A Garside map on M is simply an element $\Delta \in M$ (corresponding to $\Delta(\bullet) : \bullet \to \bullet$) which satisfies

- The sets of right- and left-divisors of Δ in M coincide.
- The set of divisors of Δ in M is finite and generates M.
- For all $m \in M$, the elements m and Δ have a left-gcd.

A Garside category which is a monoid will be called a *Garside monoid*.

The notion of Garside monoid was introduced in [DP99] with a slightly more restrictive definition (namely that Δ should be the lcm of the atoms of M). Examples of Garside monoids notably include Artin-Tits monoids of spherical type [DDGKM, Section IX.1.2].

2.1.2 Greedy normal form

One of the most elementary consequence of the existence of a Garside map on a category $\mathcal{C}$ is that it gives rise to a distinguished decomposition of the elements of $\mathcal{C}$. This decomposition (the greedy normal form) is at the core of Garside theory. For instance, the more general notion of a Garside family is actually defined by the fact that it gives rise to a similar decomposition. In our case, the greedy normal form will be a starting point for most of the constructions we make later on.

In this section, we fix a Garside category $(\mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ.

Definition 2.1.9 (Head, tail). Let $u \in \mathrm{Ob}(\mathcal{C})$. The *head* of $f \in \mathcal{C}(u, -)$ is defined as $\alpha(f) := f \wedge \Delta(u)$. Since $\mathcal{C}$ is left-cancellative, we can define the *tail* of $f \in \mathcal{C}(u, -)$ as the unique element $\omega(f)$ such that $f = \alpha(f)\omega(f)$.

Remark 2.1.10. Let $u \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathcal{C}(u, -)$. By Remark 2.1.7, we have $\alpha(\phi(f)) = \phi(\alpha(f))$ and $\omega(\phi(f)) = \phi(\omega(f))$ by left-cancellativity.

We have the following elementary property, which will prove very important later on.

Lemma 2.1.11. *[DDGKM, Proposition IV.1.50] For $f, g \in \mathcal{C}$ such that fg is defined, we have $\alpha(fg) = \alpha(f\alpha(g))$.*

Actually, [DDGKM, Proposition IV.1.50] covers the more general concept of head functions obeying the sharp $\mathcal{H}$-law. Obtaining the above lemma then boils down to stating that α is a head function in the sense of [DDGKM, Definition IV.1.47], which is proven in [DDGKM, Proposition 5.2.32 (2)].

Using the notion of head, we can define greedy paths. The philosophy is that a greedy path should be a product of elements of $\mathcal{S}$ where the leftmost term of each subpath should be as big as possible (relative to left-divisibility). This condition can be made explicit using the head function.

Definition 2.1.12 (Greedy path). A path $s_1 \cdots s_r$ of simple morphisms in $\mathcal{C}$ is *greedy* if $s_i = \alpha(s_i s_{i+1})$ for all $i \in [\![1, r-1]\!]$ and if s_r is nontrivial.

As we said, this notion is at the heart of Garside theory. A first thing we can deduce from Lemma 2.1.11 is that a path $s_1 \cdots s_r$ is greedy if and only if s_r is nontrivial and if $s_i = \alpha(s_i \cdots s_r)$ for all $i \in [\![1, r-1]\!]$. However, the definition of greediness given above is "local" in the sense that a path $s_1 \cdots s_r$ is greedy if and only if each subpath $s_i s_{i+1}$ is greedy for $i \in [\![1, r-1]\!]$. Thus we will often be interested in describing greedy paths of length 2. The following lemma gives a characterization of such paths in terms of left-gcds.

Lemma 2.1.13. *[DDGKM, Corollary V.1.54] Let $g \in \mathcal{C}$ and let $s \in \mathcal{S}$ be such that sg is defined. Let also $\overline{s}$ denotes the unique morphism such that $s\overline{s} = \Delta(u)$ (u is the source of s). The gcd $g \wedge \overline{s}$ exists, and we have*

$$\alpha(sg) = s \Leftrightarrow g \wedge \overline{s} \text{ is trivial.}$$

In particular, if g is also a nontrivial simple morphisms, then the path sg is greedy if and only if the left-gcd of g and $\overline{s}$ exists and is trivial.

We will later show that all gcds exist in $\mathcal{C}$, thus simplifying the above statement. A first corollary of Lemma 2.1.13 is that the Garside automorphism ϕ preserves greediness:

Corollary 2.1.14 (Garside automorphism preserves greediness). *[DDGKM, Proposition V.2.18] Let $s_1 \cdots s_r$ be a path of simple morphisms in $\mathcal{C}$. The path $s_1 \cdots s_r$ is greedy if and only if the path $\phi(s_1) \cdots \phi(s_r)$ is greedy.*

Proposition 2.1.15 (Second domino rule). *[DDGKM, Proposition V.1.52]*
Consider a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
\cdot \arrow[r, "s_1"] \arrow[d, "t_0"'] & \cdot \arrow[r, "s_2"] \arrow[d, "t_1"'] & \cdot \arrow[d, "t_2"'] \\
\cdot \arrow[r, "s'_1"'] & \cdot \arrow[r, "s'_2"'] & \cdot
\end{tikzcd}$$

If both $s_1 s_2$ and $t_1 s'_2$ are greedy paths, then $s'_1 s'_2$ is also greedy.

Using these results, it is possible to show that any morphism in $\mathcal{C}$ can be written uniquely as a greedy path, called its *greedy normal form*.

Proposition 2.1.16 (Greedy normal form). *Every morphism $f \in \mathcal{C}$ can be written as a unique greedy path. This unique path is given by $f = s_1 \cdots s_r$ with $s_i = \alpha(\omega^i(f))$ for $i \in [\![1, r]\!]$. We call $s_1 \cdots s_r$ the* greedy normal form *of f.*

This proposition blends together several results of [DDGKM]. By [DDGKM, Proposition V.2.25], the family $\mathcal{S} = \mathrm{Div}(\Delta)$ is a Garside family in $\mathcal{C}$, and such a family always gives rise to normal decompositions in the sense of [DDGKM, Definition III.1.17]. Furthermore, normal decompositions are characterized in term of our head function by [DDGKM, Proposition V.2.32].

Combining this proposition with Corollary 2.1.14, we obtain that the greedy normal form of a morphism is preserved under ϕ:

Corollary 2.1.17. *Let $f \in \mathcal{C}$ have greedy normal form $f = s_1 \cdots s_r$. The greedy normal form of $\phi(f)$ is given by $\phi(s_1) \cdots \phi(s_r)$.*

Since the greedy normal form of an element f of $\mathcal{C}$ is canonical, the length of the greedy decomposition of f is a well-defined integer attached to f. We call this integer the *supremum* of the morphism f.

Definition 2.1.18 (Supremum). Let $f \in \mathcal{C}$, the *supremum* of f is defined as the number $\sup(f)$ of terms in the greedy normal form of f.

The following proposition relates the supremum of the composition of two morphisms with the supremums of the factors.

Proposition 2.1.19 (Inequalities for $\sup$**).** *[DDGKM, Corollary V.1.56]*
Let f, g be two composable morphisms in $\mathcal{C}$, we have

$$\sup(f), \sup(g) \leqslant \sup(fg) \leqslant \sup(f) + \sup(g).$$

Actually, this result has the following corollary (see the proof of [DDGKM, Proposition III.1.61]), which will be useful later on.

Corollary 2.1.20. *Let $f \in \mathcal{C}$ and let $s \in \mathcal{S}$ be such that fs is defined. If s_r is the last term of the greedy normal form of f, we have*

$$\sup(fs) = \begin{cases} \sup(f) & \text{if } s_r s \in \mathcal{S}, \\ \sup(f) + 1 & \text{if } s_r s \notin \mathcal{S}. \end{cases}$$

2.1.3 Category theory consequences

The assumptions we made in the definition of Garside category seem far weaker than those appearing in earlier works (see for instance [DP99, Section 2] or [Bes07, Definition 2.4]). In particular, we do not require the existence of gcds and lcms for all elements, and we make no assumptions regarding Noetherianity. We show in this section that these classical assumptions always holds for Garside categories and can actually be seen as theorems. The results in this section are mostly taken from [DDGKM, Chapter V], (note however that they often require several preliminaries, done in the general context of Garside families).

Again, we fix for this section a Garside category $(\mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ.

First, we have that Garside categories, which are left-cancellative by definition, are also right-cancellative, and thus cancellative.

Lemma 2.1.21 (Cancellativity). *[DDGKM, Corollary V.1.37]*
Garside categories are cancellative.

Notation 2.1.22. Let $s \in \mathcal{S}(u, v)$. By definition, there are simple morphisms $s^* \in \mathcal{S}(\phi^{-1}(v), u)$ and $\overline{s} \in \mathcal{S}(v, \phi(u))$ such that $s^* s = \Delta(\phi^{-1}(u))$ and $s\overline{s} = \Delta(u)$. By cancellativity, s^* (resp. $\overline{s}$) is unique, we call it the *left-complement* of s in Δ (resp. the *right-complement* of s in Δ). Unless specified otherwise, the notation s^* and $\overline{s}$ will denote the left- and right-complements of a simple s in $\mathcal{S}$ from now on.

Proposition 2.1.23 (Strong Noetherianity). *Let $f \in \mathcal{C}(u,v)$. The set* $\mathrm{Div}(f)$ *(resp.* $\mathrm{Div}_R(f)$*) of left-divisors of f in $\mathcal{C}(u,-)$ (resp. of right-divisors of f in $\mathcal{C}(-,v)$) is finite. In particular, $\mathcal{C}$ is a strongly Noetherian category.*

Proof. Let g be a left- or right-divisor of f. By Proposition 2.1.19, we have $\sup(g) \leqslant \sup(f)$. However, as $\mathcal{S}$ is finite, there is a finite number of products of at most $\sup(f)$ simples. In particular there is a finite number of greedy such products, and g can take a finite number of values.

Since the set of divisors of any given morphism in $\mathcal{C}$ is finite, an immediate induction proves that the number of decompositions of a given morphism in $\mathcal{C}$ is also finite, hence $\mathcal{C}$ is strongly Noetherian. □

Remark 2.1.24. Consider a left-cancellative category $\mathcal{C}$, and a map $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$. If Δ is a Garside map, then $\mathcal{C}$ must be Noetherian by the above proposition. By [DDGKM, Corollary II.2.59], the atoms of $\mathcal{C}$ must then be contained in $\mathrm{Div}(\Delta)$ and $\mathrm{Div}_R(\Delta)$. This provides a first check one can perform in order to prove that a map Δ is a Garside map.

It remains to show that all gcds and lcms exist in a Garside category. To do this we need an intermediate result, which will be useful several times in this thesis. This results blends together [DDGKM, Section V.1.4] and [DDGKM, Section V.2.2], along with [DDGKM, Proposition III.1.36].

Proposition 2.1.25 (Powers of a Garside map). *For $u \in \mathrm{Ob}(\mathcal{C})$ and $m \geqslant 1$, we denote by $\Delta^m(u)$ the product $\Delta(u)\Delta(\phi(u))\cdots\Delta(\phi^{m-1}(u))$. For all integer $m \geqslant 1$, Δ^m is a Garside map in $\mathcal{C}$. Furthermore, we have*

$$\mathrm{Div}(\Delta^m) = \mathcal{S}^m := \{s_1 \cdots s_m \in \mathcal{C} \mid s_1, \ldots, s_m \in \mathcal{S}\}.$$

A first easy consequence of this proposition is the existence of common multiples in $\mathcal{C}$ (which could also be easily proven directly).

Lemma 2.1.26. *[DDGKM, Proposition V.3.44 and Corollary V.2.14] Let $u \in \mathrm{Ob}(\mathcal{C})$. Two morphisms in $\mathcal{C}(u,-)$ always have a common right-multiple. Likewise, two morphisms in $\mathcal{C}(-,u)$ always have a common left-multiple.*

In particular, we deduce from this a structural result on Garside categories which we will use in Section 2.3.

Corollary 2.1.27 (Ore category). *Garside categories are both left-Ore and right-Ore categories.*

Proof. Garside categories are cancellative by Lemma 2.1.21, they admit both left and right common-multiples by Lemma 2.1.26 □

More precisely, Proposition 2.1.25 can be used to show that lcms and gcds exist in Garside categories.

Proposition 2.1.28 (Existence of gcds and lcms). *[DDGKM, Proposition V.2.35]*
The category $\mathcal{C}$ admits all gcds and all lcms. In particular, for $u \in \mathrm{Ob}(\mathcal{C})$, the posets $(\mathcal{C}(u,-), \preccurlyeq)$ and $(\mathcal{C}(-,u), \succcurlyeq)$ are lattices.

In particular, we see that the apparent chirality of the definition of a Garside map was a mere illusion.

Corollary 2.1.29 (Opposed Garside category). *[DDGKM, Proposition V.3.44]*
The category $\mathcal{C}^{\mathrm{op}}$ is left-cancellative, and the map $\Delta^{\mathrm{op}} : \mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) \to \mathcal{C}^{\mathrm{op}}$, defined by $\Delta^{\mathrm{op}}(u) := \Delta(\phi^{-1}(u))$ is a Garside map in $\mathcal{C}^{\mathrm{op}}$.

All the above results show that, for $u \in \mathrm{Ob}(\mathcal{C})$, the monoid $\mathcal{C}(u,u)$ is a cancellative strongly Noetherian monoid. However, it may or may not be a Garside monoid. Mostly "because" it may not admit lcms and gcds (see Example 2.3.3).

2.1.4 Right-complements and presentations

As we will see below, the notion of Garside germ will provide explicit presentations of Garside categories. However, presentations induced by a (Garside) germ tend to be too big for computational purposes. For instance the presentation obtained in Example 1.2.22 using a germ required 10 generators and 12 relations (excluding the identities), whereas a presentation using the atoms only used 4 generators and 2 relations.

The purpose of this short section is to define a categorical presentation attached to a Garside category using the notion of right-complement, and to prove that this presentation is indeed a presentation of the given category.

Again, we fix for this section a Garside category $(\mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ.

Definition 2.1.30 (Complement). [DDGKM, Definition II.2.11]
Let $u \in \mathrm{Ob}(\mathcal{C})$ and let $f, g \in \mathcal{C}(u, -)$. The unique morphism $x \in \mathcal{C}$ such that $fx = f \vee g$ is denoted by $f \backslash g$ and is called the *right-complement* of f in g.
Likewise, if $f, g \in \mathcal{C}(-, u)$, then the unique morphism $y \in \mathcal{C}$ such that $yf = f \vee g$ is denoted by g/f and is called the *left-complement* of f in g.

Note that the uniqueness of $f \backslash g$ and g/f is given by cancellativity of $\mathcal{C}$. If $f \preccurlyeq g$ in $\mathcal{C}$, then $g = f \vee g$ and $f(f \backslash g) = g$. In particular, for $s \in \mathcal{S}(u, v)$ a simple morphism, we have $s \backslash \Delta(u) = \overline{s}$ and $\Delta(\phi^{-1}(v))/s = s^*$. Thus there is no conflict with Notation 2.1.22.

The complement operation will be rather important later on in Section 5.2.2, when we try and construct parabolic closures. We state some of its basic properties here. First, the fact that the concatenation of pushout diagrams is a pushout diagram can be rephrased in terms of complements.

Lemma 2.1.31 (Iteration of complement). *[DDGKM, Corollary II.2.13]*
Let f_1, f_2 be composable morphisms in $\mathcal{C}$, and let $g \in \mathcal{C}$. If g and f_1 share the same source, then we have
$$(f_1 f_2) \backslash g = f_2 \backslash (f_1 \backslash g) \text{ and } g \backslash (f_1 f_2) = (g \backslash f_1)((f_1 \backslash g) \backslash f_2).$$
If g and f_2 share the same target, then we have
$$(f_1 f_2)/g = (f_1/(g/f_2))(f_2/g) \text{ and } g/(f_1 f_2) = (g/f_2)/f_1.$$

Lemma 2.1.32 (Triple lcm). *[DDGKM, Proposition II.2.15]*
Let $f, g, h \in \mathcal{C}$. If f, g, h share the same source, then we have
$$f \backslash (g \vee h) = (f \backslash g) \vee (f \backslash h) \text{ and } (g \vee h) \backslash f = (g \backslash h) \backslash (g \backslash f) = (h \backslash g) \backslash (h \backslash f).$$
If f, g, h share the same target, then we have
$$(g \vee_L h)/f = (g/f) \vee_L (h/f) \text{ and } f/(g \vee_L h) = (f/g)/(f/g) = (f/h)/(g/h).$$

By Proposition 2.1.23, Garside categories are Noetherian. We then have that a subset $\mathcal{X}$ of $\mathcal{C}$ generates $\mathcal{C}$ if and only if it contains the atoms by [DDGKM, Corollary II.2.59]. In particular, the set of atoms of $\mathcal{C}$ is a generating set of $\mathcal{C}$.

Definition 2.1.33 (Right-lcm selector). [DDGKM, Definition 2.28]
Let $\mathcal{X}$ be a generating set of $\mathcal{C}$. A *right-lcm selector* on $\mathcal{X}$ in $\mathcal{C}$ is a map

$$\theta : \{(a, b) \in \mathcal{X} \mid a, b \text{ share the same source}\} \to \mathcal{X}^*$$

such that, for all $a, b \in \mathcal{X}$ sharing the same source, the path $\theta(a, b)$ expresses $a\backslash b$ in $\mathcal{C}$.

Let $\mathcal{X} \subset \mathcal{C}$ be a generating set, and let θ be a right-lcm selector on $\mathcal{X}$. By construction, we have $a\theta(a, b) = b\theta(b, a) = a \vee b$ in $\mathcal{C}$. Since $a\backslash a$ is trivial for all $a \in \mathcal{X}$, and since $\mathcal{C}$ has no nontrivial invertible elements, we have that $\theta(a, a)$ is the empty path.

Proposition 2.1.34. *[DDGKM, Proposition IV.3.21] Let $\mathcal{X}$ be a generating set of $\mathcal{C}$, and let θ be a right-lcm selector on $\mathcal{X}$ in $\mathcal{C}$. We have $\mathcal{C} = \langle \mathcal{X} \mid R_\theta \rangle^+$, where R_θ is the set of relations of the form $a\theta(a, b) = b\theta(b, a)$ for $a, b \in \mathcal{X}$.*

Dually, one can define the notion of left-lcm selector and prove an analogue of the above proposition by working on $\mathcal{C}^{\mathrm{op}}$.

Remark 2.1.35. In the case of a monoid, [Deh02, Théorème B'] gives an explicit criterion on a presentation of the form $\langle X \mid R_\theta \rangle^+$ so that it induces a Garside monoid. The arguments could a priori be adapted to the categorical context. This approach is algorithmic and allows one to prove that a given presentation of a monoid induces a Garside monoid. We choose to skip this theorem here, as we will instead use the notion of germ, as introduced in the next Section.

2.2 Germs

Up until now, we have given several consequences of the definition of a Garside category, but we have not yet introduced any tool that could be used to prove that a given category admits a Garside map. Another thing we also want to be able to verify is whether or not a presented category is left-cancellative.

The word reversing method, which is a natural follow up to the content of Section 2.1.4, gives a possible answer to these questions. This was the first approach considered in [DP99]. Another approach explain in [DDGKM, Section VI.2] consists in using germs. This is the approach we will use later on.

Recall from Section 1.2.2 that a germ $(\mathcal{S}, \cdot)$ is an oriented graph $\mathcal{S}$, endowed with an associative partial product $\cdot$. It naturally comes equipped with relations $\preccurlyeq_{\mathcal{S}}, \succcurlyeq_{\mathcal{S}}$, which are partial orders when $\mathcal{S}$ is cancellative and has no nontrivial invertible elements. A germ $(\mathcal{S}, \cdot)$ induces a category $\mathcal{C}(\mathcal{S})$, and we have an injective morphism of oriented graphs $\mathcal{S} \to \mathcal{C}(\mathcal{S})$.

2.2.1 Germs from Garside, Garside from germs

The purpose of this section is to show that a Garside category is somehow characterized by a germ made of its simple morphisms. Conversely, we can abstract the properties of such a germ in the notion of Garside germ, and show that a Garside germ generates a Garside category.

Definition 2.2.1 (Germ of simples). [DDGKM, Proposition VI.1.11]
Let $(\mathcal{C}, \Delta)$ be a Garside category. The *germ of simples* attached to $(\mathcal{C}, \Delta)$ is the germ $(\mathcal{S}, \cdot)$ where

- $\mathcal{S}$ is the graph of simples of $(\mathcal{C}, \Delta)$.
- For $s, t \in \mathcal{S}$, $s \cdot t$ is defined if s, t are composable and $st \in \mathcal{S}$. In this case $s \cdot t := st \in \mathcal{S}$.

Proposition 2.2.2 (Germ from Garside). *[DDGKM, Lemma VI.3.10 and VI.3.11] Let $(\mathcal{C}, \Delta)$ be a Garside category. The germ of simples $(\mathcal{S}, \cdot)$ attached to $(\mathcal{C}, \Delta)$ is a finite cancellative germ with no nontrivial invertible element. Furthermore, we have*

(a) For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u)$ is the maximum of $(\mathcal{S}(u, -), \preccurlyeq_{\mathcal{S}})$.

(b) For all $u \in \mathrm{Ob}(\mathcal{S})$, $\Delta(\phi^{-1}(u))$ is the maximum of $(\mathcal{S}(-, u), \succcurlyeq_{\mathcal{S}})$.

(c) All left-gcds exist in $\mathcal{S}$.

We can now define a Garside germ roughly as a germ which satisfies the conclusion of Proposition 2.2.2.

Definition 2.2.3 (Garside germ). A germ $(\mathcal{S}, \cdot)$ is called a *Garside germ* if

- $\mathcal{S}$ is finite, cancellative, and has no nontrivial invertible element.
- $\forall u \in \mathrm{Ob}(\mathcal{S})$, the poset $(\mathcal{S}(u, -), \preccurlyeq_{\mathcal{S}})$ admits a maximum $\Delta(u)$. The target of $\Delta(u)$ is denoted by $\phi(u)$.
- $\forall v \in \mathrm{Ob}(\mathcal{S})$ there is some $u \in \mathrm{Ob}(\mathcal{C})$ such that $\Delta(u)$ is a maximum in $(\mathcal{S}(-, v), \succcurlyeq_{\mathcal{S}})$.
- $\mathcal{S}$ admits left-gcds. That is, any two elements of $\mathcal{S}$ sharing the same source admit a meet for $\preccurlyeq_{\mathcal{S}}$.

One can reformulate Proposition 2.2.2 by saying that the germ of simples of a Garside category is a Garside germ. While this is fairly easy to prove, an important (and harder to prove) fact is that the converse also holds.

Theorem 2.2.4 (Garside from germ). *Let $(\mathcal{S}, \cdot)$ be a Garside germ. The category $\mathcal{C}(\mathcal{S})$ is left-cancellative, and the map $\Delta : \mathrm{Ob}(\mathcal{S}) = \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}(\mathcal{S})$ sending u to the maximum $\Delta(u)$ of $(\mathcal{S}(u, -), \preccurlyeq_{\mathcal{S}})$ is a Garside map. The germ $(\mathcal{S}, \cdot)$ is then identified with the germ of simples of $(\mathcal{C}(\mathcal{S}), \Delta)$.*

The proof given in [DDGKM] is intricate and relies on many distinct results. The idea is to construct the head and tail function on $\mathcal{C}(\mathcal{S})$ starting from the germ $(\mathcal{S}, \cdot)$, and to show that they have the required properties.

Let $(\mathcal{S}, \cdot)$ be a Garside germ, that we fix for the remainder of this section. Let $s \in \mathcal{S}(u, v)$. By definition of a Garside germ, there are elements $r, t \in \mathcal{S}$ such that $r \cdot s = \Delta(\phi^{-1}(v))$ and $s \cdot t = \Delta(u)$. By cancellativity, the elements r and t are unique. We write $\overline{s} := t$ and $s^* := r$ as in Notation 2.1.22.

Lemma 2.2.5. *[DDGKM, Lemma VI.3.6] Let $s, t \in \mathcal{S}$ be such that the source of t is the target of s. The product $s \cdot t$ is defined if and only if $t \preccurlyeq_{\mathcal{S}} \overline{s}$. More precisely, we have $t \cdot \overline{s \cdot t} = \overline{s}$.*

Definition 2.2.6. Let $s, t \in \mathcal{S}$ be such that the source of t is the target of s. We define $\widehat{\alpha}(s, t) := s \cdot (\overline{s} \wedge t)$. Since $\mathcal{S}$ is cancellative, we can also define $\widehat{\omega}(s, t)$ by $(\overline{s} \wedge t) \cdot \widehat{\omega}(s, t) = t$.

By construction, if $s, t \in \mathcal{S}$ are such that the source of t is the target of s, then we have $s \cdot t = \widehat{\alpha}(s, t) \cdot \widehat{\omega}(s, t)$. The function $\widehat{\alpha}$ and $\widehat{\omega}$ are then used to construct the left-gcds required in the definition of a Garside map, and to prove that $\mathcal{C}(\mathcal{S})$ is left-cancellative.

Using the notation of [DDGKM], the map $\widehat{\alpha}$ is a $\mathscr{I}$-function. Showing that it satisfies the sharp $\mathscr{I}$-law is enough to prove that $\mathcal{S}$ is a Garside family in $\mathcal{C}(\mathcal{S})$ [DDGKM, Section VI.2.2]. This result is then strengthened to show that the Garside family $\mathcal{S}$ is bounded in the sense of [DDGKM, Definition V.2.3], and we have Theorem 2.2.4. In particular, if $s, t \in \mathcal{S}$ are composable in the category $\mathcal{C}(\mathcal{S})$, then the head function in $\mathcal{C}(\mathcal{S})$ seen as a Garside category is given by $\alpha(st) = \widehat{\alpha}(s,t)$.

2.2.2 An application: interval monoids

We will use Theorem 2.2.4 throughout this thesis (notably in Chapter 4) to construct various Garside structures. A first general example we give is that of interval Garside monoids. This concept was introduced by Jean Michel (see for instance [Mic04, Section 10]). A generalization to groupoids is given in [DDGKM, Section VI.2.4].

In this section, we fix a group W, along with a positive generating set $T \subset W$ (i.e. such that the submonoid of W generated by T equals W).

The *T-length* of $w \in W$ is defined as the minimal length $\ell_T(w)$ of a T-word expressing g in W. By definition, we have

- $\ell_T(w) = 0 \Leftrightarrow w = 1$,
- $\ell_T(w) = 1 \Leftrightarrow w \in T$,
- $\forall w, v \in W, \ell_T(wv) \leqslant \ell_T(w) + \ell_T(v)$.

Using the T-length, we can define two binary relations $\preccurlyeq_T$ and $\succcurlyeq_T$ on W by

$$\forall x, y \in W, \begin{cases} x \preccurlyeq_T y \Leftrightarrow \ell_T(x) + \ell_t(x^{-1}y) = \ell_T(y), \\ y \succcurlyeq_T x \Leftrightarrow \ell_Z(x) + \ell_T(yx^{-1}) = \ell_T(y). \end{cases}$$

It is an easy exercise to show that $\preccurlyeq_T$ and $\succcurlyeq_T$ are partial orders in the group W. Of course these partial orders do not coincide in general. This leads us to considering balanced elements of W, relative to these orders.

Definition 2.2.7 (Balanced element). For $c \in W$, we write

$$I_L(c)_T = \{w \in W \mid w \preccurlyeq_T c\} \text{ and } I_R(c)_T := \{w \in W \mid c \succcurlyeq_T w\}.$$

An element $c \in W$ is *balanced* if $I_L(c)_T = I_R(c)_T$, in which case this set is simply denoted by $I(c)_T$ (or simply $I(c)$ if the set T is clear by context).

In the sequel, we will often consider situations in which the set T is globally stable under conjugacy. In which case we can apply the following lemma:

Lemma 2.2.8. *If the set T is globally stable under conjugacy in W, then the T-length function is constant on conjugacy classes in W. Furthermore, every element of W is balanced.*

Proof. Let $w, g \in W$. We claim that $\ell_T(w^g) = \ell_T(g^{-1}wg) \leqslant \ell_T(w)$. Indeed, let $s_1 \cdots s_r = w$ be a T-reduced expression of w. We have $s_i \in T$ for $i \in [\![1, r]\!]$ by definition. We then have $w^g = s_1^g s_2^g \cdots s_r^g$ with $s_i^g \in T$ for $i \in [\![1, r]\!]$ by assumption. We deduce that $\ell_T(w^g) \leqslant r$ by definition of ℓ_T. By symmetry, we also have $\ell_T(w) \leqslant \ell_T(w^g)$, whence the first result. Let now $w, v \in W$. By the first part of the proof, we have $\ell_T(w^{-1}v) = \ell_T(vw^{-1})$ since these two elements are conjugate. We easily deduce that $w \preccurlyeq_T v$ if and only if $v \succcurlyeq_T w$. $\square$

Definition 2.2.9 (Interval monoid). Let $c \in W$ be a balanced element. The *interval germ* attached to the data (W, T, c) is the germ $(I(c)_T, \cdot)$, where $s \cdot t$ is defined and equal to $u \in I(c)_T$ if and only if $st = u$ in W and $\ell_T(s) + \ell_T(t) = \ell_T(u)$.

The *interval monoid* attached to (W, T, c) is the monoid $M(c) := M(I(c)_T)$. The *interval group* attached to (W, T, c) is the enveloping group $G(c)$ of $M(c)$.

Let $c \in W$ be a balanced element. One readily checks that $(I(c)_T, \cdot)$ is indeed a germ, with $1 \in I(c)_T$ as its unit. In order to avoid confusions, we will sometimes write $\mathcal{S}$ to denote $I(c)_T$ when seen as a subset of $M(c)$, rather than as a subset of W. By construction, the relation of left-$\mathcal{S}$-divisibility (resp. right-$\mathcal{S}$-divisibility) on $\mathcal{S}$ is the same as the relations $\preccurlyeq_T$ and $\succcurlyeq_T$.

By construction, the identity map $\mathcal{S} \to I(c)_T$ induces a morphism of monoids $M(c) \to W$, which extends into a group morphism $G(c) \to W$.

Interval germs are quite particular, and they already have several interesting properties which require no further assumptions on (W, T, c).

Lemma 2.2.10. *Let $c \in W$ be a balanced element. The interval germ $(I(c)_T, \cdot)$ is cancellative and contains no nontrivial invertible element. The monoid $M(c)$ is homogeneous and its atoms set is $T \cap I(c)_T$.*

Proof. First, the set $I(c)_T$ is endowed with the length function ℓ_T. By definition of the germ structure on $I(c)_T$, this length function extends to a length morphism on $M(c)$, making it into a homogeneous monoid ($M(c)$ is generated by $I(c)_T \setminus \{1\}$, which is entirely made of elements of positive length).

Let then $s, t, t' \in I(c)_T$ such that $s \cdot t$ and $s \cdot t'$ are defined and equal. In this case, we have in particular $st = st'$ in W and $t = t'$. Thus $(I(c)_T, \cdot)$ is left-cancellative. Likewise, we show that $(I(c)_T, \cdot)$ is right-cancellative. Next, we show that $(I(c)_T, \cdot)$ has no nontrivial invertible element. If $x \in I(c)_T$ is invertible, then there is $x' \in I(c)_T$ such that $x \cdot x' = 1$. This implies that $\ell_T(x) + \ell_T(x') = 0$, thus $\ell_T(x) = \ell_T(x') = 0$ and $x = x' = 1$.

Lastly, the statement on atoms comes from Lemma 1.2.21. Indeed, an element of $I(c)_T$ has no proper left-$I(c)_T$-divisor if and only if it lies in T. □

We now reach the target of this section, which is to show that interval germs are Garside germs under some partial lattice condition. Our reference for this is [Mar21, Chapter 10], we reproduce the proof here.

Theorem 2.2.11 (Garside interval monoid). *Let W be a group, endowed with a positive generating set T and a balanced element c. Assume that $I(c)_T$ is finite and that every couple $a, b \in T \cap I(c)_T$ admits both a join in $(I(c)_T, \preccurlyeq_T)$ and in $(I(c)_T, \succcurlyeq_T)$. Then the interval germ $(I(c)_T, \cdot)$ is a Garside germ, with Garside element $c \in I(c)_T$.*

Proof. By Lemma 2.2.10, we have that $(I(c)_T, \cdot)$ is a finite cancellative germ with no nontrivial invertible element. Furthermore, c is the maximum of both $(I(c)_T, \preccurlyeq_T)$ and $(I(c)_T, \succcurlyeq_T)$ by definition. It remains to show that $I(c)_T$ admits left-gcds.

Let $r \geqslant 0$, we write $D_r := \{s \in I(c)_T \mid \ell_T(s) \leqslant r\}$. We endow it with the restrictions of $\preccurlyeq_T$ and $\succcurlyeq_T$. By definition, we have $I(c)_T = \bigcup_{r \geqslant 0} D_r$. The main part of the proof is to show that the following property holds for all $r \geqslant 0$ by induction on r.

$$(P_r): \ \forall s, t \in D_r, \ (\exists w \in D_r \mid s, t \preccurlyeq_T w) \Rightarrow (s, t \text{ have a join in } \in D_r).$$

In other words, any pair of elements in D_r which admits an upper bound in D_r admits a join in D_r. Note that we do not require the join of s and t to be a join in $I(c)_T$ a priori.

Since c is the maximum of the poset $(I(c)_R, \preccurlyeq_T)$, the property $(P_{\ell(c)})$ is simply the existence of joins in $D_{\ell(c)} = I(c)_T$. The properties P_0 and P_1 are obviously true, as $D_0 = \{1\}$, and as two distinct elements of length 1 cannot have an upper bound of length less than 2.

Let now $r > 0$ and assume that $P_1, \ldots, P_{r-1}$ holds. Let $s, t \in D_r$, and assume that $w \in D_r$ is an upper bound of s, t. Let us write $s = a \cdot s_1$ and $t = b \cdot t_1$, where $a, b \in T$. We are going to construct a diagram of this form, where each square denotes a join.

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] & \bullet \arrow[r, "s_1"] \arrow[d, "a'"] & \bullet \arrow[d, "a''"] \\
\bullet \arrow[r, "b'"'] \arrow[d, "t_1"'] & \bullet \arrow[r, "s_2"'] \arrow[d, "t_2"] & \bullet \arrow[d, "t_3"] \\
\bullet \arrow[r, "b''"'] & \bullet \arrow[r, "s_3"'] & \bullet
\end{tikzcd}$$

By assumption, we can consider the join $a \vee b$ of a and b in $I(c)_T$. We also define $a', b' \in I(c)_T$ such that $a \cdot a' = a \vee b = b \cdot b'$ in $(I(c)_T, \cdot)$. By definition, we have $a \preccurlyeq_T s \preccurlyeq_T w$ and $b \preccurlyeq_T t \preccurlyeq_T w$, and thus $a \vee b \preccurlyeq_T w$ and $\ell_T(a \vee b) \leqslant \ell_T(w)$. In particular we have $a \vee b \in D_r$ and $a', b' \in D_{r-1}$.

Then, we want to consider the join of a' and s_1. But in order to do so, we need to show that a' and s_1 have an upper bound in D_{r-1}. We have $\ell_T(s_1) = \ell_T(s) - 1 \geqslant r - 1$ and we already saw that $a' \in D_{r-1}$. Since $s, a \vee b \preccurlyeq_T w$, we can consider $k, k' \in I(c)_T$ such that $s \cdot k = w = (a \vee b) \cdot k'$. We then have

$$a \cdot s_1 \cdot k = w = a \cdot a' \cdot k',$$

and $s_1 \cdot k = a' \cdot k'$ by cancellativity. This is an upper bound of s_1 and a' of length $\ell_T(w) - 1 \preccurlyeq r - 1$. By applying P_{r-1}, we obtain that s_1 and a' have a join $s_1 \cdot a'' = a' \cdot s_2 \in D_{r-1}$. Likewise, we obtain that t_1 and b' have a join $t_1 \cdot b'' = b' \cdot t_2 \in D_{r-1}$.

Now, we want to construct the join of t_2, s_2. First, note that, as $a' \cdot s_2, b' \cdot t_2 \in D_{r-1}$, we have $s_2, t_2 \in D_{r-1}$. Now, since $s_1 \cdot k = a' \cdot k'$ is an upper bound of s_1, a' in D_{r-1}, there is some $p \in I(c)_T$ such that $a'' \cdot p = k$ and $s_2 \cdot p = k'$. Likewise, there is some $p' \in I(c)_T$ such that $b'' \cdot p' = k$ and $t_2 \cdot p' = k'$. In particular, $s_2 \cdot p = t_2 \cdot p'$ is a upper bound of s_2, t_2 of length at most $r - 1$, and we can consider the join $t_2 \cdot s_3 = s_2 \cdot t_3$ of s_2, t_2 in D_{r-1}.

We obtain that $w_0 := s \cdot a'' \cdot t_3 = (a \vee b) \cdot (s_2 \cdot t_2) = t \cdot b'' \cdot s_3$ is an upper bound of s and t such that $w_0 \preccurlyeq w$ (in particular we have $w_0 \in D_r$). Furthermore, if w' is another upper bound of s, t in D_r, then carrying out the same construction proves that $w_0 \preccurlyeq w'$, and thus w_0 is the join of s and t in D_r, which proves (P_r).

Now that we know that joins exist in $(I(c)_T, \preccurlyeq_T)$, it is easy to prove that left-gcds exist. Indeed, let $s, t \in I(c)_T$, and let $J = \{x \in I(c)_T \mid x \preccurlyeq_T s, t\}$. This set is finite since $I(c)_T$ is finite, and we can consider the join d of all its elements. One immediately obtains that d is the left-gcd of s and t in $I(c)_T$, which terminates the proof. $\square$

Note that the conditions of Theorem 2.2.11 are verified in particular if W is finite and if $(I(c)_T, \preccurlyeq_T)$ and $(I(c)_T, \succcurlyeq_T)$ are both lattices.

We finish this section by detailing two examples of Garside interval monoids, arising from one fixed group.

Example 2.2.12. Consider the symmetric group $G := \mathfrak{S}_3$. We consider the two following subsets of G

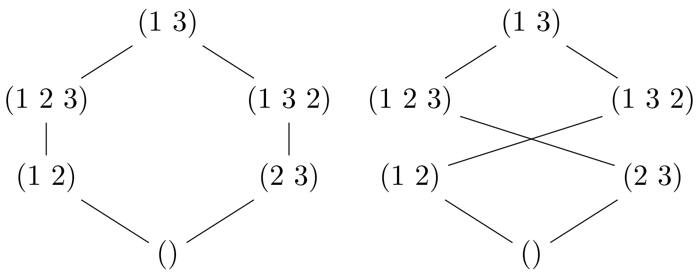
$$S := \{(1\ 2), (2\ 3)\},\ \ R := \{(1\ 2), (2\ 3), (1\ 3)\}.$$

These sets are known to positively generate G. The length of the elements of G relative to these S and R is given by

g	()	(1 2)	(2 3)	(1 3)	(1 2 3)	(1 3 2)
$\ell_S(g)$	0	1	1	3	2	2
$\ell_R(g)$	0	1	1	1	2	2

We have $(1\ 2) \preccurlyeq_S (1\ 2\ 3)$ but not $(1\ 2\ 3) \succcurlyeq_S (1\ 2)$, thus $(1\ 2\ 3)$ is not a balanced element for the set S, even though both $I_R((1\ 2\ 3))_S = \{(), (1\ 2), (1\ 2\ 3)\}$ and $I_L((1\ 2\ 3))_S = \{(), (2\ 3), (1\ 2\ 3)\}$ are lattices. On the other hand, the set R is stable under conjugacy, thus every element of G is balanced for R by Lemma 2.2.8.

The element $(1\ 3)$ is balanced for S, with $I((1\ 3))_S = G$, the respective Hasse diagrams of $I((1\ 3))_S$ for $\preccurlyeq_S$ and $\succcurlyeq_S$ are given by

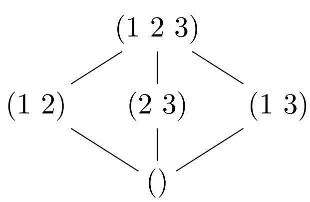


We see that both $(I((1\ 3))_S, \preccurlyeq_S)$ and $(I((1\ 3))_S, \succcurlyeq_S)$ are lattices. We then obtain a Garside monoid

$$\begin{aligned} M_1 &= \langle s, t, a, b, \Delta_1 \mid st = a, ts = b, sb = bt = as = ta = \Delta_1 \rangle^+ \\ &= \langle s, t \mid sts = tst \rangle^+ \end{aligned}$$

with Garside element $\Delta_1 = sts$.

The element $(1\ 2\ 3)$ is balanced for R, with $I((1\ 2\ 3))_R = \{(), (1\ 2), (2\ 3), (1\ 3)\}$, the Hasse diagrams of $I((1\ 2\ 3))_R$ for $\preccurlyeq_R$ and $\succcurlyeq_R$ are equal and given by



We see that both $(I((1\ 2\ 3))_R, \preccurlyeq_R)$ and $(I((1\ 2\ 3))_R, \succcurlyeq_R)$ are lattices. We then obtain a Garside monoid

$$\begin{aligned} M_2 &= \langle \alpha, \beta, \gamma, \Delta_2 \mid \alpha\beta = \beta\gamma = \gamma\alpha = \Delta_2 \rangle^+ \\ &= \langle \alpha, \beta, \gamma \mid \alpha\beta = \beta\gamma = \gamma\alpha \rangle^+, \end{aligned}$$

with Garside element $\Delta_2 = \alpha\beta$.

The element (1 3 2) is also balanced for R, but it is a conjugate of (1 2 3). Since R is stable under conjugacy, the interval germs obtained from (1 2 3) and (1 3 2) are isomorphic and induce isomorphic Garside monoids.

2.3 Garside groupoids

Garside monoids originally arose as a tool to study Artin groups of spherical type. That is to say, even though we considered categories and monoid until now, our true goal is to study groups and group theoretic questions. In this section, we give an introduction to the notion of Garside groupoid, and how the work above on Garside categories can be used to describe and understand Garside groupoids.

Definition 2.3.1 (Garside groupoid). Let $(\mathcal{C}, \Delta)$ be a Garside category. The enveloping groupoid $\mathcal{G}$ of $\mathcal{C}$ is called a *Garside groupoid*. If $\mathcal{C}$ has only one object (i.e. is a Garside monoid), then we call $\mathcal{G}$ a Garside group. A group G which is equivalent (as a category) to a Garside groupoid is called a *weak Garside group*

One should keep in mind that "being a Garside groupoid" actually refers to an additional structure on a groupoid $\mathcal{G}$, rather than to an intrinsic property. In order to reflect this, we will often denote a Garside groupoid by $(\mathcal{G}, \mathcal{C}, \Delta)$, meaning that $(\mathcal{C}, \Delta)$ is a Garside category, with enveloping groupoid $\mathcal{G}$.

Remark 2.3.2. A fixed groupoid can be the enveloping groupoid of several distinct Garside monoids. Consider the two monoids

$$M_1 := \langle s, t \mid sts = tst \rangle^+ \text{ and } M_2 := \langle \alpha, \beta, \gamma \mid \alpha\beta = \beta\gamma = \gamma\alpha \rangle^+.$$

These monoids are non isomorphic as they have a different number of atoms. We saw in Example 2.2.12 that they are both Garside interval monoids, with respective sets of simples

$$\mathcal{S}_1 := \{1, s, t, st, ts, sts\} \text{ and } \mathcal{S}_2 := \{1, \alpha, \beta, \gamma, \alpha\beta\},$$

and respective Garside elements $\Delta_1 = sts$ and $\Delta_2 = \alpha\beta$.

However, the enveloping groups of M_1 and M_2 are isomorphic. An isomorphism from $G(M_2)$ to $G(M_1)$ is given by

$$\varphi : \begin{cases} \alpha \mapsto s \\ \beta \mapsto t \\ \gamma \mapsto t^{-1}st \end{cases}.$$

We then have two distinct Garside structures (G, M_1, Δ_1) and (G, M_2, Δ_2) on the same group $G \simeq G(M_1) \simeq G(M_2)$.

The terminology "weak Garside group" was introduced by Bessis in [Bes07]. Of course Garside groups are particular examples of weak Garside groups, but it seems unreasonable to expect that all weak Garside groups are Garside groups, at least not in a "natural way".

Example 2.3.3. [Bes07, Section 13] Consider the category $\mathcal{C}$ generated by the following oriented graph $\mathcal{S}$

$$\begin{tikzcd} u \arrow[rr, bend left=60, "a"] \arrow[rr, bend right=30, "b"'] & & v \arrow[ll, bend right=30, "\alpha"'] \arrow[ll, bend left=60, "\beta"] \end{tikzcd}$$

endowed with the relations $a\alpha a = b\beta b, \alpha a\alpha = \beta b\beta$. This is the category of Example 1.2.10. It is also the category presented by the germ $(\mathcal{S}, \cdot)$ introduced in Example 1.2.22. One readily checks that $\mathcal{S}$ is a Garside germ, thus $\mathcal{C}$ is a Garside category by Theorem 2.2.4 (Garside from germ).

The length functor on $\mathcal{C}$ restricts to a length morphism on $M := \mathcal{C}(u, u)$, which is then a homogeneous monoid. The monoid M is generated by $x := a\alpha, z := a\beta, t := b\alpha, y := b\beta$. (this is easily seen by induction on the length). However, M is not a Garside monoid, as it doesn't admit right-lcms. Indeed, we have two common right-multiples of x and y

$$x^2 = a\alpha a\alpha = b\beta b\alpha = yt \text{ and } xz = a\alpha a\beta = b\beta b\beta = y^2.$$

If x and y admit a right-lcm m in $\mathcal{C}(u, u)$, then $\ell(m)$ must be even (as $\mathcal{C}(u, u)$ is generated by elements of even length), nonzero, and strictly inferior to 4 as $m \prec x^2, xz$. This implies that $\ell(m) = 2$, and $x = m = y$, which is contradictory. Note that x and y have a right-lcm in $\mathcal{C}$, but this right-lcm doesn't lie in $\mathcal{C}(u, u)$.

With that being said, we showed in Example 1.5.6 that $\mathcal{G}(u, u) = \langle x, y \mid (xyx)^2 = yxy \rangle$, where $\mathcal{G}$ is the enveloping groupoid of $\mathcal{C}$. By [Pic00, Monoïdes de type $(2, 1)$], the monoid given by the same presentation is a Garside monoid, thus $\mathcal{G}(u, u)$ is a Garside group, even if $\mathcal{C}(u, u)$ is not a Garside monoid.

We know by Corollary 2.1.27 that Garside categories are (left- and right-)Ore categories, we can apply in particular the results of Section 1.3 to get the following result:

Theorem 2.3.4 (Garside groupoid of fractions). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid.*

(a) The natural functor $\mathcal{C} \to \mathcal{G}$ is faithful (thus we identify $\mathcal{C}$ with its image in $\mathcal{G}$).

(b) Every morphism in $\mathcal{G}$ can be written as a left fraction $a^{-1}b$, with $a, b \in \mathcal{C}$.

(c) Two left-fractions $a^{-1}b$, $a'^{-1}b'$ represent the same element in $\mathcal{G}$ if and only if there are morphisms h, h' in $\mathcal{C}$ such that $(ha, hb) = (h'a', h'b')$.

(d) The composition of two left-fractions $a^{-1}b$ and $c^{-1}d$ is given by $a^{-1}bc^{-1}d = a^{-1}x^{-1}yd = (xa)^{-1}yd$, where $xb = yc$ is a common left-multiple of b and c in $\mathcal{C}$.

Since Garside categories admit not only common multiples, but lcms and gcds, we can strengthen point (c) of the above theorem:

Lemma 2.3.5. *[DDGKM, Lemma III.2.18] Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. Every element $f \in \mathcal{G}$ can be represented by a unique left-fraction $f = a^{-1}b$, where $a \wedge b$ is trivial in $\mathcal{C}$.*

Remark 2.3.6. Since the definition of a Garside category is self-dual, we can apply the above results to $(\mathcal{C}^{\mathrm{op}}, \Delta^{\mathrm{op}})$. This proves that the enveloping groupoid of a Garside category can also be described as a groupoid of right-fractions, and that every element in the enveloping groupoid of a Garside category can be written uniquely as a right-fraction xy^{-1} with $x \wedge_R y$ trivial.

Remark 2.3.7. Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $u \in \mathrm{Ob}(\mathcal{C})$. We saw in Example 2.3.3 that $\mathcal{C}(u,u)$ is not a Garside monoid in general. However, it is still both a right-Ore and a left-Ore monoid. First, $\mathcal{C}(u,u)$ is cancellative because $\mathcal{C}$ is cancellative. Then for $a, b \in \mathcal{C}(u,u)$, by Lemma 2.1.26, a and b have a common right-multiple of the form $\Delta^k(\phi^{-k}(u))$ for some positive k. Since ϕ has finite order, there is some $k' > k$ such that $\Delta^{k'}(\phi^{-k'}(u))$ has source u. The element $\Delta^{k'}(u) \in \mathcal{C}(u,u)$ is then a common right-multiple of a and b in $\mathcal{C}(u,u)$. We show similarly that $\mathcal{C}(u,u)$ admits common left-multiples. The natural functor $\mathcal{C} \to \mathcal{G}$ then induces a monoid morphism $\mathcal{C}(u,u) \to \mathcal{G}(u,u)$, and we can actually show that $\mathcal{G}(u,u)$ is the group of fractions of $\mathcal{C}(u,u)$.

Lemma 2.3.5 gives rise to the following definition:

Definition 2.3.8 (Reduced left-fraction decomposition). Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $f \in \mathcal{G}$. The unique decomposition $f = a^{-1}b$ where $a, b \in \mathcal{C}$ are such that $a \wedge b$ is trivial is called the *reduced left-fraction decomposition* of f. We denote $D(f) := a$ (resp. $N(f) := b$) the *denominator* (resp. the *numerator*) of f. We say that $f \in \mathcal{G}$ is *positive* (resp. *negative*) if $D(f)$ (resp. $N(f)$) is trivial.

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. By construction, a morphism in $\mathcal{G}$ lies in $\mathcal{C}$ if and only if it is positive. It is negative if and only if it is the inverse of a positive element of $\mathcal{G}$.

Let $f = a^{-1}b$ be a possibly non reduced fraction, and let $d := a \wedge b$, with $da' = a$, $db' = b$. In the proof of Lemma 2.3.5, we saw that the reduced left-fraction of f is given by $a'^{-1}b'$. In particular, f is positive if and only if $d = a \preccurlyeq b$, it is negative if and only if $d = b \preccurlyeq a$.

Remark 2.3.9. Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. The Garside automorphism ϕ of $\mathcal{C}$ extends to an automorphism of $\mathcal{G}$, which we also call the Garside automorphism of $\mathcal{G}$ (and also denote ϕ). By construction, ϕ sends a left-fraction $a^{-1}b$ to $\phi(a^{-1})\phi(b) = \phi(a)^{-1}\phi(b)$. Note that, as ϕ preserves lcms and gcds in $\mathcal{C}$, it also preserves reduced left-fraction decompositions.

2.3.1 Lattice orders

As we said in Section 1.1, the divisibility relations on $\mathcal{G}$ are not informative at all: every element in $\mathcal{G}(u,-)$ left-divides any other. However, there is a much more relevant way to extend the relations $\preccurlyeq$ and $\succcurlyeq$ to the groupoid $\mathcal{G}$, which will also give rise to lattice structures.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

Definition 2.3.10 ($\mathcal{C}$-divisibility). [DDGKM, Definition II.3.14] Let $u \in \mathrm{Ob}(\mathcal{G})$.

- For $f, g \in \mathcal{G}(u,-)$, we write $f \preccurlyeq_{\mathcal{C}} g$ for $f^{-1}g \in \mathcal{C}$. We say that f *left-$\mathcal{C}$-divides* g, or that g is a *right-$\mathcal{C}$-multiple* of f.
- For $f, g \in \mathcal{G}(-,u)$, we write $g \succcurlyeq_{\mathcal{C}} f$ for $gf^{-1} \in \mathcal{C}$. We say that f *right-$\mathcal{C}$-divides* g, or that g is a *left-$\mathcal{C}$-multiple* of f.

We use the symbols $\preccurlyeq_{\mathcal{C}}$ and $\succcurlyeq_{\mathcal{C}}$ as to avoid any possible confusion between these relations and the divisibility relations on $\mathcal{G}$. By construction, we have $f \preccurlyeq_{\mathcal{C}} g$ if and only if $fh = g$ for some $h \in \mathcal{C}$, and $g \succcurlyeq_{\mathcal{C}} f$ if and only if $g = hf$ for some $h \in \mathcal{C}$, whence the name of $\mathcal{C}$-divisibility. The following Lemma is easy to prove and summarizes a few elementary properties of $\preccurlyeq_{\mathcal{C}}$ and $\succcurlyeq_{\mathcal{C}}$.

Lemma 2.3.11. *Let $u, v \in \mathrm{Ob}(\mathcal{G})$.*

(a) For $f, g \in \mathcal{G}(u, -)$, we have $f \preccurlyeq_{\mathcal{C}} g$ if and only if $f^{-1} \succcurlyeq_{\mathcal{C}} g^{-1}$.

(b) The relation $\preccurlyeq_{\mathcal{C}}$ (resp. $\succcurlyeq_{\mathcal{C}}$) is a partial order on $\mathcal{G}(u, -)$ (resp. $\mathcal{G}(-, u)$).

(c) An element $f \in \mathcal{G}(u, v)$ is positive if and only if $1_u \preccurlyeq_{\mathcal{C}} f$ or $f \succcurlyeq_{\mathcal{C}} 1_v$, it is negative if and only if $f \preccurlyeq_{\mathcal{C}} 1_u$ or $1_u \succcurlyeq_{\mathcal{C}} f$.

(d) The relation $\preccurlyeq_{\mathcal{C}}$ (resp. $\succcurlyeq_{\mathcal{C}}$) is invariant under left-multiplication in $\mathcal{G}$ (resp right-multiplication in $\mathcal{G}$).

(e) Let $f, g \in \mathcal{C}(u, -)$, we have $f \preccurlyeq g$ in $\mathcal{C}$ if and only if $f \preccurlyeq_{\mathcal{C}} g$ in $\mathcal{G}$.

(f) Let $f, g \in \mathcal{C}(-, u)$, we have $f \succcurlyeq g$ in $\mathcal{C}$ if and only if $f \succcurlyeq_{\mathcal{C}} g$ in $\mathcal{G}$.

Point (e) and (f) of the above Lemma prove that the $\mathcal{C}$-divisibility relations extend the divisibility relations in $\mathcal{C}$. We know by Proposition 2.1.28 (existence of gcds and lcms) that the latter are lattice orders in $\mathcal{C}$. This property actually extends to $\preccurlyeq_{\mathcal{C}}$ and $\succcurlyeq_{\mathcal{C}}$.

Proposition 2.3.12 (Lattice properties). *Let u be an object of $\mathcal{G}$. The posets $(\mathcal{G}(u, -), \preccurlyeq_{\mathcal{C}})$ and $(\mathcal{G}(u, -), \succcurlyeq_{\mathcal{C}})$ are lattices, and the inclusions $\mathcal{C}(u, -) \subset \mathcal{G}(u, -)$, $\mathcal{C}(-, u) \subset \mathcal{G}(-, u)$ preserve joins and meets.*

Proof. Let $f, g \in \mathcal{C}(u, -)$, we show that $f \wedge g$ (resp. $f \vee g$) is the meet (resp. join) of f and g in $\mathcal{G}(u, -)$. For $a^{-1}b \in \mathcal{G}(u, -)$, we have

$$\begin{aligned} a^{-1}b \preccurlyeq_{\mathcal{C}} f, g &\Leftrightarrow b \preccurlyeq_{\mathcal{C}} af, ag \\ &\Leftrightarrow b \preccurlyeq af, ag \\ &\Leftrightarrow b \preccurlyeq a(f \wedge g) \\ &\Leftrightarrow a^{-1}b \preccurlyeq_{\mathcal{C}} f \wedge g. \end{aligned}$$

Thus $f \wedge g$ is the meet of f and g in $(\mathcal{G}(u, -), \preccurlyeq_{\mathcal{C}})$. Likewise, we get $f, g \preccurlyeq_{\mathcal{C}} a^{-1}b \Leftrightarrow f \vee g \preccurlyeq_{\mathcal{C}} a^{-1}b$, and $f \vee g$ is the join of f and g in $(\mathcal{G}(u, -), \preccurlyeq_{\mathcal{C}})$.

Let now $f \in \mathcal{G}(v, u)$ and let $g, g' \in \mathcal{G}(u, -)$. We show that, if fg, fg' have a meet d in $\mathcal{G}(v, -)$, then g, g' have a meet in $\mathcal{G}(u, -)$. Let $x \in \mathcal{G}(u, -)$. We have

$$\begin{aligned} x \preccurlyeq_{\mathcal{C}} g, g' &\Leftrightarrow fx \preccurlyeq_{\mathcal{C}} fg, fg' \\ &\Leftrightarrow fx \preccurlyeq_{\mathcal{C}} fg \wedge fg' \\ &\Leftrightarrow x \preccurlyeq f^{-1}(fg \wedge fg'). \end{aligned}$$

Thus the meet of g and g' exists and is equal to $f^{-1}d$. Likewise, if fg, fg' have a join in $\mathcal{G}(v, -)$, then g, g' have a join in $\mathcal{G}(u, -)$.

Lastly, let $g := a^{-1}b, g' := a'^{-1}b' \in \mathcal{G}(u, -)$. We consider $m = xa = x'a'$ a common left-multiple of a and a' in $\mathcal{C}$. The two morphisms $mg = xb$ and $mg' = x'b'$ lie in $\mathcal{C}$, thus they have a meet $mg \wedge mg'$ (resp. a join $mq \vee mg'$) in $\mathcal{G}(u, -)$ by the first part of the proof. The second part then gives that g and g' have a meet (resp. a join) in $\mathcal{G}(u, -)$.

The statements on $\mathcal{C}(-,u)$ and $\mathcal{G}(-,u)$ are obtained by applying the above argument to $\mathcal{C}^{\mathrm{op}}$. $\square$

Since $\preccurlyeq_{\mathcal{C}}$ and $\succcurlyeq_{\mathcal{C}}$ extend the relations $\preccurlyeq$ and $\succcurlyeq$, we will denote the joins and meets for $\preccurlyeq_{\mathcal{C}}$ and $\succcurlyeq_{\mathcal{C}}$ in the same way than the lcms and gcds in $\mathcal{C}$. Since the relation $\preccurlyeq_{\mathcal{C}}$ (resp. $\succcurlyeq_{\mathcal{C}}$) is invariant under left-multiplication (resp. right-multiplication), we have the following corollary:

Corollary 2.3.13. *Let $u \in \mathrm{Ob}(\mathcal{G})$.*

(a) For $f \in \mathcal{G}(-,u)$, $g, g' \in \mathcal{G}(u,-)$, we have $f(g \wedge g) = fg \wedge fg'$ and $f(g \vee g') = fg \vee fg'$.

(b) For $f \in \mathcal{G}(u,-)$, $g, g' \in \mathcal{G}(-,u)$, we have $(g \wedge_R g')f = gf \wedge_R g'f$ and $(g \vee_L g')f = gf \vee_L g'f$.

As an other corollary of Proposition 2.3.12, we obtain a general group-theoretic result on the structure of weak Garside groups.

Corollary 2.3.14. *Weak Garside groups are torsion-free.*

Proof. Let G be a weak Garside group. By definition, we have a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ such that $G \simeq \mathcal{G}(u,u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$. It is the sufficient to show the result for groups of the form $\mathcal{G}(u,u)$. Let $x \in \mathcal{G}(u,u)$ have finite order n. Let $x' = 1_u \vee x \vee x^2 \vee \cdots \vee x^{n-1}$. By Corollary 2.3.13, we have $xx' = x \vee x^2 \vee x^3 \vee \cdots \vee 1_u = x'$, and thus $x = 1_u$. $\square$

Lastly, we give an analogue of Remark 2.1.7 in $\mathcal{G}$:

Lemma 2.3.15. *The Garside automorphism ϕ preserve left- and right-$\mathcal{C}$-divisibility. Furthermore, ϕ also preserves $\wedge, \vee, \wedge_R$ and $\vee_L$.*

Proof. The first assertion is an immediate consequence of the fact that ϕ is an automorphism of $\mathcal{G}$ which preserves $\mathcal{C}$. Let then $u \in \mathrm{Ob}(\mathcal{G})$, let $g, g' \in \mathcal{G}(u,-)$ and let $f \in \mathcal{G}(\phi(u),-)$. We have

$$\begin{aligned} f \preccurlyeq \phi(g \wedge g') &\Leftrightarrow \phi^{-1}(f) \preccurlyeq g \wedge g' \\ &\Leftrightarrow \phi^{-1}(f) \preccurlyeq g, g' \\ &\Leftrightarrow f \preccurlyeq \phi(g), \phi(g') \\ &\Leftrightarrow f \preccurlyeq \phi(g) \wedge \phi(g'). \end{aligned}$$

And $\phi(g \wedge g') = \phi(g) \wedge \phi(g')$. A similar reasoning deals with all the other cases. $\square$

2.3.2 Symmetric and left normal forms

Let $(\mathcal{C}, \Delta)$ be a Garside category. In Section 2.1.2, we constructed a normal form for arbitrary elements of $\mathcal{C}$. This kind of decompositions carry on to the enveloping groupoid $\mathcal{G}$ of $\mathcal{C}$ in two natural ways. Both extensions will be useful depending on the context. In this section, we study these two extensions and how they interact with one another.

For this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

The first normal form we consider is built on the reduced left-fraction decomposition of a morphism in $\mathcal{G}$.

Definition 2.3.16 (Symmetric normal form). [DDGKM, Definition III.2.9]
The *symmetric normal form* of $f \in \mathcal{G}$ is the unique expression

$$f = a_p^{-1} \cdots a_1^{-1} b_1 \cdots b_q,$$

where $a_1 \cdots a_p$ (resp. $b_1 \cdots b_q$) is the greedy normal form of the denominator (resp. the numerator) of f in $\mathcal{C}$.

Note that, if $f \in \mathcal{C}$, then $D(f)$ is trivial, and the symmetric normal form of f is simply its greedy normal form in $\mathcal{C}$.

Lemma 2.3.17. *Let $u, v \in \mathrm{Ob}(\mathcal{G})$, and let $a_1 \cdots a_p, b_1 \cdots b_q$ be two nontrivial greedy paths from v to u in $\mathcal{C}$. The expression*

$$a_p^{-1} \cdots a_1^{-1} b_1 \cdots b_q$$

is a symmetric normal form in $\mathcal{G}$ if and only if $a_1 \wedge b_1 = 1_v$.

Proof. Let $a := a_1 \cdots a_p$ and $b := b_1 \cdots b_q$. By construction, the considered expression is a symmetric normal form if and only if $a \wedge b = 1_v$. Since $a_1 \wedge b_1$ is a common left-divisor of a and b, one implication is trivial. Conversely, assume that $a_1 \wedge b_1 = 1_v$, and let d be a common left-divisor of a and b. We have $\alpha(d) \preccurlyeq a, b$, thus $\alpha(d) \preccurlyeq \alpha(a) = a_1, \alpha(b) = b_1$. Thus $\alpha(d) = 1_v$, which is possible only if $d = 1_v$. We then have $a \wedge b = 1_v$, which terminates the proof. □

We know by Corollary 2.1.17 that the Garside automorphism preserves greedy normal forms in $\mathcal{C}$. Since ϕ also preserves left-gcds, this implies in turn that ϕ preserves symmetric normal forms in $\mathcal{G}$.

Lemma 2.3.18 (Garside automorphism preserves symmetric normal forms).
Let $f \in \mathcal{G}$ have symmetric normal form $a_p^{-1} \cdots a_1^{-1} b_1 \cdots b_q$. The symmetric normal form of $\phi(f)$ is given by

$$\phi(f) = \phi(a_p)^{-1} \cdots \phi(a_1)^{-1} \phi(b_1) \cdots \phi(b_q).$$

The second normal form we consider makes a more decisive use of the Garside map Δ. In particular we will show that it is somehow sufficient to give a formal inverse to $\Delta(u)$ for $u \in \mathrm{Ob}(\mathcal{C})$ to recover the whole enveloping groupoid $\mathcal{G}$. We begin with a notation generalizing that of Proposition 2.1.25 (powers of a Garside map).

Notation 2.3.19. Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $m < 0$. We set $\Delta^m(u) := (\Delta^{-m}(\phi^m(u)))^{-1} \in \mathcal{G}(u, \phi^m(u))$. Under this notation, we have $\Delta^m(u) \in \mathcal{G}(u, \phi^m(u))$ and $(\Delta^m(u))^{-1} = \Delta^{-m}(\phi^m(u))$ for all $m \in \mathbb{Z}$ and for all $u \in \mathrm{Ob}(\mathcal{G})$.

Proposition 2.3.20. *[DDGKM, Proposition V.3.12] Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u, -)$. There is a unique $m \in \mathbb{Z}$ such that $f = \Delta^m(u) b$ with $b \in \mathcal{C}(\phi^m(u), -)$ and $\Delta(\phi^m(u)) \not\preccurlyeq b$.*

Using this proposition, we can give another distinguished decomposition of an element of $\mathcal{G}$. Like in Definition 2.1.18 (supremum), this distinguished decomposition gives rise to two integer parameters naturally attached to an element of $\mathcal{G}$.

Definition 2.3.21 (Left normal form). [DDGKM, Definition V.3.17]
Let $u \in \mathrm{Ob}(\mathcal{G})$. The *left normal form* of $f \in \mathcal{G}(u, -)$ is the unique expression

$$f = \Delta^k(u) s_1 \cdots s_r,$$

where $s_1 \neq \Delta(\phi^k(u))$, and the path $s_1 \cdots s_r$ is greedy in $\mathcal{C}$. We denote $\inf(f) := k$ and $\sup(f) := k + r$. We call $\inf(f)$ the *infimum* of f.

This definition is justified by Proposition 2.3.20. The left normal form of f just consists in writing down the greedy decomposition of the unique $b \in \mathcal{C}$ such that $f = \Delta^k(u)b$ (with k maximal). Note that, if $f \in \mathcal{C}$, then the left-normal form of f is a greedy path in $\mathcal{C}$, thus equal to the greedy normal form of f by Proposition 2.1.16. In this case $\sup(f)$ is, by definition, the number of terms in the greedy normal form of f, as in Definition 2.1.18 (supremum).

We saw in 2.1.17 that the Garside automorphism ϕ preserves greedy normal forms of elements of $\mathcal{C}$. Since ϕ also preserves Δ and divisibility, we obtain that the left normal form is preserved under the Garside automorphism ϕ, as was the symmetric normal form.

Lemma 2.3.22 (Garside automorphism preserves left normal form). *Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u,-)$ have symmetric normal form $\Delta^k(u)s_1\cdots s_r$. The left normal form of $\phi(f)$ is given by*

$$\phi(f) = \Delta^k(\phi(u))\phi(s_1)\cdots\phi(s_r).$$

Lemma 2.3.23. *[DDGKM, Proposition V.3.24] Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u,-)$. We have*

$$\inf(f) = \max\{n \in \mathbb{Z} \mid \Delta^n(u) \preccurlyeq_{\mathcal{C}} f\} \text{ and } \sup(f) = \inf\{n \in \mathbb{Z} \mid f \preccurlyeq_{\mathcal{C}} \Delta^n(u)\}.$$

In this case, the left normal form of f is its greedy normal form and $\sup(f)$ coincides with the value of Definition 2.1.18.

The integers $\inf(f)$ and $\sup(f)$ will prove quite important in handling the conjugacy problem later on. For now we can at least see that they encode the positivity/negativity of f.

Corollary 2.3.24. *Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u,-)$. We have $\sup(f) \geqslant \inf(f)$, with equality if and only if f has the form $\Delta^m(u)$ for $m \in \mathbb{Z}$. Furthermore, f is positive (resp. negative) if and only if $\inf(f) \geqslant 0$ (resp. $\sup(f) \leqslant 0$). Otherwise we have $\inf(f) < 0 < \sup(f)$.*

Proof. Let $f = \Delta^k(u)s_1\cdots s_r$ be the left normal form of f. We have $r = \sup(f) - \inf(f) \geqslant 0$, with equality if and only if f is a power of $\Delta(u)$. Now, for $f \in \mathcal{G}(u,-)$, we have

$$f \text{ positive} \Leftrightarrow 1_u = \Delta^0(u) \preccurlyeq_{\mathcal{C}} f \Leftrightarrow \exists n \geqslant 0 \mid \Delta^n(u) \preccurlyeq f \Leftrightarrow \inf(f) \geqslant 0,$$

$$f \text{ negative} \Leftrightarrow f \preccurlyeq_{\mathcal{C}} \Delta^0(u) = 1_u \Leftrightarrow \exists n \leqslant 0 \mid f \preccurlyeq_{\mathcal{C}} \Delta^n(u) \Leftrightarrow \sup(f) \leqslant 0.$$

Lastly, if f is neither positive nor negative, we have $\inf(f) > 0$ and $\sup(f) < 0$ by the above characterization. □

As a matter of fact, $\inf(f)$ and $\sup(f)$ can be read using the symmetric normal form of $f \in \mathcal{G}$, and not only using the left normal form. In order to see this, we need to understand how we can go from the left normal form to the symmetric normal form of a given morphism and vice versa. This is the goal of the following two propositions:

Proposition 2.3.25 (Symmetric to left normal form). *[DDGKM, Proposition V.3.35] Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u,v)$. Let $f = a_p^{-1}\cdots a_1^{-1}b_1\cdots b_q$ be the symmetric normal form of f, with $a := a_1\cdots a_p$ and $b = b_1\cdots b_q$.*

(a) If f is positive, i.e. if $a = 1_u$, then the left normal form of f is given by

$$f = \Delta^k(u)b_{k+1}\cdots b_q,$$

where $k = \inf(b)$.

(b) If f is negative, i.e. if $b = 1_v$, then the left normal form of f is given by

$$f = \Delta^{-p}(u)\phi^{-p}(\overline{a_p}) \cdots \phi^{-(k+2)}(\overline{a_{k+2}})\phi^{-(k+1)}(\overline{a_{k+1}}),$$

where $k = \inf(a)$.

(c) If f is neither positive nor negative, then the left normal form of f is given by

$$f = \Delta^{-p}(u)\phi^{-p}(\overline{a_p}) \cdots \phi^{-2}(\overline{a_2})\phi^{-1}(\overline{a_1})b_1 \cdots b_q.$$

Proposition 2.3.26 (Left to symmetric normal form). *[DDGKM, Proposition V.3.35] Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u, v)$. Let $f = \Delta^{-k}(u)s_1 \cdots s_r$ be the left normal form of f, with $s = s_1 \cdots s_r$.*

(a) If f is positive, i.e. if $k \leqslant 0$, then the symmetric normal form of f is given by

$$f = \Delta(u)\Delta(\phi(u)) \cdots \Delta(\phi^{-k-1}(u))s_1 \cdots s_r.$$

(b) If f is negative, i.e. if $r \leqslant k$, then the symmetric normal form of f is given by

$$f = \phi^{k-1}(\overline{s_1})^{-1} \cdots \phi^{k-r+1}(\overline{s_{r-1}})^{-1}\phi^{k-r}(\overline{s_r})^{-1}\Delta^{-1}(\phi^{k-r}(v)) \cdots \Delta^{-1}(\phi(v)).$$

(c) If f is neither positive nor negative, i.e. if $0 < k < r$, then the symmetric normal form of f is given by

$$f = \phi^{k-1}(\overline{s_1})^{-1} \cdots \phi(\overline{s_{k-1}})^{-1}\overline{s_k}^{-1}s_{k+1} \cdots s_r.$$

As mentioned above, these two propositions allow us to compute $\inf(f)$ and $\sup(f)$ using just the symmetric normal form of f.

Corollary 2.3.27 ($\inf$ and $\sup$ from symmetric normal form). *Let $f \in \mathcal{G}$ be written as a reduced left-fraction $f = a^{-1}b$.*

(a) If f is positive, then $\inf(f) = \inf(b)$ and $\sup(f) = \sup(b)$.

(b) If f is negative, then $\inf(f) = -\sup(a)$ and $\sup(f) = -\inf(a)$.

(c) If f is neither positive nor negative, then $\inf(f) = -\sup(a)$ and $\sup(f) = \sup(b)$.

Proof. Let $u \in \mathrm{Ob}(\mathcal{G})$ be the source of f, and let $f = a_p^{-1} \cdots a_1^{-1}b_1 \cdots b_q$ be the symmetric normal form of f. If f is positive, then the result is an immediate consequence of Proposition 2.3.25.(a). If f is negative, then Proposition 2.3.25.(b) gives $\inf(f) = -\sup(a)$ and $\sup(f) = -p + (p - k) = -k = -\inf(a)$. Lastly, if f is neither positive nor negative, then we have $\inf(f) = -\sup(a)$ and $\sup(f) = -p + p + q = q = \sup(b)$ by Proposition 2.3.25.(c). □

Another consequence of the above propositions is a general formula for the left normal form of the inverse of an element $f \in \mathcal{G}$.

Corollary 2.3.28 ($\inf$ and $\sup$ of inverse). *[DDGKM, Proposition V.3.26 and Corollary V.3.28] Let $u, v \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u, v)$ have left normal form $f = \Delta^k(u)s_1 \cdots s_r$. The left normal form of f^{-1} is given by*

$$f^{-1} = \Delta^{-k-r}(v)\phi^{-k-r}(\overline{s_r}) \cdots \phi^{-k-1}(\overline{s_1}).$$

In particular we have $\inf(f^{-1}) = -\sup(f)$ and $\sup(f^{-1}) = -\inf(f)$.

Using these results, we can give general inequalities involving inf and sup. The following proposition can be seen as an upgrade to Proposition 2.1.19 (inequalities for sup).

Proposition 2.3.29 (Inequalities for inf **and** sup**).** *[DDGKM, Proposition V.3.30] Let $f, g \in \mathcal{G}$ be such that fg is defined. We have*

$$\begin{aligned} \inf(f)+\inf(g) \leqslant \inf(fg) \leqslant \inf(f)+\sup(g), \sup(f)+\inf(g), \\ \sup(f)+\inf(g), \inf(f)+\sup(g) \leqslant \sup(fg) \leqslant \sup(f)+\sup(g). \end{aligned}$$

We finish this section by giving some insights about the situation in the opposed groupoid $\mathcal{G}^{\mathrm{op}}$. We know that $(\mathcal{C}^{\mathrm{op}}, \Delta^{\mathrm{op}})$ is a Garside category. The enveloping groupoid of $\mathcal{C}^{\mathrm{op}}$ is identified with $\mathcal{G}^{\mathrm{op}}$. We can then apply the earlier results in this section to get that every morphism f in $\mathcal{G}$ can be written uniquely as a right-fraction cd^{-1}, with $c \wedge_R d$ trivial. We call cd^{-1} the *reduced right-fraction decomposition* of f, and we denote $D_R(f) := c$ and $N_R(f) := d$. For $f, g \in \mathcal{G}^{\mathrm{op}}(u, -) = \mathcal{G}(-, u)$, we have

$$f \preccurlyeq_{\mathcal{C}^{\mathrm{op}}} g \Leftrightarrow f^{-1}g \in \mathcal{C}^{\mathrm{op}} \Leftrightarrow gf^{-1} \in \mathcal{C} \Leftrightarrow g \succcurlyeq_{\mathcal{C}} f.$$

Likewise, for $f, g \in \mathcal{G}^{\mathrm{op}}(-, u) = \mathcal{G}(u, -)$, we have $g \succcurlyeq_{\mathcal{C}^{\mathrm{op}}} f \Leftrightarrow f \preccurlyeq_{\mathcal{C}} g$.

Let $f \in \mathcal{G}$, we can define $\inf^{\mathrm{op}}(f)$ and $\sup^{\mathrm{op}}(f)$ as the values of $\inf(f)$ and $\sup(f)$ in the Garside groupoid $(\mathcal{G}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}, \Delta^{\mathrm{op}})$. These values actually coincide with the corresponding values in $\mathcal{G}$.

Lemma 2.3.30. *Let $f \in \mathcal{G}$, we have $\inf^{\mathrm{op}}(f) = \inf(f)$ and $\sup^{\mathrm{op}}(f) = \sup(f)$.*

Proof. Let $u, v \in \mathrm{Ob}(\mathcal{G})$ be the source and target of f, respectively. We have $f \in \mathcal{G}^{\mathrm{op}}(v, u)$. By Lemma 2.3.23, we have

$$\inf^{\mathrm{op}}(f) = \max\{n \in \mathbb{Z} \mid (\Delta^{\mathrm{op}})^n(v) \preccurlyeq_{\mathcal{C}^{\mathrm{op}}} f\}.$$

By definition, we have $(\Delta^{\mathrm{op}})^n(v) = \Delta^n(\phi^{-n}(v))$, and

$$\begin{aligned} \Delta^n(\phi^{-n}(v)) \preccurlyeq_{\mathcal{C}^{\mathrm{op}}} f &\Leftrightarrow f \succcurlyeq_{\mathcal{C}} \Delta^n(\phi^{-n}(v)) \\ &\Leftrightarrow \exists h \in \mathcal{C} \mid f = h\Delta^n(\phi^{-n}(u)) = \Delta^n(v)\phi^n(h) \\ &\Leftrightarrow \exists h' = \phi^n(h) \in \mathcal{C} \mid f = \Delta^n(u)h \\ &\Leftrightarrow \Delta^n(u) \preccurlyeq_{\mathcal{C}} f. \end{aligned}$$

And $\inf^{\mathrm{op}}(f) = \inf(f)$. A similar argument shows the second claim. □

Using this Lemma, we get the following result on lcms in the category $\mathcal{C}$:

Corollary 2.3.31. *Let $a, b \in \mathcal{C}(u, -)$ be such that $a \wedge b = 1_u$, and let $ax = by = a \vee b$. We have $\sup(x) = \sup(b)$ and $\sup(y) = \sup(a)$.*

Proof. We can assume that neither a nor b is trivial, otherwise the result is trivial. Let $f = a^{-1}b$. By assumption, $a^{-1}b$ is the reduced left-fraction decomposition of f. By definition, we also have $f = xy^{-1}$. We claim that xy^{-1} is the reduced right-fraction decomposition of f. Let $d = x \wedge_R y$, with $x = x'd$ and $y = y'd$. We have $ax'd = ax = by = by'd$, and $ax' = by'$ by cancellativity of $\mathcal{C}$. Since $ax = by$ is the right-lcm of a and b, we get that d is trivial, and xy^{-1} is a reduced right-fraction. Since neither a nor b is trivial, this is also the case of x and y. By Corollary 2.3.27 and Lemma 2.3.30, we have

$$\sup(x) = \sup{}^{\mathrm{op}}(x) = \sup{}^{\mathrm{op}}(f) = \sup(f) = \sup(b),$$

$$\sup(y) = \sup{}^{\mathrm{op}}(y) = -\inf{}^{\mathrm{op}}(f) = -\inf(f) = \sup(a).$$

$\square$

Chapter 3

Conjugacy in Garside groupoids

In this chapter, we present tools for studying conjugacy in Garside groupoids. Sections 3.1, 3.2 and 3.4 contain no new material and are taken from [DDGKM, Chapter VIII].

Section 3.3 is devoted to the adaptation of the notion of swap, recently defined for Garside groups by González-Meneses and Marin in [GM22], to the categorical context. We study in particular the behavior of the swap when applied to powers, allowing us to prove that powers of recurrent elements for swaps are again recurrent, a result which does not appear in [GM22].

Contents

One of the original purposes of Garside in [Gar69] was to give a solution to the conjugacy problem in the usual braid group. More recent developments in the Garside theory were geared towards giving general solutions to the conjugacy problem in Garside groups, and later in Garside groupoids. These developments led to efficient algorithms for solving the conjugacy problem, which almost all rely on efficiently computing small finite subsets characterizing conjugacy classes.

In this chapter, we give an introduction to this subject. In the first section, we introduce the general framework we will use when considering conjugacy in a category, along with the first elementary results occurring when considering conjugacy in Garside groupoids. In the second and third sections, we study two ways to solve the conjugacy problem in Garside groupoids. While the first is already known, the second is an adaptation to groupoids of a method previously only defined for Garside groups, and which is useful in handling more theoretical aspects of conjugacy.

Lastly, we consider in the fourth section a particular class of elements in Garside groupoids, called periodic elements. Their conjugacy is rich and they will play a particular role in the context of complex braid groups (see Chapter 9).

3.1 Conjugacy in categories and groupoids

In this section, we present the general framework for studying conjugacy in a category. Although the main definitions require no assumptions on the ambient category, we will soon restrict our attention to Garside categories and Garside groupoids, in order to be able to state more interesting results.

In this section, we fix an arbitrary category $\mathcal{C}$.

Definition 3.1.1 (Conjugacy in a category). [DDGKM, Definition VIII.1.1]
Let x, x' be two endomorphisms in $\mathcal{C}$. An element $f \in \mathcal{C}$ *conjugates* x to x' in $\mathcal{C}$ if $xf = fx'$ holds in $\mathcal{C}$. The endomorphisms x and x' are *conjugate* if there is some $f \in \mathcal{C}$ which conjugates x to x'. The set of all conjugates of an endomorphism x in $\mathcal{C}$ is called the *conjugacy class* of x in $\mathcal{C}$ and is denoted by $\mathrm{Cl}_{\mathcal{C}}(x)$.

When the context is clear, we will denote $\mathrm{Cl}(x)$ instead of $\mathrm{Cl}_{\mathcal{C}}(x)$.

In the case where $\mathcal{C} = G$ is a group (seen as a category with one object), one recovers the classical notion of conjugacy in a group. However, in this case, for x, g in G, there is always a unique element x' in G such that g conjugates x to x', namely $x' = g^{-1}xg$. In a category (or in a monoid), this is false in general: the conjugate of an element by another may not exist, or have several distinct values, as in the following examples:

Example 3.1.2.

- Consider the free monoid $M := \langle a, b\rangle^+$. There is no element $c \in M$ such that a conjugates b to c. In the enveloping group $G(M)$ of M, we can of course take $c = a^{-1}ba$, which does not lie in $M \subset G(M)$.
- Consider the monoid $M = \langle a, b, c, d \mid ba = ac = ad\rangle^+$. It is given by a homogeneous presentation, and we can see that $c \neq d$ in M. However, we can see that a conjugates b to both c and d. The enveloping group $G(M)$ of M is given by
$$G(M) = \langle a, b, c, d \mid a^{-1}ba = c = d\rangle = \langle a, b\rangle,$$
which is again a free group on two generators.

As usual, a cancellativity condition on $\mathcal{C}$ ensures that the conjugate of an element by another, should it exist, is unique.

Lemma 3.1.3. *Assume that $\mathcal{C}$ is left-cancellative, and let x, x' be endomorphisms in $\mathcal{C}$. If $f \in \mathcal{C}$ conjugates x to x' then x' is the unique element such that $xf = fx'$, and we denote it by x^f. We then have*
$$\mathrm{Cl}_{\mathcal{C}}(x) = \{x^f \mid f \in \mathcal{C}(u, -) \text{ such that } f \preccurlyeq xf\}.$$

Proof. Let x' and x'' both be endomorphisms such that g conjugates x to both x' and x''. We have $xf = fx' = fx''$ and $x' = x''$ by left-cancellativity. The assertion on the conjugacy class is an obvious consequence. □

In particular, if $\mathcal{C} = \mathcal{G}$ is a groupoid, then $\mathcal{C}$ is cancellative, and we have $\mathrm{Cl}_{\mathcal{G}}(x) = \{x^f \mid f \in \mathcal{G}(u, -)\}$ as one would expect.

The general concept of conjugacy in the category $\mathcal{C}$ is itself encoded in a category, where the objects are the endomorphisms in $\mathcal{C}$, and the morphisms are the conjugating elements in $\mathcal{C}$ between two endomorphisms. This category is called the conjugacy category attached to $\mathcal{C}$:

Definition 3.1.4 (Conjugacy category). [DDGKM, Definition VIII.1.1]
The *conjugacy category* $\mathrm{Conj}(\mathcal{C})$ of $\mathcal{C}$ is the category defined as follows

- The objects of $\mathrm{Conj}(\mathcal{C})$ are the endomorphisms of $\mathcal{C}$.
- For x, x' two endomorphisms of $\mathcal{C}$, the morphisms from x to x' in $\mathrm{Conj}(\mathcal{C})$ are the $f \in \mathcal{C}$ which conjugate x to x'. We will denote $\mathrm{Conj}_{\mathcal{C}}(x, x')$ or $\mathrm{Conj}(x, x')$ instead of $\mathrm{Conj}(\mathcal{C})(x, x')$. In order to avoid confusion between f seen as an element of $\mathcal{C}$ or as an element of $\mathrm{Conj}(\mathcal{C})$ going from x to x', we will sometimes denote $x \xrightarrow{f} x'$ the element f seen in $\mathrm{Conj}(x, x')$.

For x an endomorphism in $\mathcal{C}$, the set $C_{\mathcal{C}}(x) := \mathrm{Conj}(x, x)$ is called the *centralizer* of x in $\mathcal{C}$.

Note that, when $\mathcal{C}$ is left-cancellative, morphisms in $\mathrm{Conj}(\mathcal{C})$ all have the form $x \xrightarrow{f} x^f$, and we can simply write f_x for $x \xrightarrow{f} x^f$, omitting the reference to x^f.

Of course, $\mathrm{Conj}(\mathcal{C})$ is indeed a category, where composition is given by the formula

$$\left(x \xrightarrow{f} x'\right)\left(x' \xrightarrow{g} x''\right) = x \xrightarrow{fg} x''.$$

Lemma 3.1.5. *[DDGKM, Lemma VIII.1.4] The map $\pi : \mathrm{Conj}(\mathcal{C}) \to \mathcal{C}$ defined by $\pi(x) = u$ for $x \in \mathcal{C}(u, u)$ and $\pi(x \xrightarrow{f} x') = f$ is a surjective functor from $\mathrm{Conj}(\mathcal{C})$ onto $\mathcal{C}$. Furthermore, for $x \in \mathcal{C}(u, u)$, we have*

$$\pi(\mathrm{Conj}_{\mathcal{C}}(x, -)) = \{f \in \mathcal{C}(u, -) \mid f \preccurlyeq xf\},$$
$$\pi(\mathrm{Conj}_{\mathcal{C}}(-, x)) = \{f \in \mathcal{C}(-, u) \mid fx \succcurlyeq f\}.$$

Let $u \in \mathrm{Ob}(\mathcal{C})$, and let $x \in \mathcal{C}(u, u)$. By construction, $C_{\mathcal{C}}(x)$ is a submonoid of $\mathcal{C}(u, u)$. In particular, if $\mathcal{C} = \mathcal{G}$ is a groupoid, then $C_{\mathcal{G}}(x)$ is the centralizer (in the usual sense) of x in the group $\mathcal{G}(u, u)$.

Let again $\mathcal{C}$ be an arbitrary category, and let $\mathcal{G}$ be the enveloping groupoid of $\mathcal{C}$. The morphism $\mathcal{C}(u, u) \to \mathcal{G}(u, u)$ induces a morphism $C_{\mathcal{C}}(x) \to C_{\mathcal{G}}(x)$, but there is no reason for $C_{\mathcal{G}}(x)$ to be the enveloping group of $C_{\mathcal{C}}(x)$, as the following example shows:

Example 3.1.6. Consider the monoid $M = \langle a, b, c, a' \mid ab = ba', ac = ca' \rangle^+$. We have that both b and c conjugate a to a' in M, and $b^{-1}c$ obviously centralizes a in the enveloping group of M. However, we can show that the centralizer of a in M is generated by a.

Indeed, let $m \in M$ be such that $am = ma$. Since M is homogeneous, one can show by induction on the length ℓ of m that $m = a^\ell$. If $\ell = 1$, then we either have $m = a$, $m = b$ or $m = c$. Again, since M is homogeneous, it is easy to compute all the representatives of the same element of M. We see in particular that ab (resp. ac) has no representative whose last letter is a. Thus we have $m = a$ as claimed.

Now, assume that $\ell > 1$ is the length of m. No relations in the defining presentation of M can change the last letter of am into a. This proves that the last letter of any representative of m is a. We can then write $m = m'a$, and the equality $am = ma$ becomes $am'a = m'aa$. Again, since no relation in the defining presentation of M involve a word with last letter a, this implies that $am' = m'a$. By induction hypothesis, we obtain that $m' = a^\ell$ and $m = a^{\ell+1}$.

We now restrict our attention to the case where $\mathcal{C}$ is a Garside category. We first notice that $\mathrm{Conj}(\mathcal{C})$ inherits most of the structural properties of $\mathcal{C}$.

Lemma 3.1.7. *[DDGKM, Lemma VIII.1.10] Let $(\mathcal{C},\Delta)$ be a Garside category. The category* $\mathrm{Conj}(\mathcal{C})$ *is cancellative, strongly Noetherian, and it has no nontrivial invertible elements. Furthermore, for f_x and g_y in* $\mathrm{Conj}(\mathcal{C})$*, we have*

(a) $f_x \preccurlyeq g_y$ if and only if $x = y$ and $f \preccurlyeq g$ in $\mathcal{C}$,

(b) $g_y \succcurlyeq f_x$ if and only if $x^f = y^g$ and $g \succcurlyeq f$ in $\mathcal{C}$.

In particular, $\mathrm{Conj}(\mathcal{C})$ *admits all gcds and all lcms.*

Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid. The above lemma proves in particular that $\mathrm{Conj}(\mathcal{C})$ is an Ore category. It is easily checked that the groupoid of fractions $\mathcal{G}(\mathrm{Conj}(\mathcal{C}))$ attached to $\mathrm{Conj}(\mathcal{C})$ is identified with the full subcategory of $\mathrm{Conj}(\mathcal{G})$ whose objects are the endomorphism in $\mathcal{C}$. In the sequel, we will also be interested in the conjugacy of endomorphisms in $\mathcal{G}$ (and not only in $\mathcal{C}$). The following lemma ensures that studying the $\mathcal{C}$-conjugacy of an endomorphism in $\mathcal{G}$ is the same as studying its $\mathcal{G}$-conjugacy.

Lemma 3.1.8. *Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid, and let x,x' be endomorphisms in $\mathcal{G}$. Then x,x' are conjugate by an element of $\mathcal{G}$ if and only if they are conjugate by an element of $\mathcal{C}$. In particular for an endomorphism $x \in \mathcal{C}$ we have $\mathrm{Cl}_{\mathcal{C}}(x) = \{y \in \mathrm{Cl}_{\mathcal{G}}(x) \mid y \in \mathcal{C}\}$.*

Proof. Of course if x and x' are conjugate by some $g \in \mathcal{C}$, then they are conjugate by $g \in \mathcal{G}$. Conversely, let $g \in \mathcal{G}$ be such that $xg = gx'$. Since the Garside automorphism ϕ of $\mathcal{G}$ has finite order, there is a positive k such that $\Delta^k(u) \in Z(\mathcal{G}(u,u))$, where u is the source of x. Up to taking a multiple of k, we can assume that $k + \inf(g) \geqslant 0$. In this case, $g' := \Delta^k(u)g$ is an element of $\mathcal{C}$ such that $x^{g'} = x'$. $\square$

This lemma motivates the following definition:

Definition 3.1.9 (Positive conjugacy category). Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid. The *positive conjugacy category* $\mathrm{Conj}^+(\mathcal{G})$ of $(\mathcal{G},\mathcal{C},\Delta)$ is the subcategory of $\mathrm{Conj}(\mathcal{G})$ whose morphisms are the f_x with $f \in \mathcal{C}$.

In other words, $\mathrm{Conj}^+(\mathcal{G}) = \pi^{-1}(\mathcal{C})$, where $\pi : \mathrm{Conj}(\mathcal{G}) \to \mathcal{G}$ is the functor of Lemma 3.1.5. Following the arguments of the proof of [DDGKM, Lemma VIII.1.10] yields the following result:

Lemma 3.1.10. *Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid. The category* $\mathrm{Conj}^+(\mathcal{G})$ *is cancellative, strongly Noetherian, and it has no nontrivial invertible elements. Furthermore, for f_x and g_y in* $\mathrm{Conj}^+(\mathcal{G})$*, we have*

(a) $f_x \preccurlyeq g_y$ if and only if $x = y$ and $f \preccurlyeq g$ in $\mathcal{C}$,

(b) $g_y \succcurlyeq f_x$ if and only if $x^f = y^g$ and $g \succcurlyeq f$ in $\mathcal{C}$.

In particular, $\mathrm{Conj}^+(\mathcal{C})$ *admits all gcds and all lcms.*

Again, we deduce from this lemma that $\mathrm{Conj}^+(\mathcal{G})$ is both a left- and right-Ore category. The enveloping groupoid of $\mathrm{Conj}^+(\mathcal{G})$ is identified with $\mathrm{Conj}(\mathcal{G})$, and $\mathrm{Conj}(\mathcal{C})$ is the full subcategory of $\mathrm{Conj}^+(\mathcal{G})$ whose objects are the endomorphisms of $\mathcal{C}$.

Remark 3.1.11. Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid. A priori, the category $\mathrm{Conj}^+(\mathcal{G})$ admits an infinite number of objects, and as such cannot be endowed with a Garside map. However, the above lemma proves that this is the only obstruction. Actually, the map $\Delta_{\mathrm{Conj}} : \mathrm{Ob}(\mathrm{Conj}^+(\mathcal{G})) \to \mathrm{Conj}^+(\mathcal{G})$ which sends $g \in \mathcal{G}(u,u)$ to $(\Delta(u))_g : g \to \phi(g)$ is a quasi-Garside map, and $\mathrm{Conj}^+(\mathcal{G})$ is a quasi-Garside category.

In Section 4.3, we will consider some full subgroupoids of $\mathrm{Conj}(\mathcal{G})$, which we will be able to endow with a Garside groupoid structure.

3.2 Super-summit sets

The original solution by Garside of the conjugacy problem in the usual braid groups involves a finite subset of the conjugacy class of a given element x -the *summit set*- defined as the set of all conjugates of x for which the value of inf is maximal. Although this set is finite, it may be quite big in practice. This led to the introduction by El-Rifai and Morton in [EM94] of the *super-summit set*, a smaller subset of the summit set. This work was generalized in [DDGKM, Section VIII.2] to the general context of categories endowed with convenient Garside families. We state here the main results of *loc. cit.* in the context of Garside groupoids following definition 2.1.5.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

Since the definition of the super-summit sets involves inf and sup, we start with the following easy lemma, which will allow us to set aside the (trivial) case of powers of the Garside map Δ.

Lemma 3.2.1. *Let x be an endomorphism in $\mathcal{G}$, and assume that x is conjugate in $\mathcal{G}$ to an element of the form $\Delta^m(u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$. We have* $\inf(x) \leqslant m \leqslant \sup(x)$.

Proof. Let $f \in \mathcal{G}$ be such that $\Delta^m(u)f = fx$. By Proposition 2.3.29 (inequalities for inf and sup), we have

$$\begin{aligned}\inf(f) + \inf(x) \leqslant \inf(\Delta^m(u)f) &= m + \inf(f)\\ &\leqslant m + \sup(f) = \sup(\Delta^m(u)f) \leqslant \sup(f) + \sup(x).\end{aligned}$$

which give the desired result. □

Let $x \in \mathcal{G}$ is an endomorphism which is conjugate to some power $\Delta^m(u)$ of the Garside map (for $u \in \mathrm{Ob}(\mathcal{G})$). An easy consequence of the above lemma is that there is a finite subset of $\mathrm{Cl}_{\mathcal{G}}(x)$, on which both inf and sup take their maximal possible values (namely, m). This set is simply the set of elements $\Delta^m(v)$ which are conjugate to x.

This situation is in fact general and will give the definition of the super-summit set. On top of being finite, the super-summit set of a given endomorphism is effectively computable using the cycling and decycling operations, which will allow us to increase the infimum (resp. decrease the supremum) whenever it is possible.

Definition 3.2.2 (Initial factor, cycling). [DDGKM, Definition VIII.2.3]
Let $u \in \mathrm{Ob}(\mathcal{G})$, and let $x \in \mathcal{G}(u, u)$ be an endomorphism with left normal form $x = \Delta^k(u)s_1 \cdots s_r$. The *initial factor* $\mathrm{init}(x)$ of x is defined as $\phi^{-k}(s_1)$. The *cycling* of x is then defined as

$$\mathrm{cyc}(x) = x^{\mathrm{init}(x)} = \Delta^k(\phi^{-k}(v))s_2 \cdots s_r\phi^{-k}(s_1),$$

where v is the target of s_1.

Note that this definition makes sense only if $\sup(x) - \inf(x) > 0$. Replacing the definition of $\mathrm{init}(x)$ by $\mathrm{init}(x) = \alpha(x\Delta^{-k}(u))$ (with $k = \inf(x)$), we obtain a more uniform definition, which gives $\mathrm{init}(\Delta^m(u)) = 1_u$ and $\mathrm{cyc}(\Delta^m(u)) = \Delta^m(u)$ for $m \neq 0$.

By construction, we see that $\inf(\operatorname{cyc}(x)) \geqslant \inf(x)$ and $\sup(\operatorname{cyc}(x)) \leqslant \sup(x)$. We actually have the following much stronger results:

Proposition 3.2.3 (Cycling and $\inf$**).** *[DDGKM, Proposition VIII.2.5]*
Let $x \in \mathcal{G}$ be an endomorphism. Assume that x has a conjugate x' in $\mathcal{G}$ such that $\inf(x') > \inf(x)$. Then for n big enough, we have

$$\inf(\operatorname{cyc}^n(x)) > \inf(x).$$

Definition 3.2.4 (Final factor, decycling). [DDGKM, Definition VIII.2.8]
Let $x \in \mathcal{G}(u,u)$ be an endomorphism, and let $x = \Delta^k(u)s_1 \cdots s_r$ be the left normal form of x. The *final factor* $\operatorname{fin}(x)$ if x is defined as s_r. The *decycling* of x is then defined as

$$\operatorname{dec}(x) = {}^{\operatorname{fin}(x)}x := (\operatorname{fin}(x))x \operatorname{fin}(x)^{-1} = s_r \Delta^k(u) s_1 \cdots s_{r-1}.$$

Again, this definition makes sense only if $\sup(x) - \inf(x) > 0$. Otherwise we extend this definition by setting $\operatorname{fin}(\Delta^m(u)) = \Delta(\phi^{-1}(u))$ and $\operatorname{dec}(\Delta^m(u)) = \Delta^m(\phi(u))$. We choose this convention in order to get uniform results on cycling and decycling (even though these results are quite vacuous when applied to powers of the Garside map).

The decycling operation is related to the cycling operation, and actually consists in cycling the inverse (up to an application of the Garside automorphism). By [DDGKM, Discussion after Lemma VIII.2.9], we have the following lemma:

Lemma 3.2.5. *Let $x \in \mathcal{G}$ be an endomorphism. We have $\overline{\operatorname{fin}(x)} = \operatorname{init}(x^{-1})$ and $\operatorname{dec}(x) = \phi^{-1}(\operatorname{cyc}(x^{-1})^{-1})$.*

The following corollary can then be easily deduced from Proposition 3.2.3 and Corollary 2.3.28 (inf and sup of inverse).

Corollary 3.2.6 (Decycling and $\sup$**).** *[DDGKM, Corollary VIII.2.10]*
Let $x \in \mathcal{G}$ be an endomorphism. Assume that x has a conjugate x' in $\mathcal{G}$ such that $\sup(x') < \sup(x)$. Then, for n big enough, we have

$$\sup(\operatorname{dec}^n(x)) < \sup(x).$$

Combining Proposition 3.2.3 with Corollary 3.2.6 yields the following corollary:

Corollary 3.2.7. *[DDGKM, Proposition VIII.2.16] Let $x \in \mathcal{G}$ be an endomorphism. There are finite positive integers $m, n > 0$ such that for all $z \in \operatorname{Cl}_{\mathcal{G}}(x)$, we have $\inf(z) \leqslant \inf(y)$ and $\sup(z) \geqslant \sup(y)$, with $y = \operatorname{dec}^m(\operatorname{cyc}^n(x))$.*

This corollary proves that for any endomorphism $x \in \mathcal{G}$, the subset of $\operatorname{Cl}_{\mathcal{G}}(x)$ made of elements with maximal inf and minimal sup is nonempty. This set is by definition the super-summit set of x.

Definition 3.2.8 (Super-summit set). [DDGKM, Definition VIII.2.13]
Let $x \in \mathcal{G}$ be an endomorphism. The *super-summit set* of x is defined as

$$\operatorname{SSS}(x) = \{y \in \operatorname{Cl}_{\mathcal{G}}(x) \mid \forall z \in \operatorname{Cl}_{\mathcal{G}}(z),\ \inf(y) \geqslant \inf(z) \text{ and } \sup(y) \leqslant \sup(z)\}.$$

For $y \in \operatorname{SSS}(x)$, we write $\inf(\operatorname{SSS}(x)) := \inf(y)$ and $\sup(\operatorname{SSS}(x)) := \sup(y)$.

By Corollary 3.2.7, one can always reach an element of the super-summit set of a given endomorphism by applying a finite sequence of cycling, followed by a finite sequence of decycling. We also notice that the super-summit set is always a finite subset of the conjugacy class, which depends only on the conjugacy class, and not on the choice of an element of this class. We deduce that for endomorphisms $x, y \in \mathcal{G}$, we have $y \in \mathrm{Cl}_{\mathcal{G}}(x)$ if and only if $\mathrm{SSS}(x) = \mathrm{SSS}(y)$. The ability to effectively compute super-summit sets then gives a solution to the conjugacy problem in $\mathcal{G}$. The following lemma is key in making super-summit sets computable in practice. It will also be important in endowing super-summit sets with a conjugacy graph structure (see Section 4.3).

Lemma 3.2.9. *[DDGKM, Lemma VIII.2.19] Let $x \in \mathcal{G}$ be an endomorphism, and let $y \in \mathrm{SSS}(x)$. If $f \in \mathcal{C}$ is such that $y^f \in \mathrm{SSS}(x)$, then $y^{\alpha(f)} \in \mathrm{SSS}(x)$.*

Let us also mention the existence of a distinguished subset of the super-summit set, introduced by Gebhardt and González-Meneses [GG10b] in the case of Garside groups, and called the set of *sliding circuits*. Instead of considering the two operations of cycling and decycling, they consider one operation: the sliding operation.

Definition 3.2.10 (Preferred prefix, sliding). [DDGKM, Definition VIII.2.29]
Let $x \in \mathcal{G}$ be an endomorphism. The *preferred prefix* of x is $\mathfrak{p}(x) := \mathrm{init}(x) \wedge \overline{\mathrm{fin}(x)}$, and the *sliding* $\mathrm{sl}(x)$ of x is the conjugate $x^{\mathfrak{p}(x)}$ of x.

A first strong property of sliding is that it is ultimately periodic, and that it always ends up in the super-summit set.

Proposition 3.2.11. *[DDGKM, Proposition VIII.2.30] Let $x \in \mathcal{G}$ be an endomorphism. There are integers $0 \leqslant n < m$ such that $\mathrm{sl}^m(x) = \mathrm{sl}^n(x)$, and $\mathrm{sl}^k(x) \in \mathrm{SSS}(x)$ for $k \geqslant n$.*

Definition 3.2.12 (Set of sliding circuits). Let $x \in \mathcal{G}$ be an endomorphism. The *set of sliding circuits* $\mathrm{SC}(x)$ of x is the set

$$\mathrm{SC}(x) := \{y \in \mathrm{Cl}_{\mathcal{G}}(x) \mid \exists n > 0,\ \mathrm{sl}^n(y) = y\}.$$

Let $x \in \mathcal{G}$ be an endomorphism. By the above proposition, the set $\mathrm{SC}(x)$ is a nonempty subset of $\mathrm{SSS}(x)$. In particular it is finite. It can also be computed effectively [DDGKM, Algorithm VIII.2.39 and VIII.2.41], and this gives another (simpler) solution to the conjugacy problem in $\mathcal{G}$. In general, the set of sliding circuits is a strict subset of the super-summit set, as in Example 3.2.14 below.

In the sequel (Section 9.2 and Section B.1.2), we will be interested in a particular class of elements, called rigid elements. Since the definition of rigid elements is related to sliding, we give it now. Following [BGG07a, Definition 3.1], we say that an endomorphism x in $\mathcal{G}$ is *rigid* is the preferred prefix $\mathfrak{p}(x)$ is trivial. If we write the left normal form of x as $\Delta^k(u)s_1 \cdots s_r$, then x is rigid if and only if either $r = 0$ or if the word $\mathrm{init}(x)\,\mathrm{fin}(x)$ is greedy. If this is the case, then we can easily compute the left normal form of powers of x, which is given by

$$\forall m > 0,\ x^m = \Delta^{mk}(u)\phi^{(m-1)k}(s_1)\cdots\phi^{(m-1)k}(s_r)\phi^{(m-2)k}(s_1)\cdots\phi^{(m-2)k}(s_r)\cdots s_1\cdots s_r.$$

In the general case, the above formula, while still true, does not give the left normal form of x^m.

By definition, a rigid endomorphism x in $\mathcal{G}$ is such that $\mathrm{sl}(x) = x$. In particular, we have that $x \in \mathrm{SSS}(x)$ if x is rigid. Moreover, we have the following lemma:

Lemma 3.2.13. *Let $x, y \in \mathcal{G}$ be rigid endomorphisms. For any positive integer n, we have* $\mathrm{cyc}(x)^n = \mathrm{cyc}(x^n)$, $\mathrm{dec}(x)^n = \mathrm{dec}(x^n)$ *and* $\phi(x^n) = \phi(x)^n$. *Moreover, if $x^n = y^n$ for some positive integer n, then $x = y$.*

Proof. The result is trivial if $n = 1$, we assume that $n \geqslant 2$ from now on. Let $x = \Delta^k(u)s_1 \cdots s_r$ be the left normal form of x. Since x is rigid, the left normal form of x^n is

$$x^n = \Delta^{nk}(u)\phi^{(n-1)k}(s_1 \cdots s_r) \cdots \phi^k(s_1 \cdots s_r)s_1 \cdots s_r.$$

We note that the initial factor (resp. the final factor) of x^n is the same as that of x, hence the result on $\mathrm{cyc}(x^n)$ and $\mathrm{dec}(x^n)$. The result on $\phi(x^n)$ is immediate by definition of ϕ.

Moreover, we have $\inf(x^n) = n \inf(x)$ and $\sup(x^n) = n \sup(x)$. The left-weighted factorization of x can be recovered using that of x^n by taking $\Delta^{\frac{\inf(x^n)}{n}}$, followed by the last $\frac{\sup(x^n)}{n}$ terms of the left-weighted factorization of x^n. Since this depends only on the left-weighted factorization of x^n, we obtain that $x^n = y^n$ implies $x = y$ if y is another rigid element. □

Example 3.2.14. Consider the monoid $M := \langle s, t, u \mid sts = tst, tut = utu, su = us\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. Consider the element x given in left normal form by

$$x := \Delta^{-1}tsuts \cdot stu \cdot u.$$

We have $\mathrm{init}(x) = tustu$, and $\mathrm{cyc}(x) = \Delta^{-1}stu \cdot u \cdot tustu = uts$. We have $stu \cdot sts = \Delta$ thus stu is a simple morphism. In particular it lies in its own super-summit set. The super-summit set $\mathrm{SSS}(x)$ is made of the simples of M which are conjugate to stu, we easily compute that

$$\mathrm{SSS}(x) = \{stu, ust, tus, uts\}.$$

We compute the sliding of each element of the super-summit set

- We have $\mathrm{init}(stu) = \mathrm{fin}(stu) = stu$, thus $\mathfrak{p}(stu) = stu \wedge \overline{stu} = stu \wedge sts = st$ and $\mathrm{sl}(stu) = ust$.
- We have $\mathrm{init}(ust) = \mathrm{fin}(ust) = ust$, thus $\mathfrak{p}(ust) = ust \wedge \overline{ust} = ust \wedge ust = ust$ and $\mathrm{sl}(ust) = ust$.
- We have $\mathrm{init}(tus) = \mathrm{fin}(tus) = tus$, thus $\mathfrak{p}(tus) = tus \wedge \overline{tus} = tus \wedge tus = tus$ and $\mathrm{sl}(tus) = tus$.
- We have $\mathrm{init}(uts) = \mathrm{fin}(uts) = uts$, thus $\mathfrak{p}(uts) = uts \wedge \overline{uts} = uts \wedge utu = ut$ and $\mathrm{sl}(uts) = ust$.

The set of sliding circuits of x is then $\mathrm{SC}(x) = \{ust, tus\}$.

3.3 Circuits for swap

The computation of super-summit sets or other finite subsets of conjugacy classes (ultra-summit sets, sets of sliding circuits...) gives efficient solutions to the conjugacy problem in Garside groupoids. In [GM22] is introduced a new simple procedure to treat more theoretical aspects of conjugacy in Garside groups. The goal of this section is to state and prove the results in [GM22, Section 4.4 and 4.5] in the case of Garside groupoids (following Definitions 2.1.5 and 2.3.1).

3.3.1 Swap, recurrent elements

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

Definition 3.3.1 (Swap function). Let $x \in \mathcal{G}$ be an endomorphism, and let $x = a^{-1}b$ be the reduced left-fraction decomposition of x. The *left-swap* (or just the *swap*) of x is defined as $\mathrm{sw}(x) = ba^{-1}$.

Notice that saying that $x = f^{-1}g \in \mathcal{G}$ is an endomorphism amounts to saying that f and g have same target and same source. Thus the morphism gf^{-1} is defined and is an endomorphism in $\mathcal{G}$. Its source is the target of f.

Definition 3.3.2 (Recurrent element). Let $x \in \mathcal{G}$ be an endomorphism. We say that x is *recurrent for swap* (or just *recurrent*) if $\mathrm{sw}^m(x) = x$ for some $m > 0$.

This is the direct adaptation of the notion of swap in Garside groups (cf. [GM22, Definition 4.11]) to the case of a Garside groupoid.

It is worth noticing that if x is positive then it is recurrent, as its left-denominator is trivial and then $\mathrm{sw}(x) = x$. The same happens if x is negative, as in this case the left-numerator of x is trivial and then $\mathrm{sw}(x) = x$.

Remark 3.3.3. Note that, unlike cycling, decycling and sliding, the swap operation does not depend on the choice of a Garside structure, but only on the reduced left-fraction decomposition, which comes from intrinsic properties of $\mathcal{G}$ and of $\mathcal{C}$, as it only relies on lcms and gcds, which do not depend on the choice of a Garside structure.

Lemma 3.3.4. *Let $x \in \mathcal{G}$ be an endomorphism. We have* $\inf(\mathrm{sw}(x)) \geqslant \inf(x)$ *and* $\sup(\mathrm{sw}(x)) \leqslant \sup(x)$.

Proof. If x is either positive or negative, then we have $\mathrm{sw}(x) = x$ by definition, and the result is trivial. Otherwise, let $x = a^{-1}b$ be the reduced left-fraction decomposition of x. We write $p := \sup(a)$ and $q := \sup(b)$. By Corollary 2.3.27 (inf and sup from symmetric normal form), we have $\inf(x) = -p$ and $\sup(x) = q$. Let $c := b/a$ and $d := a/b$, so that $ca = db = a \vee_L b$. Let $\delta := c \wedge d$, with $\delta x = c$ and $\delta y = d$. By cancellativity, we have that $xa = yd$ is a common left-multiple of a and b. Since $ca = db$ is the left-lcm of a and b, this implies that δ is trivial. Thus $\mathrm{sw}(x) = ba^{-1} = d^{-1}c$ is the reduced left-fraction decomposition of $\mathrm{sw}(x)$. An easy induction using Lemma 2.1.31 (iteration of complement) proves that $\sup(c) \leqslant q$ and $\sup(d) \leqslant p$.

- If $\mathrm{sw}(x)$ is positive (i.e. if d is trivial), then $\inf(\mathrm{sw}(x)) \geqslant 0 \geqslant -p = \inf(x)$ and $\sup(\mathrm{sw}(x)) = \sup(c) \leqslant q = \sup(x)$.
- If $\mathrm{sw}(x)$ is negative (i.e. if c is trivial), then $\inf(\mathrm{sw}(x)) = -\sup(d) \geqslant -p = \inf(x)$ and $\sup(\mathrm{sw}(x)) = -\inf(d) \leqslant 0 = \sup(c) \leqslant q = \sup(x)$.
- If $\mathrm{sw}(x)$ is neither positive nor negative, then $\inf(\mathrm{sw}(x)) = -\sup(d) \geqslant -p = \inf(x)$ and $\sup(\mathrm{sw}(x)) = \sup(c) \leqslant q = \sup(x)$.

□

Just like for sliding, we can show that every endomorphism in $\mathcal{G}$ is pre-periodic for the swap operation. In other words, repeatedly applying swaps to any endomorphism always ends up with a recurrent endomorphism.

Lemma 3.3.5. *For every endomorphism x of $\mathcal{G}$, there are integers $0 \leqslant m < n$ such that $\mathrm{sw}^m(x) = \mathrm{sw}^n(x)$. The elements in the set $\{\mathrm{sw}^m(x), \dots, \mathrm{sw}^{n-1}(x)\}$ are all recurrent, and the set will be called a* circuit for swap.

Proof. This is a parallel of the proof of [GM22, Proposition 4.13]. The second claim is trivial, so we just need to show the first one. Let $x := a^{-1}b$ be the reduced left-fraction decomposition of x. If x is either positive or negative, then we have $\mathrm{sw}(x) = x$ and x is already recurrent. We can then assume that neither a nor b is trivial.

By Remark 3.3.3, the statement does not depend on the choice of a Garside map on $\mathcal{C}$. We can then replace the Garside map Δ by one of its powers to assume that both a and b are nontrivial simple morphisms (see Proposition 2.1.25 (powers of a Garside map)). In this case, we have $\inf(x) = -1$ and $\sup(x) = 1$ by Corollary 2.3.27 (inf and sup from symmetric normal form).

By Lemma 3.3.4, we have $\inf(\mathrm{sw}^m(x)) \geqslant -1$ and $\sup(\mathrm{sw}^m(x)) \leqslant 1$ for all $m > 0$. If $\inf(\mathrm{sw}^m(x)) > -1$ for some $m > 0$, then $\mathrm{sw}^m(x)$ is positive and in particular recurrent. If $\sup(\mathrm{sw}^m(x)) < 0$ for some $m > 0$, then $\mathrm{sw}^m(y)$ is negative and in particular recurrent. We can then assume that $\inf(\mathrm{sw}^m(x)) = -1$ and that $\sup(\mathrm{sw}^m(x)) = 1$ for all $m > 0$. Again by Corollary 2.3.27, we obtain that the symmetric normal form of $\mathrm{sw}^m(x)$ has the form $a_m^{-1}b_m$ for some $a_m, b_m \in \mathcal{S}$. Since $\mathcal{S}$ is finite, there is a finite number of such products, and the sequence $\{\mathrm{sw}^m(x)\}_{m \geqslant 0}$ must become periodic. □

An important statement related to Lemma 3.3.5 is the particular case of endomorphisms which are conjugate to positive endomorphisms. In this case, not only is the swap ultimately periodic, but it ends up on a positive endomorphism in $\mathcal{G}$ (in particular, it is ultimately constant).

Lemma 3.3.6. *Let $x \in \mathcal{G}$ be an endomorphism. If x is conjugate to a positive element, then $\mathrm{sw}^m(x) \in \mathcal{C}$ for some $m \geqslant 0$.*

Proof. This is a parallel of the proof of [GM22, Lemma 4.15]. Let $x := a^{-1}b$ be the reduced left-fraction decomposition of x, and let u (resp. v) be the source of x (resp. of a). As in the proof of Lemma 3.3.5, we can replace the Garside map Δ by one of its powers to assume that both a and b are simple morphisms. By Proposition 2.3.25 (symmetric to left normal form), we have that

$$x = a^{-1}b = \Delta^{-1}(u)a^*b$$

is the left normal form of x.

We are assuming that x is conjugate to a positive element, so there exists $c \in \mathcal{G}$ such that $cxc^{-1} \in \mathcal{C}$. By Lemma 3.1.8, we can assume that c is positive. We then have $cxc^{-1} = c\Delta^{-1}(u)a^*bc^{-1} \in \mathcal{C}$ and

$$\Delta^{-1}(w)\phi^{-1}(c)a^*b \in \mathcal{C},$$

where w is the source of c. This means that $\Delta(\phi^{-1}(w)) \preccurlyeq \phi^{-1}(c)a^*b$. Since $\Delta(\phi^{-1}(w))$ is a simple morphism, we deduce that

$$\Delta(\phi^{-1}(w)) \preccurlyeq \alpha(\phi^{-1}(c)a^*b) = \alpha(\phi^{-1}(c)\alpha(a^*b)) = \alpha(\phi^{-1}(c)a^*) \preccurlyeq \phi^{-1}(c)a^*.$$

From this we deduce that $\phi^{-1}(c)a^* \succcurlyeq \Delta(\phi^{-1}(v))$, and that

$$\Delta(\phi^{-1}(w))c = \phi^{-1}(c)\Delta(\phi^{-1}(u)) = \phi^{-1}(c)a^*a \succcurlyeq \Delta(\phi^{-1}(v))a \Rightarrow c\Delta(u) \succcurlyeq a\Delta(u) \Rightarrow a \succcurlyeq f.$$

Therefore, we have shown that if c is a positive morphism such that cxc^{-1} is positive, then $c \succcurlyeq a$. We can then decompose $c = c_1 a$, and we obtain $cxc^{-1} = c_1 a(a^{-1}g)a^{-1}c_1^{-1} = c_1(ga^{-1})c_1^{-1} = c_1 \operatorname{sw}(x)c_1^{-1}$, where $1 \preccurlyeq c_1 \prec c$.

If $\operatorname{sw}(x)$ is positive, we are done. Otherwise, we repeat the process and we will find $1 \preccurlyeq c_2 \prec c_1 \prec c$ such that $c_2 \operatorname{sw}^2(x)c_2^{-1}$ is positive, so we can repeat the process again with $\operatorname{sw}^2(x)$. This process must stop by Noetherianity, hence $\operatorname{sw}^m(x) \in \mathcal{C}$ for some $m \geqslant 0$, as we wanted to show. □

Remark 3.3.7. It is interesting to point out a consequence of the above proof: If x is conjugate to a positive element, and $\operatorname{sw}^i(x) = a_i^{-1}b_i$ is the reduced left-fraction decomposition of $\operatorname{sw}^i(x)$ for each $i \geqslant 0$, then $c = a_{m-1}\dots a_1 a_0$ is the minimal positive element, with respect to $\succcurlyeq$ such that cxc^{-1} is positive (provided m is the smallest nonnegative integer such that $\operatorname{sw}^m(x)$ is positive). In other words, iterated swaps conjugate x to a positive element in the fastest possible way (conjugating by positive elements on the left). If one prefers to use $\preccurlyeq$ and to conjugate by positive elements on the right, one can parallel the same arguments just by using reduced right-fractions and right-swaps, which are defined in the symmetric way.

We can now characterize the sets of recurrent elements in a conjugacy class, in two particular cases. Given $x \in \mathcal{G}$ and endomorphism, let $\mathsf{R}(x)$ be the set of recurrent elements conjugate to x, which coincides with the set of circuits for swap in the conjugacy class of x. Let $C^+(x)$ be the set of positive elements conjugate to x and let $C^-(x)$ be the set of negative elements conjugate to x (any of the two latter sets could be empty).

Proposition 3.3.8. *Let $x \in \mathcal{G}$ be an endomorphism. One has*

(a) If x is conjugate to a positive element, then $\mathsf{R}(x) = C^+(x)$.

(b) If x is conjugate to a negative element, then $\mathsf{R}(x) = C^-(x) = (C^+(x^{-1}))^{-1}$.

Proof. This is a parallel of the proof of [GM22, Proposition 4.17].

(a) Suppose that x is conjugate to a positive morphism. If $y \in C^+(x)$ then y is conjugate to x and $\operatorname{sw}(y) = y$, hence $y \in \mathsf{R}(x)$. Conversely, let $y \in \mathsf{R}(x)$. Since y is conjugate to x, it is conjugate to a positive element. By Lemma 3.3.6, $\operatorname{sw}^m(y)$ is positive for some $m \geqslant 0$, and the $\operatorname{sw}^{m+k}(y) = \operatorname{sw}^m(y)$ for every $k \geqslant 0$. But y is recurrent, so $\operatorname{sw}^n(y) = y$ for some $n > 0$, and this implies that $y = \operatorname{sw}^{nm}(y) = \operatorname{sw}^m(y)$. Hence y is positive, and the orbit of y under ϕ consists of just one element, namely y.

(b) If x is conjugate to a negative element, x^{-1} is conjugate to a positive element, hence $\mathsf{R}(x^{-1}) = C^+(x^{-1})$ by the previous property. Now recall that, if $z = d^{-1}c$ is a reduced left-fraction, then $z^{-1} = c^{-1}d$ is a reduced left-fraction. Hence $\operatorname{sw}(z^{-1}) = \operatorname{sw}(z)^{-1}$. Therefore, taking inverses commutes with sw and also preserves conjugations and conjugating elements. It follows that $\mathsf{R}(x^{-1}) = \mathsf{R}(x)^{-1}$, that $(C^-(x))^{-1} = C^+(x^{-1})$, and then $\mathsf{R}(x) = (\mathsf{R}(x^{-1}))^{-1} = (C^+(x^{-1}))^{-1} = C^-(x)$.

□

3.3.2 Swap and powers

At the end of Section 3.2, we saw that cycling, decycling and sliding behave pretty well relative to powers when considering rigid endomorphisms. However, the rigidity assumptions is very restrictive and not very reasonable in practice. In this section, we show that the behavior of the swap relative to powers is much easier to understand, and only requires considering recurrent elements.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix $x \in \mathcal{G}$ a recurrent endomorphism with source $u_0 \in \mathrm{Ob}(\mathcal{G})$.

We write $a_0^{-1} b_0 = x$ for the reduced left-fraction decomposition of x. For $k \geqslant 0$, we define u_k as the source of $\mathrm{sw}^k(x)$, and $a_k^{-1} b_k$ as the reduced left-fraction decomposition of $\mathrm{sw}^k(x)$. Since $\mathrm{sw}^n(x) = x$ for some integer n, we can extend these definition to negative integers by declaring $u_{-1} := u_{n-1}$ and $\mathrm{sw}^{-1}(x) := \mathrm{sw}^{n-1}(x)$. We fix this notation for the remainder of this section.

Lemma 3.3.9. *For any integer $m \geqslant 1$, we have $x^m = (a_{m-1} \cdots a_0)^{-1} b_{m-1} \cdots b_0$.*

Proof. First, an immediate recurrence gives that, for all $p \geqslant 1$, we have

$$b_{p-1} \cdots b_0 a_0^{-1} = a_p^{-1} b_p \cdots b_1.$$

From this, we can deduce the result by an easy recurrence on m. First, if $m = 1$, the result is immediate, if $m = 2$, then we have

$$x^2 = a_0^{-1} b_0 a_0^{-1} b_0 = a_0^{-1} \,\mathrm{sw}(x) b_0 = a_0^{-1} a_1^{-1} b_1 b_0.$$

Generally, if the results holds for some integer $m \geqslant 1$, then we have

$$\begin{aligned} x^{m+1} &= a_0^{-1} \cdots a_{m-1}^{-1} b_{m-1} \cdots b_0 a_0^{-1} b_0 \\ &= a_0^{-1} \cdots a_{m-1}^{-1} a_m^{-1} b_m \cdots b_1 b_0. \end{aligned}$$

and the result holds for $m + 1$. □

Our main goal is to show that $x^m = (a_{m-1} \cdots a_0)^{-1} b_{m-1} \cdots b_0$ is actually the reduced left-fraction decomposition of x^m. Our argument is inspired by the proof of [CGGW19, Theorem 8.2] and [BGG07a, Theorem 2.9].

Theorem 3.3.10 (Reduced left-fraction decomposition of powers). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $x \in \mathcal{G}$ be a recurrent endomorphism. For $k \geqslant 0$, let $a_k^{-1} b_k$ be the reduced left-fraction decomposition of* $\mathrm{sw}^k(x)$*. For $m > 0$ an integer, the reduced left-fraction decomposition of x^m is given by*

$$x^m = (a_{m-1} \cdots a_0)^{-1} (b_{m-1} \cdots b_0).$$

Before we get to the rather intricate proof of this result, we state a corollary which precise the interactions between the swap operation and taking powers of endomorphisms.

Corollary 3.3.11 (Swap and powers). *Let $x \in \mathcal{G}$ be a recurrent endomorphism, and let m be a nonzero integer. We have* $\mathrm{sw}(x^m) = (\mathrm{sw}^{|m|}(x))^m$*. In particular, x^m is a recurrent endomorphism.*

Proof. First, we prove the result for $m = -1$. Let $x = a^{-1} b$ be the reduced left-fraction decomposition of x. We have that $b^{-1} a$ is the reduced left-fraction decomposition of x^{-1}. Thus, we obtain

$$\mathrm{sw}(x^{-1}) = \mathrm{sw}(b^{-1} a) = a b^{-1} = (b a^{-1})^{-1} = (\mathrm{sw}(x))^{-1}.$$

Now, if $m > 0$ is a positive integer, then with the notation of Theorem 3.3.10, the reduced left-fraction decomposition of x^m is given by $x^m = (a_{m-1} \cdots a_0)^{-1}(b_{m-1} \cdots b_0)$. We then have

$$\begin{aligned} \mathrm{sw}(x^m) &= b_{m-1} \cdots b_0 a_0^{-1} \cdots a_{m-1}^{-1} \\ &= a_m^{-1} b_m \cdots b_1 a_1^{-1} \cdots a_{m-1}^{-1} \\ &= a_m^{-1} a_{m+1}^{-1} \cdots a_{2m-1}^{-1} b_{2m-1} \cdots b_m \\ &= (\mathrm{sw}^m(x))^m, \end{aligned}$$

as we wanted to show (see Lemma 3.3.9). Lastly, if m is a negative integer, then we have

$$\mathrm{sw}(x^m) = \mathrm{sw}((x^{-m})^{-1}) = \left(\mathrm{sw}(x^{-m})\right)^{-1} = \left(\left(\mathrm{sw}^{-m}(x)\right)^{-m}\right)^{-1} = \left(\mathrm{sw}^{-m}(x)\right)^{m},$$

which terminates the proof of the first claim. For the second claim, let n be a positive integer such that $\mathrm{sw}^n(x) = x$. By an immediate induction, we obtain that $\mathrm{sw}^n(x^m) = (\mathrm{sw}^{nm}(x))^m = x^m$, and x^m is recurrent for swap. □

Now, let us proceed with the proof of Theorem 3.3.10. We begin by giving some reductions. First, if $x = b_0$ is positive, then we have $b_k = b_0$ for all $k \geqslant 0$, and $x^m = b_0^m = b_{m-1} \cdots b_0$. Likewise, the result is easily obtained when $x = a_0^{-1}$ is negative. Also, both the definition of $\mathrm{sw}^k(x)$ and the statement in the Theorem depend only on $\mathcal{C}$ as a category, and not on the Garside map Δ (see Remark 3.3.3). In particular, we can replace Δ by some of its powers to assume that both a_0 and b_0 are simple morphisms.

It remains to show the result when a_0 and b_0 are both nontrivial simples, which we assume from now on. Since x is recurrent, we have that $\mathrm{sw}^k(x)$ is neither positive nor negative for all $k \in \mathbb{Z}$. By Corollary 2.3.27 (inf and sup from symmetric normal form), we have $\sup(\mathrm{sw}^k(x)) = \sup(b_k)$ and $\inf(\mathrm{sw}^k(x)) = -\sup(a_k)$. By Lemma 3.3.4, we deduce that both a_k and b_k are nontrivial simples for all $k \in \mathbb{Z}$.

We will need a handful of intermediate definitions and results. First, if p, q are two positive integers, we define

$$\begin{aligned} C_{[p,q]} &:= a_p^{-1} \cdots a_{p+q-1}^{-1} \Delta^q(u_{p+q}) : u_p \to \phi^q(u_{p+q}). \\ R_{[p,q]} &:= \phi^q(b_{p+q-1} \cdots b_p) : \phi^q(u_{p+q}) \to \phi^q(u_p). \end{aligned}$$

We can describe powers of $\mathrm{sw}^p(x)$ using $C_{[p,q]}$ and $R_{[p,q]}$. Indeed, we have

$$\begin{aligned} C_{[p,q]} R_{[p,q]} &= a_p^{-1} \cdots a_{p+q-1}^{-1} \Delta^q(u_{p+q}) \phi^q(b_{p+q-1} \cdots b_p) \\ &= a_p^{-1} \cdots a_{p+q-1}^{-1} b_{p+q-1} \cdots b_p \Delta^q(u_p) \\ &= (\mathrm{sw}^p(x))^q \Delta^q(u_p). \end{aligned}$$

We will show that $\Delta^m(u_0) \wedge x^m \Delta^m(u_0) = C_{[0,m]}$. But first, we have to show that the supremum of $C_{[0,m]}$ is m. The argument is adapted from [BGG07a, Lemma 2.4].

Lemma 3.3.12. *Let $m \geqslant 1$ be an integer, and let $F_m = \mathrm{fin}(C_{[0,m]})$ be the final factor of $C_{[0,m]}$. We have $\sup(C_{[0,m]}) = m$ and $\phi^{-m}(F_m) \succcurlyeq b_m$.*

Proof. By an easy induction, we have that $C_{[0,m]} = \overline{a_0}\phi(\overline{a_1}) \cdots \phi^m(\overline{a_m})$ is a product of m simple morphisms. We show the result by induction on m. First, in general, for $k \geqslant 0$, we have

$$\mathrm{sw}^{k+1}(x) = \Delta^{-1}(u_{k+1})\phi^{-1}(b_k)a_k^* = \Delta^{-1}(u_{k+1})a_{k+1}^* b_{k+1}.$$

In particular, the greedy normal form of $\phi^{-1}(b_k)a_k^*$ is $a_{k+1}^*b_{k+1}$.

Now, assume that $m=1$. We have $C_{[0,1]}=\overline{a_0}$ and $\sup(C_{[0,1]})=1$. Now, since the greedy normal form of $\phi^{-1}(b_0)a_0^*$ is $a_1^*b_1$, we have

$$a_0^*=\phi^{-1}(\overline{a_0})=\phi^{-1}(F_1)\succcurlyeq b_1.$$

Now, assume that the result is obtained for some integer $m\geqslant 1$. The product $\phi^{-1}(b_m)a_m^*$ is not simple since its greedy normal form is $a_{m+1}^*b_{m+1}$. By applying ϕ, we obtain that $b_m\overline{a_m}$ is not simple. Since $\phi^{-m}(F_m)\succcurlyeq b_m$, we obtain that $\phi^{-m}(F_m)\overline{a_m}\succcurlyeq b_m\overline{a_m}$ is also not simple. Applying ϕ^m yields that $F_m\phi(\overline{a_m})$ is not simple. By Corollary 2.1.20, we deduce that $\sup(C_{m+1})=\sup(C_m\phi^m(\overline{a_m}))=\sup(C_m)+1=m+1$. Lastly, we have

$$\phi^{-m}(F_{m+1})=\operatorname{fin}(\phi^{-m}(\overline{a_0}\phi(\overline{a_1})\cdots\phi^m(\overline{a_m})))=\operatorname{fin}(\phi^{-m}(F_m)\overline{a_m}).$$

Since $\phi^{-m}(F_m)\succcurlyeq b_m$ we have $\phi^{-1}(F_m)\overline{a_m}\succcurlyeq b_m\overline{a_m}$. In general, if s,s',t are two simples such that $s\succcurlyeq s'$, and $\sup(st)=\sup(s't)=2$ we have $\operatorname{fin}(st)\succcurlyeq\operatorname{fin}(s't)$. In particular, we have

$$\operatorname{fin}(\phi^{-m}(F_m)\overline{a_m})\succcurlyeq\operatorname{fin}(b_m\overline{a_m})=\phi(b_{m+1}),$$

whence the result. □

In fact, since $C_{[p,q]}$ is $C_{[0,q]}$ defined for $\mathrm{sw}^p(x)$ instead of x, we also obtain that $\sup(C_{[p,q]})=q$ for any pair of positive integers p,q.

Lemma 3.3.13. *For two integers p,q, we have*

$$\alpha(\phi^{q-1}(\overline{a_{p+q-1}})R_{[p,q]})=\phi^{q-1}(\overline{a_{p+q-1}}).$$

Proof. Let us show the result by induction on $q\geqslant 1$. For $q=1$, we have to show that $\overline{a_p}R_{[p,1]}\wedge\Delta(u_p)=\overline{a_p}\phi(b_p)\wedge\Delta(u_p)=\overline{a_p}$. This is a consequence of the definition of a reduced left-fraction. Indeed, we have

$$\begin{aligned}b_p\wedge a_p=1&\Rightarrow\phi(b_p)\wedge\phi(a_p)=1\\&\Rightarrow\overline{a_p}\phi(b_p)\wedge\Delta(u_p)=\overline{a_p}.\end{aligned}$$

Let now $q\geqslant 2$ be an integer and assume that the result holds for $q-1$. We have

$$\phi^{q-2}(\overline{a_{p+q-2}})R_{[p,q-1]}\wedge\Delta(\phi^{q-2}(u_{p+q-2}))=\phi^{q-2}(\overline{a_{p+q-2}}).$$

Multiplying $\phi^{q-2}(\overline{a_{p+q-2}})R_{[p,q-1]}$ on the left by $\phi^{q-2}(b_{p+q-2})$ yields

$$\begin{aligned}\phi^{q-2}(b_{p+q-2}\overline{a_{p+q-2}})R_{p,q-1}&=\phi^{q-2}\left(b_{p+q-2}a_{p+q-2}^{-1}\Delta(u_{p+q-1})\phi(b_{p+q-2}\cdots b_p)\right)\\&=\phi^{q-2}\left(b_{p+q-2}a_{p+q-2}^{-1}b_{p+q-2}\cdots b_p\Delta(u_p)\right)\\&=\phi^{q-2}\left(a_{p+q-1}^{-1}b_{p+q-1}b_{p+q-2}\cdots b_p\Delta(u_p)\right)\\&=\phi^{q-2}\left(a_{p+q-1}^{-1}\Delta(u_{p+q})\phi(b_{p+q-1}b_{p+q-2}\cdots b_p)\right)\\&=\phi^{q-2}\left(\overline{a_{p+q-1}}\phi(b_{p+q-1}b_{p+q-2}\cdots b_p)\right)\\&=\phi^{q-2}(\overline{a_{p+q-1}})\phi^{-1}(R_{[p,q]})\end{aligned}$$

We then have

$$\begin{aligned}\phi^{q-2}(b_{p+q-2})\phi^{q-2}(\overline{a_{p+q-2}}) &= \phi^{q-2}(b_{p+q-2})\left(\phi^{q-2}(\overline{a_{p+q-2}})R_{[p,q-1]} \wedge \Delta(\phi^{q-2}(u_{p+q-2}))\right)\\ &= \phi^{q-2}(\overline{a_{p+q-1}})\phi^{-1}(R_{[p,q]}) \wedge \phi^{q-2}(b_{p+q-2})\Delta(\phi^{q-2}(u_{p+q-2})).\end{aligned}$$

Since ϕ preserves gcds, we can apply ϕ to the above equality to get

$$\phi^{q-1}(b_{p+q-2}\overline{a_{p+q-2}}) = \phi^{q-1}(\overline{a_{p+q-1}})R_{[p,q]} \wedge \phi^{q-1}(b_{p+q-2})\Delta(\phi^{q-1}(u_{p+q-2})).$$

Since $\phi^{q-1}(b_{p+q-2})\Delta(\phi^{q-1}(u_{p+q-2})$ is left-divisible by $\Delta(\phi^{q-1}(u_{p+q-1}))$, taking the gcd with $\Delta(\phi^{q-1}(u_{p+q-1}))$ in the above equality yields

$$\begin{aligned}\phi^{q-1}(b_{p+q-2}\overline{a_{p+q-2}}) \wedge \Delta(\phi^{q-1}(u_{p+q-1})) &= \phi^{q-1}(\overline{a_{p+q-1}})R_{[p,q]} \wedge \Delta(\phi^{q-1}(u_{p+q-1}))\\ \alpha(\phi^{q-1}(b_{p+q-2}\overline{a_{p+q-2}})) &= \alpha(\phi^{q-1}(\overline{a_{p+q-1}}R_{[p,q]})).\end{aligned}$$

Since, by definition, the greedy normal form of $b_{p+q-2}\overline{a_{p+q-2}}$ is given by $\overline{a_{p+q-1}}b_{p+q-1}$ we obtain

$$\alpha(\phi^{q-1}(\overline{a_{p+q-1}})R_{p,q}) = \phi^{q-1}(\overline{a_{p+q-1}}).$$

which is the desired result. □

Corollary 3.3.14. *For any integer $m \geqslant 1$, we have $x^m\Delta^m(u_0) \wedge \Delta^m(u_0) = C_{[0,m]}$.*

Proof. We actually show that $\Delta^{m-k}(u_0) \wedge C_{[0,m-k]}\phi^{-k}(R_{[-k,m]}) = C_{[0,m-k]}$ for all $k \in [\![0, m-1]\!]$. Since $\sup(C_{[0,m-k]}) = m-k$ by Lemma 3.3.12, this is equivalent to saying that the path $\operatorname{fin}(C_{[0,m-k]})\operatorname{init}(\phi^{-k}(R_{[-k,m]})$ is greedy. We proceed by descending induction on k. For $k = m-1$, we compute $C_{[0,1]}\phi^{m-1}(R_{[1-m,m]}) \wedge \Delta(u_0)$. We have $C_{[0,1]} = \overline{a_0}$ and the result is a direct consequence of Lemma 3.3.13. Assume that the result holds for the integer $k+1$ and let us show that it also holds for k. By assumption, we have

$$\Delta^{m-k-1}(u_0) \wedge C_{[0,m-k-1]}\phi^{-k-1}(R_{[-k-1,m]}) = C_{[0,m-k-1]}.$$

We multiply $C_{[0,m-k-1]}\phi^{-k-1}(R_{[-k-1,m]})$ on the right by $\phi^{m-k-1}(\overline{a_{k-1}})$. First, we have

$$\phi^{-k-1}(R_{[-k-1,m]})\phi^{m-k-1}(\overline{a_{k-1}}) = \phi^{m-k-1}(\overline{a_{m-k-1}})\phi^{-k}(R_{[-k,m]}),$$

and $\phi^{m-k-1}(\overline{a_{m-k-1}}) = \alpha\left(\phi^{m-k-1}(\overline{a_{m-k-1}}))\phi^{-k}(R_{[-k,m]})\right)$ by Lemma 3.3.13. As in [DDGKM, Algorithm III.1.60], we can compute the greedy normal form of a product of the form $s_1\cdots s_r s$ (where $s_1\cdots s_r$ is greedy) by replacing $s_r s$ with its normal form, then replacing $s_{r-1}\alpha(s_r s)$ with its normal form and so on. In particular (and by induction hypothesis), we have

$$\begin{aligned}&\Delta^{m-k}(u_0) \wedge \left(C_{[0,m-l]}\phi^{-k}(R_{[-k,m]})\right)\\ =&\Delta^{m-k}(u_0) \wedge \left(C_{[0,m-k-1]}\phi^{m-k-1}(\overline{a_{m-k-1}})\phi^{-k}(R_{[-k,m]})\right)\\ =&\Delta^{m-k}(u_0) \wedge \left(C_{[0,m-k-1]}\alpha(\phi^{m-k-1}(\overline{a_{m-k-1}})\phi^{-k}(R_{[-k,m]}))\right)\\ =&\Delta^{m-k}(u_0) \wedge C_{[0,m-k-1]}\phi^{m-k-1}(\overline{a_{m-k-1}})\\ =&\Delta^{m-k}(u_0) \wedge C_{[0,m-k]} = C_{[0,m-k]}.\end{aligned}$$

By Lemma 3.3.12. Thus $\Delta^{m-k}(u_0) \wedge C_{[0,m-k]}\phi^{-k}(R_{[-k,m]}) = C_{[0,m-k]}$ for all $k \in [\![0, m-1]\!]$. Applying this to $k = 0$ yields

$$\Delta^m(u_0) \wedge C_{[0,m]}R_{[0,m]} = \Delta^m(u_0) \wedge x^m\Delta^m(u_0) = C_{[0,m]}.$$

□

Using Corollary 3.3.14, we are finally able to show Theorem 3.3.10. Indeed, multiplying the equality $x^m \wedge \Delta^m(u_0) \wedge \Delta^m(u_0)$ by $\Delta^{-m}(u_0)$ yields

$$\phi^m(x^m) \wedge 1 = \Delta^{-m}(u_0)C_{[0,m]} = \phi^m(a_0^{-1} \cdots a_{m-1}^{-1}),$$

and the denominator of x^m is $a_{m-1} \cdots a_0$, which is precisely Theorem 3.3.10, since we already know that the denominator of x^m divides $a_{m-1} \cdots a_0$ on the right.

3.3.3 Transport for swap and convexity

In this section, we will explain some important properties of sets of circuits for swap, following [GM22, Section 4.5]. Unfortunately, we do not know whether the set of recurrent conjugates of a given endomorphism in a Garside groupoid is always finite, even in the case in which it consists only of positive endomorphisms. In this later case, the set is finite if the considered groupoid is homogeneous. But in a general Garside groupoid, this set could a priori be infinite.

In any case, we will see that the set of circuits for swap satisfies the same properties as, for instance, the super-summit set, namely that it is a conjugacy set in the sense of Lemma 4.3.1. Although this can be shown by comparing the set $\mathsf{R}(x)$ with other sets like the ultra summit set of x, we will proceed to show all details avoiding the use of Garside normal forms.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We start with a very basic property.

Lemma 3.3.15. *Let $x \in \mathcal{G}$ be an endomorphism. We have $\mathrm{sw}(\phi(x)) = \phi(\mathrm{sw}(x))$. In particular, the set $\mathsf{R}(x)$ is globally stable under ϕ.*

Proof. Since ϕ preserves the lattice structures on $\mathcal{G}$, it follows that it preserves the the reduced left-fraction decomposition, hence $\mathrm{sw}(\phi(x)) = \phi(\mathrm{sw}(x))$. That is, applying a left swap commutes with ϕ.

Given $y \in \mathsf{R}(x)$, there is some $m > 0$ such that $\mathrm{sw}^m(y) = y$. By the argument in the previous paragraph, we have $\mathrm{sw}^m(\phi(y)) = \phi(\mathrm{sw}^m(\phi(y)) = \phi(y)$. Therefore $\phi(y) \in \mathsf{R}(x)$. □

Our purpose is to extend the swap operation to conjugating elements between two endomorphism. In [GM22], this is done via the notion of *transport for swap*, which is reminiscent of the notion of transport for sliding [GG10b, Definition 1.14]. In [DDGKM, Proposition VIII.2.33], the notion of transport of sliding is presented as a functor on some subcategory of $\mathrm{Conj}^+(\mathcal{G})$. We adopt a similar point of view here.

Recall from Definition 2.3.8 (reduced left-fraction decomposition) that, for $f \in \mathcal{G}$, $D(f)$ (resp. $N(f)$) is the denominator (resp. the numerator) of f, so that $f = D(f)^{-1}N(f)$ is the reduced left-fraction decomposition of f.

Proposition 3.3.16 (Swap functor). *We define* sw *on the category $\mathrm{Conj}^+(\mathcal{G})$ by mapping x to $\mathrm{sw}(x)$ for $x \in \mathrm{Ob}(\mathrm{Conj}^+(\mathcal{G}))$, and f_x to $(D(x)fD(x^f)^{-1})_{\mathrm{sw}(x)}$ for $f_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, x^f)$.*

- *(a) The maps* sw *define a functor from $\mathrm{Conj}^+(\mathcal{G})$ to itself. In particular, for $f_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, x^f)$, we have $\mathrm{sw}(f_x) \in \mathrm{Conj}^+_{\mathcal{G}}(\mathrm{sw}(x), \mathrm{sw}(x^f))$.*
- *(b) The functor* sw *preserves left-divisibility and left-gcds.*
- *(c) Let $x \in \mathcal{G}$ be an endomorphism with source u. We have $\mathrm{sw}(\Delta(u)_x) = \Delta(v)_{\mathrm{sw}(x)}$, where v is the source of $\mathrm{sw}(x)$.*

(d) The functor sw *preserves the preimage of* $\mathcal{S}$ *under the functor* $\pi : \mathrm{Conj}^+(\mathcal{G}) \to \mathcal{C}$*.*

Proof. (a) Let $x \in \mathcal{G}$ be an endomorphism, and let $x := a^{-1}b$ be its reduced left-fraction decomposition. Let also $f_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, x^f)$, and let $x^f = c^{-1}d$ be the reduced left-fraction decomposition of x^f. By definition, we have $\mathrm{sw}(f_x) = (afc^{-1})_{\mathrm{sw}(x)}$. We then have have

$$\mathrm{sw}(x)^{afc^{-1}} = cf^{-1}a^{-1}ba^{-1}afc^{-1} = cf^{-1}a^{-1}bfc^{-1} = cc^{-1}dc^{-1} = \mathrm{sw}(x^f),$$

and thus $\mathrm{sw}(f_x) \in \mathrm{Conj}_{\mathcal{G}}(\mathrm{sw}(x), \mathrm{sw}(x^f))$.

In order to show that $\mathrm{sw}(f_x) \in \mathrm{Conj}^+_{\mathcal{G}}(\mathrm{sw}(x), \mathrm{sw}(x^f))$, we still have to show that $afc^{-1} \in \mathcal{C}$. We actually have

$$afc^{-1} = bfd^{-1} = af \wedge bf.$$

Indeed, we have $x^f = (af)^{-1}(bf)$ by definition. The reduced left-fraction decomposition of x^f is obtained by simplifying the left-gcd $\delta := af \wedge bf$ from both af and bf. That is, $c = \delta^{-1}af$, $d = \delta^{-1}bf$ and $\delta = afc^{-1} = bfd^{-1}$ as claimed. In particular, we obtain that afc^{-1} is positive as the left-gcd of $af, bf \in \mathcal{C}$. Since sw is obviously compatible with compositions of conjugating elements, we obtain a functor from $\mathrm{Conj}^+(\mathcal{G})$ to itself.

(b) Let $x \in \mathcal{G}$ be an endomorphism, with reduced left-fraction decomposition $x = a^{-1}b$, and let $f_x, g_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, -)$. If $f_x \preccurlyeq g_x$, then $f \preccurlyeq g$ by Lemma 3.1.10. We then have $af \preccurlyeq ag$, $bf \preccurlyeq bg$ and $af \wedge bf \preccurlyeq ag \wedge bg$, thus $\mathrm{sw}(f_x) \preccurlyeq \mathrm{sw}(f_y)$, again by Lemma 3.1.10.

Furthermore, Lemma 3.1.10 also proves that $f_x \wedge g_x = (f \wedge g)_x$. We then have

$$a(f \wedge g) \wedge b(f \wedge g) = af \wedge ag \wedge bf \wedge bg = (af \wedge bf) \wedge (ag \wedge bg),$$

and $\mathrm{sw}((f \wedge g)_x) = \mathrm{sw}(f_x) \wedge \mathrm{sw}(g_x)$, thus sw preserves left-gcds.

(c) Let us write $x = a^{-1}b$ for the reduced left-fraction decomposition of x. The source v of $\mathrm{sw}(x)$ is the source of a. We have $\mathrm{sw}(\Delta(u)_x) = a\Delta(u)\phi^{-1}(a)^{-1} = \Delta(v)$.

(d) By Lemma 3.1.10, the preimage of $\mathcal{S}$ under π consists of the morphisms in $\mathrm{Conj}^+(\mathcal{G})$ which left-divides some morphism of the form $\Delta(u)_x$ for some $x \in \mathcal{G}(u, u)$. We then obtain the result by combining points (b) and (c). □

The next remarkable property of the swap functor is that it is "ultimately periodic". More precisely, we show the following result, stating in particular that the swap functor acts on the full subcategory of $\mathrm{Conj}^+(\mathcal{G})$ whose objects are the recurrent endomorphisms in $\mathcal{G}$.

Proposition 3.3.17. *Let* $x \in \mathcal{G}$ *be a recurrent endomorphism, and let* $f_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, -)$ *be such that* $y := x^f$ *is also recurrent. There is an integer* $n > 0$ *such that* $\mathrm{sw}^n(f_x) = f_x$ *(and in particular* $\mathrm{sw}^n(x) = x$, $\mathrm{sw}^n(y) = y$*).*

Proof. Since x and y are recurrent, we can consider integers p and q such that $\mathrm{sw}^p(x) = x$ and $\mathrm{sw}^q(y) = y$. By considering the lcm N of p and q, we obtain $\mathrm{sw}^N(x) = x$ and $\mathrm{sw}^N(y) = y$.

We then have $\mathrm{sw}^N(f_x) \in \mathrm{Conj}^+_{\mathcal{G}}(x, y)$. Furthermore, since sw preserves divisibility, we have $f_x \preccurlyeq \Delta^r(u)_x$ and $\mathrm{sw}^N(f_x) \preccurlyeq \mathrm{sw}^N(\Delta^r(u)_x) = \Delta^r(u)_x$, where $r = \sup(f)$. Thus $\sup(\pi(\mathrm{sw}^N(f_x))) \leqslant r$. However, there is a finite number of morphisms in $\mathcal{C}$ with supremum inferior to r. Thus, the sequence $\pi(\mathrm{sw}^{iN}(f_x))$ for $i \geqslant 0$ can only take a finite number of values, and there are some integers $m > n \geqslant 0$ such that $\mathrm{sw}^{mN}(f_x) = \mathrm{sw}^{nN}(f_x)$.

Note that $\mathrm{sw} : \mathrm{Conj}^+_{\mathcal{G}}(x, y) \to \mathrm{Conj}^+_{\mathcal{G}}(\mathrm{sw}(x), \mathrm{sw}(y))$ is injective by construction. Thus, $\mathrm{sw}^{mN}(f_x) = \mathrm{sw}^{nN}(f_x)$ implies $\mathrm{sw}^{(m-n)N}(f_x) = f_x$, which finishes the proof. □

Corollary 3.3.18 (Swap is ultimately periodic). *The functor* sw *on* $\mathrm{Conj}^+(\mathcal{G})$ *is ultimately periodic. That is, for every* $f_x \in \mathrm{Conj}^+(\mathcal{G})$*, there exist* $m > n \geqslant 0$ *satisfying* $\mathrm{sw}^m(f_x) = \mathrm{sw}^n(f_x)$.

Proof. By Lemma 3.3.5 we can consider p (resp. q) such that $\mathrm{sw}^p(x)$ (resp. $\mathrm{sw}^q(y)$) is recurrent. By considering the lcm N of p and q , we obtain that both $\mathrm{sw}^N(x)$ and $\mathrm{sw}^N(y)$ are recurrent. By Proposition 3.3.17, there is an integer $k > 0$ such that $\mathrm{sw}^n(\mathrm{sw}^N(f_x)) = \mathrm{sw}^N(f_x)$. □

We can then show a very important property, analogous to Lemma 3.2.9, and which is shared by all sets which are used in general to solve the conjugacy problem in Garside groupoids.

Proposition 3.3.19. *Let* $x \in \mathcal{G}$ *be an endomorphism, and let* $y \in \mathsf{R}(x)$*. If* $f \in \mathcal{C}$ *is such that* $y^f \in \mathsf{R}(x)$*, then* $y^{\alpha(f)} \in \mathsf{R}(x)$.

Proof. By Proposition 3.3.17, we know that there is a positive integer N such that $\mathrm{sw}^N(f_y) = f_y$ and $\mathrm{sw}^N(y) = y$. We then have $\mathrm{sw}^N(\Delta(u)_y) = \Delta(u)_y$ by Proposition 3.3.16 (swap functor), where u is the source of x. By Proposition 3.3.16, we have

$$\mathrm{sw}^n(\alpha(f)_y) = \mathrm{sw}^N(f_y \wedge \Delta(u)_y) = \mathrm{sw}^N(f_y) \wedge \mathrm{sw}^N(\Delta(u)_y) = f_y \wedge \Delta(u)_y = \alpha(f)_y.$$

Since $\mathrm{sw}^N(\alpha(f)_y)$ conjugates $\mathrm{sw}^N(y)$ to $\mathrm{sw}^N(y^{\alpha(f)})$ by definition of the swap functor, and $\alpha(f)$ obviously conjugates y to $y^{\alpha(f)}$, it follows that $\mathrm{sw}^N(y^{\alpha(f)}) = y^{\alpha(f)}$, that is, $y^{\alpha(f)} \in \mathsf{R}(x)$, as we wanted to show. □

3.4 Periodic elements

In this section, we study a particular class of endomorphisms in Garside groupoids, called periodic elements. These elements were studied by Lee and Lee in [LL11] in the case of Garside groups, and by Bessis in [Bes07] in the case of Garside groupoids. As we said earlier, periodic elements will play a particular role in studying complex braid groups (this was also one of the key motivations of Bessis).

This section is mostly a short retelling of [DDGKM, Section VIII.3], culminating in Theorem 3.4.4, which completely describes the conjugacy of periodic elements in a Garside groupoid. This theorem already appears in [Bes07], but the proof is mostly left to reader as a “translation exercise” from the previous paper to the context of categories, and the first complete proof that we know of (for Garside groupoids) appears in [DDGKM].

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

Definition 3.4.1 (Periodic element). Let p, q be integers, and let $u \in \mathrm{Ob}(\mathcal{G})$. An element $\rho \in \mathcal{G}(u, u)$ is called (p, q)-*periodic* if $\rho^p = \Delta^q(u)$.

Let $u \in \mathrm{Ob}(\mathcal{G})$. Since the group $\mathcal{G}(u, u)$ is torsion-free, there are no $(p, 0)$ periodic elements for $p \neq 0$. Thus, if $\rho \in \mathcal{G}(u, u)$ is (p, q) periodic for some integers p, q, then we either have $p = q = 0$, in which case $\rho = 1_u$, or $p \neq 0$ and $q \neq 0$. In the sequel, we will only consider (p, q) periodic elements for $p, q \neq 0$.

Remark 3.4.2. Any (p,q)-periodic element in $\mathcal{G}$ is also $(-p,-q)$-periodic, and the inverse of a (p,q)-periodic element is $(p,-q)$-periodic. Thus, in the study of periodic elements up to conjugacy, we can restrict our attention to (p,q) periodic elements where p and q are both positive.

The following lemma shows that conjugating a periodic element yields another periodic element. Although the parameters can change under such conjugacy, the ratio between the parameters is preserved.

Lemma 3.4.3. *[DDGKM, Lemma VIII.3.4 and VIII.3.33] Let p,q be nonzero integers, and let $\rho \in \mathcal{G}$ be a (p,q)-periodic element. If ρ is also (a,b)-periodic for some nonzero integers a,b, then $\frac{p}{q}=\frac{a}{b}$. Furthermore, any conjugate of ρ in $\mathcal{G}$ is (p',q') periodic for some nonzero integers p',q' such that $\frac{p}{q}=\frac{p'}{q'}$.*

We obviously see that, if $\rho \in \mathcal{G}$ is (p,q) periodic, then it is also (np,nq)-periodic for all integer n. The converse statement is not true per say. In the monoid $\langle a,b \mid aba=bab\rangle$, the element ba^2 is $(2,2)$-periodic, but is is not $(1,1)$-periodic. However, we see that $(ba^2)^{a^{-1}}=aba=\Delta$ is $(1,1)$-periodic. The following Theorem states that this is actually a general situation.

Theorem 3.4.4 (Conjugacy of periodic elements). *[DDGKM, Proposition 3.34]*
Let $(\mathcal{G},\mathcal{C},\Delta)$ be a Garside groupoid, and let $p,q>0$ be integers with gcd d. Let also $p':=\frac{p}{d}$ and $q':=\frac{q}{d}$. Every (p,q)-periodic element in $\mathcal{G}$ is conjugate to a (p',q')-periodic element $\rho \in \mathcal{C}(u,u)$ for some $u \in \mathrm{Ob}(\mathcal{C})$. Furthermore, there is some $s \in \mathcal{S}$ such that $\phi^{q'}(s)=s$ and that, for any positive integers λ,μ with $p\lambda-q\mu=d$, we have

$$\begin{cases} s=\rho^{-\mu}\Delta^{\lambda}(u), \\ s\phi^{\lambda}(s)\cdots\phi^{(p'-1)\lambda}(s)=\Delta(u), \\ s\phi^{\lambda}(s)\cdots\phi^{(q'-1)\lambda}(s)=\rho. \end{cases}$$

The proof of this theorem is the purpose of [DDGKM, Section VIII.3] using a generalization of the geometric approach of M. Bestvina [Bes99]. It is rather long and intricate and we skip it here. We give a few corollaries obtained by applying Theorem 3.4.4 to particular values of p and q.

Corollary 3.4.5 ((p,np)-periodic elements). *Let $p,n>0$ be integers. Every (p,np)-periodic element in $\mathcal{G}$ is conjugate to an element of the form $\Delta^n(u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$.*

In this case, we can take $\lambda=n+1$ and $\mu=1$. The element s of Theorem 3.4.4 is then simply $\Delta(u)$, which is invariant under ϕ^n since $\Delta^n(u)$ has target u.

Corollary 3.4.6 ((nq,q)-periodic elements). *Let $q,n>0$ be integers. Every (nq,q)-periodic element in $\mathcal{G}$ is conjugate to a n-th root of some $\Delta(u)$ for $u \in \mathrm{Ob}(\mathcal{G})$, which is ϕ-invariant.*

Proof. We write $\lambda=1$ and $\mu=n-1$. By Theorem 3.4.4, there is an element $s \in \mathcal{S}$ which is invariant under ϕ and such that $\rho:=s$ is a $(q,1)$-periodic element, i.e. a n-th root of Δ. □

Corollary 3.4.7 ($(p,2)$-periodic elements). *Let $p>0$ be an integer.*

(a) $p=2r$ is even, then every $(p,2)$-periodic element in $\mathcal{G}$ is conjugate to a r-th root of some $\Delta(u)$ for $u \in \mathrm{Ob}(\mathcal{G})$.

(b) $p=2r+1$ is odd, then every $(p,2)$-periodic element in $\mathcal{G}$ is conjugate to an element of the form $s\phi(s)$, where $s \in \mathcal{S}$ is such that $\phi^2(s)=s$.

Proof. Statement (a) is an immediate consequence of Corollary 3.4.6. Statement (b) is a straightforward consequence of Theorem 3.4.4, taking $\lambda = 1$ and $\mu = r$. □

Remark 3.4.8. Using the notation of Theorem 3.4.4, the assertion that

$$s\phi^{\lambda}(s)\cdots\phi^{(p'-1)\lambda}(s) = \Delta(u)$$

can be reformulated by saying that $s\Delta^{-\lambda}(v)$ is $(p', -q'\mu)$-periodic, where v is the target of s. Likewise, the assertion that $s\phi^{\lambda}(s)\cdots\phi^{(q'-1)\lambda}(s) = \rho$ can be reformulated by saying that $\rho = (s\Delta^{-\lambda}(v))^{q'}\Delta^{q'\lambda}(u)$.

Using Theorem 3.4.4, we are easily able to describe super-summit sets of periodic elements.

Corollary 3.4.9 (Super-summit set of periodic elements). *Let $p, q > 0$ be integers, and let also $q = mp + r$ be the euclidean division of q by p.*

(a) If $p|q$, then the super-summit set of any (p,q)-periodic element in $\mathcal{G}$ is made of elements of the form $\Delta^m(u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$.

(b) If $p \nmid q$, then the super-summit set of and (p,q)-periodic element in $\mathcal{G}$ is made of elements of the form $\Delta^m(u)h$ for some $u \in \mathrm{Ob}(\mathcal{G})$ and some $h \in \mathcal{S}$ nontrivial.

Proof. Let d denote the gcd of p and q, and let $p' := \frac{p}{d}$, $q' := \frac{q}{d}$ $r' = \frac{r}{d}$. We know that d divides r, and that the euclidean division of q' by p' is $q' = mp' + \frac{r}{d}$. We can consider ρ and s as in Theorem 3.4.4. If $p|q$, i.e. if $p' = 1$, one can take $\lambda = q' + 1$ and $\mu = 1$ in Theorem 3.4.4 to get

$$\begin{cases} s = \rho^{-1}\Delta^{q'+1}(v), \\ s = \Delta(v), \\ s\phi^{q'+1}(s)\cdots\phi^{(q'-1)(q'+1)} = \rho. \end{cases} \Leftrightarrow \begin{cases} s = \Delta(v), \\ \rho = \Delta^{q'}(v) = \Delta^m(v). \end{cases}$$

were v is the source of ρ. We then have

$$m \leqslant \inf(\mathrm{SSS}(\rho)) \leqslant \sup(\mathrm{SSS}(\rho)) \leqslant m,$$

and the super-summit set of a (p,q)-periodic elements is made of elements of the form $\Delta^m(u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$.

If $p \nmid q$, let λ, μ be positive integers with $p\lambda - q\mu = 1$. By Theorem 3.4.4, we have

$$\begin{aligned} \rho &= s\phi^{\lambda}(s)\cdots\phi^{(q'-1)\lambda}(s) \\ &= \left(s\cdots\phi^{(p'-1)\lambda}(s)\right)\cdots\left(\phi^{p'(m-1)\lambda}(s)\cdots\phi^{(p'm-1)\lambda}(s)\right)\phi^{p'm\lambda}(s)\cdots\phi^{(p'm+r'-1)\lambda}(s) \\ &= \Delta^m(v)\phi^{p'm}\left(s\phi^{\lambda}(s)\cdots\phi^{(r'-1)\lambda}(s)\right), \end{aligned}$$

where v is the source of ρ. Since $r' < p'$ by assumption, $s\phi^{\lambda}(s)\cdots\phi^{(r'-1)\lambda}(s) =: \phi^{-p'm}(h)$ is a simple morphism, and we have $\rho = \Delta^m(v)h$. Since $p \nmid q$, ρ is not conjugate to a power of Δ, thus we have

$$m \leqslant \inf(\mathrm{SSS}(\rho)) \leqslant \sup(\mathrm{SSS}(\rho)) - 1 \leqslant m,$$

and the super-summit set of a (p,q)-periodic element is made of elements of the form $\Delta^m(u)h$ for some $u \in \mathrm{Ob}(\mathcal{G})$ and some $h \in \mathcal{S}$ nontrivial. □

Of course, Theorem 3.4.4 is a stronger statement than Corollary 3.4.9. The set of conjugates of a periodic element given in Theorem 3.4.4 is in general smaller than the super-summit set described in Corollary 3.4.9, as we see in the following example:

Example 3.4.10. Consider the monoid $M = \langle s, t, u \mid sts = tst, tut = utu, su = us\rangle^+$ (Artin-Tits monoid of type A_3) as in Example 3.2.14. It is a Garside monoid with Garside element $\Delta = stsuts$. One easily checks that the element

$$\rho := \Delta^{-1} tsuts \cdot stu \cdot u$$

is $(4, 2)$-periodic. In Example 3.2.14, we saw that $\mathrm{SSS}(\rho) = \{stu, ust, tus, uts\}$, which is coherent with Corollary 3.4.9. However, by Corollary 3.4.7, ρ is conjugate to a ϕ-invariant square root of Δ. The set of such square roots is $\{sut, tus\}$, which is also the set of sliding circuits of ρ, as we saw in Example 3.2.14.

From the computation of the super-summit sets of periodic elements, we can also deduce a first finiteness result on periodic elements. We will show stronger results in Section 4.5.

Corollary 3.4.11. *Let $(p, q) > 0$ be integers. There is a finite number of conjugacy classes in $\mathcal{G}$ which contain (p, q)-periodic elements.*

Proof. First, we can assume that there is at least one (p, q)-periodic elements, otherwise the result is trivial. Let ρ be a (p, q)-periodic element. Since $\mathcal{S}$ is finite, there is a finite number of elements of the form $\Delta^m(u)$ for some $u \in \mathrm{Ob}(\mathcal{G})$ (resp. $\Delta^m(u)h$ for some $u \in \mathrm{Ob}(\mathcal{G})$ and some $h \in \mathcal{S}$). By Corollary 3.4.9, the super-summit set of ρ is then a subset of some finite subset of $\mathcal{G}$, in particular it can only take a finite number of values. Since the super-summit set of ρ completely characterizes its conjugacy class, the result follows. $\square$

Chapter 4

Constructions of Garside groupoids

This chapter presents several ways of producing new Garside groupoids starting from known Garside groupoids. Most of these constructions (although not standard) already appear in the literature (except the one in Section 4.2), but often with less details than given here. For instance, the concept of conjugacy graphs originates from [FG03a], and the idea of endowing conjugacy categories with Garside structures appears in [DDGKM], but the idea of endowing conjugacy graphs with a Garside structure is new. Likewise, the construction of the germ of periodic elements in Definition 4.5.7 is new, as well as the results we obtain on conjugacy classes of periodic elements.

Contents

So far, the only way we saw of producing Garside categories was through the use of Garside germs. And the only way of building new Garside germs we presented was through the concept of interval Garside germ attached to a group. Such a germ always produces a Garside *monoid*, and the reader may ask to which extent it was necessary to go through the general theory of Garside categories, rather than just sticking to the more classical Garside monoids and groups.

We present here various constructions which can be applied to a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ in order to produce a new Garside groupoid $(\mathcal{G}', \mathcal{C}', \Delta')$ (which is related to $(\mathcal{G}, \mathcal{C}, \Delta)$ in some sense). Most of these constructions may produce Garside groupoids, even when applied to a Garside group, whence the necessity of considering Garside groupoids in general. These constructions will in particular allow us to prove that the class of weak Garside groups enjoys some stability properties which are not shared (to our knowledge) by Garside groups.

Theorem. *(Corollary 4.2.4 and Corollary 4.3.11) The class of weak Garside groups is stable under taking centralizers and finite index subgroups.*

4.1 Groupoids of fixed points

A classical result of Crisp [Cri00] proves that the submonoid of fixed points under an automorphism of an Artin-Tits monoid of spherical type is again an Artin-Tits monoid of spherical type (such an automorphism corresponds to a diagram folding of the associated Coxeter diagram). This result was generalized to Garside categories in [Bes07, Section 4], showing that the subcategory of fixed points under a "convenient" automorphism of a Garside category is again a Garside category (see also [DDGKM, Section VII.4.1]).

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also fix $\psi : \mathcal{C} \to \mathcal{C}$ an automorphism of $\mathcal{C}$. Such an automorphism extends to an automorphism of $\mathcal{G}$, which we also denote by ψ. Note that we can equivalently consider an automorphism ψ of $\mathcal{G}$ which preserves $\mathcal{C}$ globally.

We are interested in describing the groupoid $\mathcal{G}^{\psi}$ (resp. the category $\mathcal{C}^{\psi}$) of ψ-invariant morphisms in $\mathcal{G}$ (resp. in $\mathcal{C}$) as in [DDGKM, Section VII.4.1]. A convenient condition will ensure that $\mathcal{G}^{\psi}$ is compatible in a sense with the Garside structure of $\mathcal{G}$. This condition can be formulated in two equivalent ways, as in the following lemma:

Lemma 4.1.1. *We have $\psi(\mathcal{S}) = \mathcal{S}$ if and only if $\psi(\Delta(u)) = \Delta(\psi(u))$ for all $u \in \mathrm{Ob}(\mathcal{C})$. In this case, we say that ψ* preserves Δ *(globally).*

Proof. Assume that $\psi(\mathcal{S}) = \mathcal{S}$, and let $u \in \mathrm{Ob}(\mathcal{C})$. We have $\psi(\Delta(u)) \in \mathcal{S}(\psi(u), -)$, thus $\psi(\Delta(u)) \preccurlyeq \Delta(\psi(u))$. By applying the same reasoning to ψ^{-1}, we get $\psi^{-1}(\Delta(\psi(u))) \preccurlyeq \Delta(u)$. By applying ψ to this equality, we get $\Delta(\psi(u)) \preccurlyeq \psi(\Delta(u))$ and $\Delta(\psi(u)) = \psi(\Delta(u))$.

Conversely, assume that $\Delta(\psi(u)) = \psi(\Delta(u))$ for $u \in \mathrm{Ob}(\mathcal{C})$. Let $s \in \mathcal{S}(u, -)$ for some $u \in \mathrm{Ob}(\mathcal{C})$. We have $\psi(s)\psi(\overline{s}) = \psi(\Delta(u)) = \Delta(\psi(u))$ and $\psi(s) \preccurlyeq \Delta(\psi(u))$, thus $\psi(\mathcal{S}) \subset \mathcal{S}$. The converse inclusion comes from finiteness of $\mathcal{S}$. □

In particular, we obtain that if ψ preserves Δ, then it must commute with the Garside automorphism ϕ.

Corollary 4.1.2. *Assume that ψ preserves Δ.*

(a) For $s \in \mathcal{S}$, we have $\psi(\overline{s}) = \overline{\psi(s)}$.

(b) The automorphisms ψ and ϕ commute with one another.

Proof. We proved point (a) in the proof of Lemma 4.1.1. Now, let $s \in \mathcal{S}$, we have $\phi(\psi(s)) = \overline{\overline{\psi(s)}} = \psi(\overline{\overline{s}}) = \psi(\phi(s))$. Thus ψ and ϕ commute when restricted to $\mathcal{S}$. Since ψ and ϕ are functors, and since $\mathcal{S}$ generated $\mathcal{G}$, we obtain the result. □

Remark 4.1.3. If ψ is a power of the Garside automorphism ϕ, then ψ preserves Δ and we can apply the results in this section.

We assume from now on that ψ preserves Δ. The groupoid $\mathcal{G}^\psi$ of ψ-invariant morphisms in $\mathcal{G}$ is a subgroupoid which may be empty or not. Likewise, the subcategory $\mathcal{C}^\psi \subset \mathcal{C}$ may be empty or not. We obviously have $\mathcal{C}^\psi \subset \mathcal{G}^\psi$, but it is not immediate that $\mathcal{C}^\psi$ generates the subgroupoid $\mathcal{G}^\psi$ in $\mathcal{G}$. In order to show this, we first show that ψ preserves normal forms.

Lemma 4.1.4. *Let $f \in \mathcal{C}$ have greedy normal form $f = s_1 \cdots s_r$. The greedy normal form of $\psi(f)$ is $\psi(s_1)\cdots\psi(s_r)$.*

Proof. We proceed by induction on $\sup(f)$, the case $\sup(f) = 1$ (i.e. $f \in \mathcal{S}$) being trivial by Lemma 4.1.1. We know by Proposition 2.1.16 (greedy normal form) that $s_1 = \alpha(f) = f \wedge \Delta(u)$, where u is the source of f. As an automorphism of a category, ψ preserves gcds and lcms. We then have

$$\alpha(\psi(f)) = \psi(f) \wedge \Delta(\psi(u)) = \psi(f) \wedge \psi(\Delta(u)) = \psi(f \wedge \Delta(u)) = \psi(\alpha(f)).$$

Thus $\psi(s_1)$ is the first term in the greedy normal form of $\psi(f)$. Since $\sup(\omega(f)) = \sup(f) - 1$ by definition. We obtain the result by using the induction hypothesis on $\omega(\psi(f)) = \psi(\omega(f))$. $\square$

Proposition 4.1.5 (Preservation of normal forms). *Let $u \in \mathrm{Ob}(\mathcal{G})$ and let $f \in \mathcal{G}(u, -)$.*

(a) If $f = a^{-1}b$ is the reduced left-fraction decomposition of f, then the reduced left fraction of $\psi(f)$ is $\psi(a)^{-1}\psi(b)$.

(b) If $f = a_p^{-1} \cdots a_1^{-1} b_1 \cdots b_q$ is the symmetric normal form of f, then the symmetric normal form of $\psi(f)$ is $\psi(f) = \psi(a_p)^{-1} \cdots \psi(a_1)^{-1}\psi(b_1)\cdots\psi(b_q)$.

(c) If $f = \Delta^k(u)s_1 \cdots s_r$ is the left normal form of f, then the left normal form of $\psi(f)$ is $\psi(f) = \Delta^k(\psi(u))\psi(s_1)\cdots\psi(s_r)$.

Proof. (a) As ψ is a functor, we have $\psi(f) = \psi(a)^{-1}\psi(b)$. Then, as an automorphism of $\mathcal{C}$, ψ preserves left-gcds. We then have that $\psi(a) \wedge \psi(b) = \psi(a \wedge b)$ is trivial, and $\psi(a)^{-1}\psi(b)$ is reduced.

(b) By definition of the symmetric normal form, this is simply applying Lemma 4.1.4 to a).

(c) As ψ is a functor, we have $\psi(f) = \psi(\Delta^k(u))\psi(s_1)\cdots\psi(s_r) = \Delta^k(\psi(u))\psi(s_1)\cdots\psi(s_r)$. Since ψ is an automorphism, we have $\psi(s_1) \neq \Delta(\phi^k(\psi(u))$ as $s_1 \neq \Delta(\phi^k(u))$. The path $\psi(s_1)\cdots\psi(s_r)$ is greedy by Lemma 4.1.4. $\square$

Using this preservation of normal forms, we easily obtain that $\mathcal{G}^\psi$ is generated by $\mathcal{C}^\psi$ in $\mathcal{G}$. Moreover, we have the following result:

Corollary 4.1.6. *The category $\mathcal{C}^\psi$ is nonempty if and only if the groupoid $\mathcal{G}^\psi$ is nonempty, and we have $\mathcal{C}^\psi = \mathcal{G}^\psi \cap \mathcal{C}$. Furthermore, if $\mathcal{G}^\psi$ is nonempty, then $\mathcal{G}^\psi$ is the subgroupoid of $\mathcal{G}$ generated by $\mathcal{C}^\psi$, and the embedding $\mathcal{C}^\psi \to \mathcal{G}$ identifies $\mathcal{G}^\psi$ with the enveloping groupoid of $\mathcal{C}^\psi$.*

Proof. The first statement is immediate, as $\mathcal{C}^\psi \neq \varnothing$ and $\mathcal{G}^\psi \neq \varnothing$ are both equivalent to $\mathrm{Ob}(\mathcal{G})^\psi \neq \varnothing$.

Then, the inclusion $\mathcal{C}^\psi \subset \mathcal{G}^\psi \cap \mathcal{C}$ is immediate. Conversely, let $f \in \mathcal{G}$ have reduced left-fraction decomposition $f = a^{-1}b$. By Proposition 4.1.5, the reduced left-fraction decomposition of $\psi(f)$ is $\psi(a)^{-1}\psi(b)$. In particular, we have $f = \psi(f)$ if and only if $a = \psi(a)$ and $b = \psi(b)$. If $f \in \mathcal{G}^\psi \cap \mathcal{C}$, then we have $f = a = \psi(a)$ and $f \in \mathcal{C}^\psi$.

Assume now that $\mathcal{G}^\psi$ is nonempty. We have $\mathcal{C}^\psi \subset \mathcal{G}^\psi$, thus the subgroupoid of $\mathcal{G}$ generated by $\mathcal{C}^\psi$ is included in $\mathcal{G}^\psi$. Conversely, let $f \in \mathcal{G}^\psi$ have reduced left-fraction decomposition $f = a^{-1}b$. We saw that $a, b \in \mathcal{C}^\psi$, and thus f belongs to the subgroupoid of $\mathcal{G}$ generated by $\mathcal{C}^\psi$.

Lastly, the embedding $\mathcal{C}^\psi \to \mathcal{G}^\psi \subset \mathcal{G}$ defines a functor from the enveloping groupoid $\mathcal{G}(\mathcal{C}^\psi)$ to $\mathcal{G}^\psi$, which is faithful and injective on objects. Conversely, since $\mathcal{G}^\psi$ is generated by $\mathcal{C}^\psi$, we obtain that this functor is full, and thus an isomorphism of groupoids. $\square$

Assume that $\mathcal{C}^\psi$ is nonempty. The groupoid $\mathcal{G}^\psi$ is identified with the groupoid of fractions of $\mathcal{C}^\psi$, and thus for $u \in \mathrm{Ob}(\mathcal{G}^\psi)$, the set $\mathcal{G}^\psi(u,-)$ (resp. $\mathcal{G}^\psi(-,u)$) is endowed with the $\mathcal{C}^\psi$ divisibility relation $\preccurlyeq_{\mathcal{C}^\psi}$ (resp. $\succcurlyeq_{\mathcal{C}^\psi}$), as in Definition 2.3.10. On the other hand, we know that $\mathcal{G}^\psi(u,-)$ (resp. $\mathcal{G}^\psi(-,u)$) is endowed with the restriction of the $\mathcal{C}$-divisibility relation on $\mathcal{G}(u,-)$ (resp. on $\mathcal{G}(-,u)$). The following lemma states that these two relations are in fact equal.

Lemma 4.1.7. *Assume that $\mathcal{G}^\psi$ is nonempty, and let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$.*

(a) Let $f, g \in \mathcal{G}^\psi(u,-)$. We have $f \preccurlyeq_{\mathcal{C}} g$ in $\mathcal{G}(u,-)$ if and only if we have $f \preccurlyeq_{\mathcal{C}^\psi} g$ in $\mathcal{G}^\psi(u,-)$.

(b) Let $f, g \in \mathcal{G}^\psi(-,u)$. We have $g \succcurlyeq_{\mathcal{C}} f$ in $\mathcal{G}(-,u)$ if and only if we have $g \succcurlyeq_{\mathcal{C}^\psi} f$ in $\mathcal{G}^\psi(-,u)$.

In particular, for $f, g \in \mathcal{C}^\psi$, we have that f left-divides (resp. right-divides) g in $\mathcal{C}^\psi$ if and only if it does in $\mathcal{C}$.

Proof. We only prove point (a), point (b) is obtained similarly. Since $\mathcal{C}^\psi$ is a subcategory of $\mathcal{C}$. We immediately have that $\mathcal{C}^\psi$-divisibility implies in particular $\mathcal{C}$-divisibility. Conversely, assume that $f, g \in \mathcal{G}^\psi(u,-)$ are such that $f \preccurlyeq_{\mathcal{C}} g$. We have $f^{-1}g \in \mathcal{C}$ by assumption, and $f^{-1}g \in \mathcal{G}^\psi$ since $f, g \in \mathcal{G}^\psi$. By Corollary 4.1.6, we have $f^{-1}g \in \mathcal{C}^\psi$ and $f \preccurlyeq_{\mathcal{C}^\psi} g$. $\square$

Using this lemma, we will denote divisibility in $\mathcal{C}^\psi$ and in $\mathcal{C}$ indifferently with the symbols $\preccurlyeq$ and $\succcurlyeq$.

Assume that $\mathcal{G}^\psi$ is nonempty. Let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$ and let $a \in \mathcal{G}(u,-)$. The automorphism ψ globally preserves $\mathcal{S}$, it then has finite order as $\mathcal{S}$ is finite. The family $F = \{\psi^n(a) \mid n \in \mathbb{Z}\} \subset \mathcal{G}(u,-)$ is then finite and we can define

$$a^\flat := \bigwedge F \text{ and } a^\# := \bigvee F,$$

where $\wedge$ and $\vee$ are the meet and join in the lattice $\mathcal{G}(u,-)$. Since ψ is an automorphism of categories, it preserves lcms and gcds in $\mathcal{C}$, and thus it preserves the lattice structures in $\mathcal{G}$. Since F is globally ψ-invariant by construction, we obtain that $a^\flat, a^\# \in \mathcal{G}^\psi(u,-)$. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}^\psi(u,-) \arrow[r, hook] & \mathcal{G}(u,-), \arrow[l, bend right, two heads, "(-)^\#"'] \arrow[l, bend left, two heads, "(-)^\flat"] \end{tikzcd}$$

where the middle map denotes the natural inclusion $\mathcal{G}^\psi(u,-) \to \mathcal{G}(u,-)$. The three maps in the above diagram respect the following adjointness property:

Lemma 4.1.8 (Adjointness). *Assume that $\mathcal{G}^\psi$ is nonempty, and let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$, $a \in \mathcal{G}(u,-)$. We have $a^\flat \preccurlyeq a \preccurlyeq a^\#$, and $a^\flat = a$ if and only if $a = a^\#$ if and only if $a \in \mathcal{G}^\psi$.*
Let $a \in \mathcal{G}(u,-), f \in \mathcal{G}^\psi(u,-)$, we have

$$a \preccurlyeq f \Leftrightarrow a^\# \preccurlyeq f \text{ and } f \preccurlyeq a \Leftrightarrow f \preccurlyeq a^\flat.$$

Proof. By definition, we have $a \in \{\psi^n(a) \mid n \in \mathbb{Z}\}$. As $a^\flat$ (resp. $a^\#$) is defined as the meet (resp. the join) of F, we have $a^\flat \preccurlyeq a \preccurlyeq a^\#$. Since $a^\flat, a^\# \in \mathcal{G}^\psi$, either $a^\flat = a$ or $a = a^\#$ imply $a \in \mathcal{G}^\psi$. Conversely, if $a \in \mathcal{G}^\psi$, then $F = \{a\}$, and $a = a^\flat = a^\#$.

Let now $u \in \mathrm{Ob}(\mathcal{G})$ be the source of a, and let $f \in \mathcal{G}(u,-)$. If $a^\# \preccurlyeq f$ (resp. $f \preccurlyeq a^\flat$), then $a \preccurlyeq f$ (resp. $f \preccurlyeq a$) is obvious by the first part. Conversely, if $a \preccurlyeq f$, then for all $n \in \mathbb{Z}$, $\psi^n(a) \preccurlyeq \psi^n(f) = f$, thus $a^\# \preccurlyeq f$ by definition. Likewise, if $f \preccurlyeq a$, then $f \preccurlyeq \psi^n(a)$ for all $n \in \mathbb{Z}$, and $f \preccurlyeq a^\flat$. □

By construction, this lemma gives that, for $a \in \mathcal{G}$, $a^\#$ is the meet of all the right-$\mathcal{C}^\psi$-multiples of a in $\mathcal{G}^\psi$, and that $a^\flat$ is the join of all the left-$\mathcal{C}^\psi$-divisors of a in $\mathcal{G}^\psi$.

The adjointness property of Lemma 4.1.8 can be used to show that the embedding $\mathcal{G}^\psi \to \mathcal{G}$ preserves the lattice structures.

Lemma 4.1.9. *Assume that $\mathcal{G}^\psi$ is nonempty, and let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$.*

(a) The poset $(\mathcal{G}^\psi(u,-), \preccurlyeq_{\mathcal{C}^\psi})$ is a lattice, and the natural inclusion $\mathcal{G}^\psi(u,-) \to \mathcal{G}(u,-)$ preserves joins and meets.

(b) The poset $(\mathcal{G}^\psi(-,u), \succcurlyeq_{\mathcal{C}^\psi})$ is a lattice, and the natural inclusion $\mathcal{G}^\psi(-,u) \to \mathcal{G}(-,u)$ preserves joins and meets.

In particular, we obtain that lcms and gcds exist in $\mathcal{C}^\psi$.

Proof. (a) Let $f, g \in \mathcal{G}^\psi(u,-)$. They have a join $f \vee g$ in $\mathcal{G}(u,-)$. By Lemma 4.1.8, we know that the map $(-)^\# : \mathcal{G}(u,-) \to \mathcal{G}^\psi(u,-)$ is a left-adjoint. As such, it preserves colimits, and in particular it preserves joins: the morphism $(f \vee g)^\# \in \mathcal{G}^\psi(u,-)$ is then a join of f and g in $(\mathcal{G}^\psi(u,-), \preccurlyeq_{\mathcal{C}^\psi})$. Likewise, $(-)^\flat$ is a right-adjoint, which preserves limits and meets. Thus $(f \vee g)^\flat$ is a meet of f and g in $(\mathcal{G}^\psi(u,-), \preccurlyeq_{\mathcal{C}^\psi})$. Point (b) is obtained similarly, by defining analogues of $(-)^\flat$ and $(-)^\#$ going from $\mathcal{G}(-,u)$ to $\mathcal{G}^\psi(-,u)$ and proving Lemma 4.1.8 in this case.

□

Another feature of the map $(-)^\#$ is that it allows us to recognize the atoms of $\mathcal{C}^\psi$.

Lemma 4.1.10 (Atoms of category of fixed points). *[DDGKM, Proposition 4.4] All atoms of $\mathcal{C}^\psi$ have the form $a^\#$, where a is an atom of $\mathcal{C}$ with source in $\mathrm{Ob}(\mathcal{C}^\psi)$.*

Proof. Let $\alpha \in \mathcal{C}^\psi$ be an atom with source $u \in \mathrm{Ob}(\mathcal{C}^\psi)$. Since $\alpha \in \mathcal{C}(u,-)$ is nontrivial, one can consider an atom $a \in \mathcal{C}(u,-)$ such that $a \preccurlyeq \alpha$. We then have $1_u \prec a^\# \preccurlyeq \alpha$ and $a^\# = \alpha$. □

Note that this does not prove that every $a^\#$, where a is an atom with source in $\mathrm{Ob}(\mathcal{C}^\psi)$, is an atom of $\mathcal{C}^\psi$. We could have $a^\# \prec b^\#$ for two distinct atoms a and b.

We are now equipped to show that, when nonempty, the groupoid of fixed points $\mathcal{G}^\psi$ is a Garside groupoid, compatible with the Garside structure on $\mathcal{G}$.

By construction, ψ induces a bijection of the germ of simples $(\mathcal{S}, \cdot)$ attached to $(\mathcal{C}, \Delta)$, and we have

$$\forall s, t \in \mathcal{S},\ \psi(s) \cdot \psi(t) = \psi(u) \Leftrightarrow s \cdot t = u.$$

Indeed, $\psi(s) \cdot \psi(t)$ is defined if and only if the composition $\psi(s)\psi(t) = \psi(st)$ exists in $\mathcal{C}$ and is equal to $\psi(u)$. Since ψ is an automorphism, this is equivalent to $st = u$ in $\mathcal{C}$.

We can then consider the *germ of fixed points* $(\mathcal{S}^\psi, \cdot)$, defined as a subgerm of the germ of simples $(\mathcal{S}, \cdot)$. That is, for $s, t, u \in \mathcal{S}^\psi$, we have $s \cdot t = u$ in $(\mathcal{S}^\psi, \cdot)$ if and only if $s \cdot t = u$ in $(\mathcal{S}, \cdot)$.

Theorem 4.1.11 (Garside groupoid of fixed points). *[DDGKM, Corollary 4.3]*
Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let ψ be an automorphism of $\mathcal{C}$ which preserves Δ and such that $\mathcal{G}^\psi$ is nonempty.
The triple $(\mathcal{G}^\psi, \mathcal{C}^\psi, \Delta)$ is a Garside groupoid with germ of simples $(\mathcal{S}^\psi, \cdot)$. Furthermore, an element $f \in \mathcal{G}$ lies in $\mathcal{G}^\psi$ if and only if all the terms of its symmetric (resp. left) normal form lie in $\mathcal{G}^\psi$. Under these conditions, the symmetric (resp. left) normal form of f in $\mathcal{G}$ and in $\mathcal{G}^\psi$ are equal.

Proof. Let $f \in \mathcal{G}$ have symmetric normal form $f = a_p^{-1} \cdots a_1^{-1} b_1 \cdots b_q$. By Proposition 4.1.5, the symmetric normal form of $\psi(f)$ is $\psi(a_p)^{-1} \cdots \psi(a_1)^{-1}\psi(b_1) \cdots \psi(b_q)$. By uniqueness of the symmetric normal form, we obtain that $f \in \mathcal{G}^\psi$ if and only if $\psi(a_i) = a_i$ for every $i \in [\![1, p]\!]$ and $\psi(b_j) = b_j$ for every $j \in [\![1, q]\!]$. The case of the left normal form is handled similarly.

We now show that $(\mathcal{G}^\psi, \mathcal{C}^\psi, \Delta)$ is a Garside groupoid. We know that $\mathcal{G}^\psi$ is the enveloping groupoid of $\mathcal{C}^\psi$ by Corollary 4.1.6, and we only have to show that $(\mathcal{C}^\psi, \Delta)$ is a Garside category. First, we have that $\mathcal{C}^\psi$ is cancellative and admits no nontrivial invertible elements as a subcategory of $\mathcal{C}$. Then, for $u \in \mathrm{Ob}(\mathcal{C}^\psi)$, we already showed that $\Delta(u)$ lies in $\mathcal{C}^\psi$ and has source u. We obtain that Δ is a balanced map in $\mathcal{C}^\psi$ by Lemma 4.1.7. Indeed we have that the (left- or right-) divisors of Δ in $\mathcal{C}^\psi$ are exactly the elements of $\mathcal{S}^\psi$. The first part of the proof gives that $\mathcal{S}^\psi$ generates $\mathcal{C}^\psi$. Lastly, Lemma 4.1.9 gives that any $g \in \mathcal{C}^\psi(u, -)$ has a left-gcd with $\Delta(u)$, and that this left-gcd is equal to $\alpha(g)$. Thus, Δ is indeed a Garside map in $\mathcal{C}^\psi$.

Let $f \in \mathcal{C}^\psi$, we show that the greedy normal form of f in $\mathcal{C}^\psi$ is the same as its greedy normal form in $\mathcal{C}$. This simply comes from Lemma 4.1.9, which gives that $\alpha(f) = f \wedge \Delta(u) \in \mathcal{C}^\psi$ and $\omega(f) \in \mathcal{C}^\psi$. From this we easily deduce that the symmetric (resp. left) normal form of $f \in \mathcal{G}^\psi$ in $\mathcal{G}^\psi$ or in $\mathcal{G}$ are equal.

We finish by showing that the germ of simples of $(\mathcal{C}^\psi, \Delta)$ is the germ of fixed points $(\mathcal{S}^\psi, \cdot)$. By definition, we have $s \cdot t = u$ in the germ of simples of $(\mathcal{C}^\psi, \Delta)$ if and only if $st = u$ in $\mathcal{C}^\psi$, with $s, t, u \in \mathcal{S}^\psi$. But $st = u$ in $\mathcal{C}^\psi$ if and only if $st = u$ in $\mathcal{C}$, which is in turn equivalent to $s \cdot t = u$ in the germ of fixed points. The two germs structures on $\mathcal{S}^\psi$ are then equal, which terminates the proof. $\square$

In particular, with the above notation, we obtain that $f \in \mathcal{C}$ lies in $\mathcal{C}^\psi$ if and only if all the terms of its greedy normal form lie in $\mathcal{C}^\psi$, and the embedding $\mathcal{C}^\psi \to \mathcal{C}$ preserves greedy normal forms.

Example 4.1.12. Consider the monoid $M = \langle s, t, u, \mid sts = tst, su = us, tut = utu\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. The set $\mathcal{S}$ of simples contains 24 elements, and the lattice $(\mathcal{S}, \preccurlyeq)$ is described in [Pic00, Example 2]. Consider the Garside automorphism ϕ of M. It acts on the atoms by swapping s and u. As in Lemma 4.1.10, the atoms of M^ψ are given by

$$t^\# = \bigvee\{t\} = t \quad \text{and} \quad s^\# = s \vee \phi(s) = su = us = u^\#.$$

More generally, we have $\mathcal{S}^\psi = \{1, t, su, sut, tsu, sutsu, tsut, \Delta\}$. The Hasse diagram of $\mathcal{S}^\psi$ is given by

$$\begin{tikzcd}[arrows=-]
 & \Delta \arrow[dl, dashed] \arrow[dr] & \\
tsut \arrow[d] & & sutsu \arrow[d, dashed] \\
tsu \arrow[d, dashed] & & sut \arrow[d] \\
t \arrow[dr] & & su \arrow[dl, dashed] \\
 & 1 &
\end{tikzcd}$$

where the plain (resp. dashed) lines represent multiplication on the right by t (resp. su). If we set $x := su$ and $y := t$, we obtain that $M^\psi = \langle x, y \mid xyxy = yxyx\rangle^+$ is the Artin-Tits monoid of type B_2.

Remark 4.1.13 (Connectedness). A category of fixed points $\mathcal{C}^\psi$ could have several connected components, even if the category $\mathcal{C}$ is connected. Taking the fixed points under some automorphism in a Garside category is one of the constructions we consider that can produce disconnected Garside categories.

Example 4.1.14. The condition that ψ preserves Δ depends on the choice of the Garside category $(\mathcal{C}, \Delta)$. Consider for instance the group $G = \langle a, b \mid abab = baba\rangle$. We can endow G with the automorphism ψ swapping a and b. Consider now the two monoids

$$M_1 = \langle a, b \mid abab = baba\rangle^+ \text{ and } M_2 = \langle a, b, c, d \mid ab = bc = cd = da\rangle^+.$$

They are Garside monoids with respective Garside elements $\Delta_1 := abab$ and $\Delta_2 := ab$. One readily checks that $G(M_1) \simeq G(M_2) \simeq G$. The set of simples of (M_1, Δ_1) is $\{1, a, b, ab, ba, aba, bab, abab\}$ and is preserved by ψ. On the other hand, ψ does not preserve $\Delta_2 = ab$, and thus it doesn't preserve the set of simples of (M_2, Δ_2).

4.2 Groupoids of cosets

Let $\mathcal{G}$ be a connected groupoid. Let also $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u, u)$ be a finite index subgroup. Recall from Section 1.4 that the groupoid of cosets $\mathcal{G}_H$ is the groupoid whose objects are the right-cosets $H\backslash\mathcal{G}$, and whose morphisms have the form $f_{[g]} : [g] \to [gf]$ for $g \in \mathcal{G}(u, v)$ and $f \in \mathcal{G}(v, v')$. The groupoid $\mathcal{G}_H$ is equivalent to H as a category. It naturally comes equipped with a functor $\pi : \mathcal{G}_H \to \mathcal{G}$, sending $f_{[g]}$ to f. We aim to construct a Garside structure on $\mathcal{G}_H$, provided that $\mathcal{G}$ is a Garside groupoid. This idea was communicated to us by Thomas Haettel, and we did not find explicit references to it in the literature (although the fact that a finite index subgroup of a weak Garside group is again a weak Garside group is an immediate consequence of [HH24, Theorem 4.7]). We thank him sincerely for this idea.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also fix an object u of $\mathcal{G}$, along with a finite index subgroup $H \subset \mathcal{G}(u, u)$.

We define the *category of cosets* $\mathcal{C}_H$ as the subcategory $\pi^{-1}(\mathcal{C})$ of the groupoid of cosets $\mathcal{G}_H$. That is, elements of $\mathcal{C}_H$ are the $f_{[g]}$ such that $f \in \mathcal{C}$. We claim that $\mathcal{C}_H$ is a Garside category with enveloping groupoid $\mathcal{G}_H$. The main ingredient of the proof is the following lemma, which allows us to relate the $\mathcal{C}_H$-divisibility in $\mathcal{G}_H$ to the $\mathcal{C}$-divisibility in $\mathcal{G}$.

Lemma 4.2.1 (Preservation of lattice structures). *Let $v \in \mathrm{Ob}(\mathcal{G})$, and let $g \in \mathcal{G}(u,v)$. The functor $\pi : \mathcal{G}_H \to \mathcal{G}$ induces poset isomorphisms*

$$(\mathcal{G}_H([g],-), \preccurlyeq_{\mathcal{C}_H}) \simeq (\mathcal{G}(v,-), \preccurlyeq_{\mathcal{C}}) \text{ and } (\mathcal{G}_H(-,[g]), \succcurlyeq_{\mathcal{C}_H}) \simeq (\mathcal{G}(-,v), \succcurlyeq_{\mathcal{C}}).$$

Proof. We know from Lemma 1.4.6 that π induces a bijection between $\mathcal{G}_H([g],-)$ and $\mathcal{G}(v,-)$, whose inverse sends $f \in \mathcal{G}(v,-)$ to $f_{[g]}$. Let $f, h \in \mathcal{G}_H([g],-)$ for some $[g] \in \mathrm{Ob}(\mathcal{G}_H)$, we have

$$f \preccurlyeq_{\mathcal{C}_H} h \Leftrightarrow f^{-1}h \in \mathcal{C}_H \Leftrightarrow \pi(f)^{-1}\pi(h) \in \mathcal{C} \Leftrightarrow \pi(f) \preccurlyeq_{\mathcal{C}} \pi(h).$$

Thus π induces the desired isomorphism of posets. A similar arguments works for the dual assumption. $\square$

Notice that, with the notation of the lemma, the functor π restricts to isomorphisms of posets $(\mathcal{C}_H([g],-), \preccurlyeq) \simeq (\mathcal{C}_H(v,-), \preccurlyeq)$ and $(\mathcal{C}_H(-,[g]), \succcurlyeq) \simeq (\mathcal{C}_H(v,-), \succcurlyeq)$.

Theorem 4.2.2 (Garside groupoid of cosets). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. Let also $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u,u)$ be a finite index subgroup.*
The triple $(\mathcal{G}_H, \mathcal{C}_H, \Delta_H)$ is a Garside groupoid, where $\Delta_H : \mathrm{Ob}(\mathcal{G}_H) \to \mathcal{G}_H$ is defined by $\Delta_H([g]) := \Delta(v)_{[g]}$ for $g \in \mathcal{G}(u,v)$. The simple elements of $(\mathcal{G}_H, \mathcal{C}_H, \Delta_H)$ are given by $\mathcal{S}_H := \pi^{-1}(\mathcal{S})$, and its Garside automorphism is given by $\phi_H(f_{[g]}) = \phi(f)_{[\Delta(u)\phi(g)]}$.

Proof. First, we have to prove that $\mathcal{C}_H$ is a left-cancellative category with no nontrivial invertible elements. Cancellativity is obvious as $\mathcal{C}_H$ is a subcategory of the groupoid $\mathcal{G}_H$, let then $f_{[g]}$ be an invertible morphism in $\mathcal{C}_H$, with $g \in \mathcal{G}(u,v)$. Since π is functor, $\pi(f_{[g]})$ is invertible in $\mathcal{C}$, thus it is trivial. By Lemma 4.2.1, this implies that $f_{[g]}$ is trivial.

Now, we show that Δ_H is a Garside map in $\mathcal{C}_H$. Let $g \in \mathcal{G}(u,v)$, and let $f_{[g]} \in \mathcal{G}_H([g],-)$. By Lemma 4.2.1, we have $f_{[g]} \preccurlyeq \Delta_H([g])$ if and only if $f \preccurlyeq \Delta(v)$. Thus $\mathcal{S}_H := \mathrm{Div}(\Delta_H) = \pi^{-1}(\mathcal{S})$. We also show that $\mathrm{Div}_R(\Delta_H) = \pi^{-1}(\mathcal{S})$ using Lemma 4.2.1. Since $\mathcal{S}$ generates $\mathcal{C}$, we have that $\pi^{-1}(\mathcal{S})$ generates $\pi^{-1}(\mathcal{C})$. Now, we know that π induces a bijection between $\mathcal{S}_H([g],-)$ and $\mathcal{S}(v,-)$. Thus $\mathcal{S}_H$ is finite since both $\mathcal{S}$ and $\mathrm{Ob}(\mathcal{G}_H)$ are finite.

Lastly, let $f_{[g]} \in \mathcal{C}_H$ for some $g \in \mathcal{G}(u,v)$. By Lemma 4.2.1, the elements f and $\Delta_H([g])$ have a left-gcd, which is simply given by $\alpha(f)_{[g]}$. Thus $(\mathcal{C}_H, \Delta_H)$ is a Garside category. For the Garside automorphism, let $g \in \mathcal{G}(u,v)$ and $f \in \mathcal{G}(v,v')$, we have

$$\begin{aligned} f_{[g]}\Delta_H([gf]) &= f_{[g]}\Delta(v')_{[gf]} \\ &= (f\Delta(v'))_{[g]} \\ &= (\Delta(v)\phi(f))_{[g]} \\ &= \Delta_H([g])\phi(f)_{[g\Delta(v)]} \\ &= \Delta_H([g])\phi(f)_{[\Delta(u)\phi(g)]}. \end{aligned}$$

Now that we know that $(\mathcal{C}_H, \Delta_H)$ is a Garside category, it only remains to show that $\mathcal{G}_H$ is the enveloping groupoid of $\mathcal{C}_H$. Let $\mathcal{G}(\mathcal{C}_H)$ be the enveloping groupoid of $\mathcal{C}_H$. As $\mathcal{C}_H$ is a

Garside category, the natural functor $\mathcal{C}_H \to \mathcal{G}(\mathcal{C}_H)$ is faithful. As the embedding $\mathcal{C}_H \to \mathcal{G}_H$ is also faithful, we get that the induced functor $\mathcal{G}(\mathcal{C}_H) \to \mathcal{G}_H$ is also faithful. It is bijective on objects by definition. It is then enough to show that $\mathcal{C}_H$ generates $\mathcal{G}_H$ to show that the functor $\mathcal{G}(\mathcal{C}_H) \to \mathcal{G}_H$ is full, and therefore an isomorphism. Let $f_{[g]} \in \mathcal{G}_H$. We can write $f = x^{-1}y$ as a fraction in $\mathcal{G}$, with $x \in \mathcal{C}(v', v)$. We then have that $f_{[g]} = (x_{[gx^{-1}]})^{-1} y_{[gx^{-1}]}$ is a fraction of two elements of $\mathcal{C}_H$. Thus $\mathcal{C}_H$ generates $\mathcal{G}_H$, and $\mathcal{G}(\mathcal{C}_H) = \mathcal{G}_H$. □

Remark 4.2.3. The assumption that H has finite index in $\mathcal{G}(u, u)$ is only essential to show that $\mathcal{S}_H$ is finite. The rest of the proof would work out the same without this assumption. We would obtain a quasi-Garside structure, respecting some "local finiteness" condition: for $[g] \in \mathrm{Ob}(\mathcal{G}_H)$, the set $\mathcal{S}_H([g], -)$ would still be finite.

A more philosophical consequence of this theorem is the following corollary:

Corollary 4.2.4. *Let G be a weak Garside group, and let $H \subset G$ be a finite index subgroup. The group H is again a weak Garside group.*

To our knowledge, no such result is known, nor conjectured, regarding Garside groups.

Corollary 4.2.5 (Preservation of normal forms). *The functor $\pi : \mathcal{C}_H \to \mathcal{C}$ preserves greedy normal forms. The functor $\pi : \mathcal{G}_H \to \mathcal{G}$ preserves symmetric normal forms and left normal forms.*

Proof. We first show that $\pi : \mathcal{C}_H \to \mathcal{C}$ preserves greedy normal forms. We saw that, tanks to Lemma 4.2.1, π preserves gcds. In particular, for $f \in \mathcal{G}_H$, we have $\alpha(\pi(f)) = \pi(\alpha(f))$ and $\pi(\omega(f)) = \omega(\pi(f))$. Since the greedy normal form of f is obtained by iterating α and ω (see Proposition 2.1.16), the result follows by an immediate induction.

Let now $f_{[g]} = (x_{[gx^{-1}]})^{-1} y_{[gx^{-1}]}$ be the reduced left-fraction decomposition of $f_{[g]}$ in $\mathcal{G}_H$. By definition, the left-gcd $x_{[gx^{-1}]} \wedge y_{[gx^{-1}]}$ is trivial. By Lemma 4.2.1, this implies that $x \wedge y$ is trivial. Thus $f = x^{-1}y$ is the reduced left-fraction decomposition of $f = \pi(f_{[g]})$. Since we already know that π preserves greedy normal form, this implies that π also preserves symmetric normal forms.

Since inf and sup are uniquely determined by the greedy normal forms of the numerator and denominator (Corollary 2.3.27), we also have that π preserves inf and sup. By definition, the left normal form of $f \in \mathcal{G}_H([g], -)$ is given by $\Delta_H^{\inf(f)}([g])$, followed by the greedy normal form of $\Delta_H^{-\inf(f)}(\phi_H^{\inf(f)}([g]))f \in \mathcal{C}_H$. Since π preserves the Garside map, the inf and greedy normal forms of elements of $\mathcal{C}_H$, we deduce that π preserves left normal forms. □

4.3 Conjugacy graphs

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. For $x \in \mathcal{G}$ an endomorphism, we already considered in Chapter 3 several subsets of the conjugacy class $\mathrm{Cl}_{\mathcal{G}}(x)$, which completely characterize $\mathrm{Cl}_{\mathcal{G}}(x)$ (super-summit set, set of sliding circuits, set of circuits for swap...). Each of these sets can be seen as the objects set of some full subcategory of the positive conjugacy category $\mathrm{Conj}^+(\mathcal{G})$. We also mentioned (see Remark 3.1.11) that the positive conjugacy category $\mathrm{Conj}^+(\mathcal{G})$ was naturally endowed with the structure of a quasi-Garside category.

The purpose of this section is to show that, under suitable assumption, the quasi-Garside structure on $\mathrm{Conj}^+(\mathcal{G})$ actually induces a Garside structure on a finite subset of $\mathrm{Cl}_{\mathcal{G}}(x)$, allowing in turn to study the centralizer $C_{\mathcal{G}}(x)$ as a weak Garside group.

4.3.1 Conjugacy sets

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also consider an endomorphism x in $\mathcal{G}$.

We begin by giving two equivalent conditions which will be at the core of the subsequent definition of a conjugacy graph.

Lemma 4.3.1 (Conjugacy sets). *Let Γ be a nonempty subset of $\mathrm{Cl}_{\mathcal{G}}(x)$ which is stable under ϕ. The following statements are equivalent:*

(i) For all $y \in \Gamma$, if $f, g \in \mathcal{G}$ are such that $y^f, y^g \in \Gamma$, then $y^{f\wedge g} \in \Gamma$.

(ii) For all $y \in \Gamma$, if $f \in \mathcal{C}$ is such that $y^f \in \Gamma$, then $y^{\alpha(f)} \in \Gamma$.

If these conditions are met, we say that Γ is a conjugacy set *(for x).*

Proof. The fact that (i) implies (ii) is quite obvious: by assumption, we have $\phi(y) = y^{\Delta(u)} \in \Gamma$, where u is the source of y in $\mathcal{G}$. Thus, we have $y^{\alpha(f)} = y^{f\wedge\Delta(u)} \in \Gamma$ by (i).

Conversely, assume (ii). Let $f \in \mathcal{C}$ be such that $y^f \in \Gamma$, and let $f = s_1 \cdots s_r$ be the greedy normal form of f. Since $s_i = \alpha(s_i \cdots s_r)$, an easy induction shows that

$$\forall i \in [\![1, r]\!],\ y^{s_1 \cdots s_i} \in \Gamma.$$

Recall from Proposition 2.1.25 (powers of a Garside map) that $\alpha_i(f) := f \wedge \Delta^i(u)$ is equal to $s_1 \cdots s_i$. We deduce that, if $y^f \in \Gamma$ for some $f \in \mathcal{C}$, then $y^{\alpha_r(f)} \in \Gamma$ for all $r \geqslant 1$.

We first show (i) when f and g are both positive. In this case, let u (resp. v) be the source (resp. the target) of f, and let $r = \sup(f)$. By Proposition 2.1.25 (powers of a Garside map), we can consider $f^* \in \mathcal{C}$ be such that $f^* f = \Delta^r(\phi^{-r}(v))$. Since ϕ has finite order, Γ is stable under ϕ^{-1} and ϕ^{-r}, thus

$$f^* y (f^*)^{-1} = \Delta^r(\phi^{-r}(v)) f^{-1} y f \Delta^{-r}(v) = \phi^{-r}(y^f) \in \Gamma.$$

Furthermore, $(f^* y (f^*)^{-1})^{f^* g} = y^g$ belongs to Γ by assumption, we deduce that

$$(f^* y (f^*)^{-1})^{\alpha_r(f^* g)} \in \Gamma.$$

However, we have

$$\alpha_r(f^* g) = \Delta^r(\phi^{-r}(v)) \wedge f^* g = f^* f \wedge f^* g = f^*(f \wedge g).$$

And thus $(f^* y (f^*)^{-1})^{\alpha_r(f^* g)} = y^{f\wedge g} \in \Gamma$ as claimed.

Now, assume that f and g are arbitrary morphisms such that $y^f, y^g \in \Gamma$. Let $m > 0$ be such that ϕ^m is trivial and $\Delta^m(\phi^{-m}(u))f$ and $\Delta^m(\phi^{-m}(u))g$ are positive (where u is the common source of x, f and g). We have $y^{\Delta^m(\phi^{-m}(u))f} = y^f \in \Gamma$ and $y^{\Delta^m(\phi^{-m}(u))g} = y^g \in \Gamma$. By the first part, we have

$$y^{\Delta^m(\phi^{-m}(u))f \wedge \Delta^m(\phi^{-m}(u))g} = y^{\Delta^m(\phi^{-m}(u))(f\wedge g)} = y^{f\wedge g} \in \Gamma,$$

whence the result. □

As an immediate consequence of this, we can show that the various subsets associated to a conjugacy class that we saw in Chapter 3 are examples of conjugacy sets.

Corollary 4.3.2. *The super-summit set* $\mathrm{SSS}(x)$*, the set of sliding circuits* $\mathrm{SC}(x)$*, and the set of recurrent conjugates* $\mathsf{R}(x)$ *of* x*, are all conjugacy sets.*

Proof. First, the super-summit set if stable under ϕ since ϕ preserves both the infimum and the supremum. It is a conjugacy set by Lemma 3.2.9. The set of sliding circuits is a conjugacy set by [DDGKM, Proposition VIII.2.33] and [DDGKM, Lemma VIII.2.36]. Lastly, the set of recurrent conjugates of x is a conjugacy set by Lemma 3.3.15 and Proposition 3.3.19 □

Later on, we will need a finiteness assumption on a conjugacy set in order to obtain a Garside category (and not only a quasi-Garside category). In practice this is not a big problem thanks to the following proposition, which shows that a conjugacy set can always be written as the union of an increasing sequence of finite conjugacy sets.

Proposition 4.3.3. *Let* Γ *be a conjugacy set for* x*. For* $m > 0$ *an integer, let*

$$\Gamma_m := \{y \in \Gamma \mid \inf(y) \geqslant -m \text{ and } \sup(y) \leqslant m\}.$$

If nonempty, the set Γ_m *is a finite conjugacy set for* x*. Furthermore,* Γ_m *is nonempty for* m *big enough.*

Proof. Let $y \in \Gamma$, we have $y \in \Gamma_m$ for $m = \max\{-\inf(y), \sup(y)\}$ and Γ_m is nonempty. Furthermore, Γ_m is always finite, and we only need to show that it is a conjugacy set. Since ϕ preserves both inf and sup, and since Γ is stable under ϕ, we obtain that Γ_m is stable under ϕ.

Let $y \in \Gamma_m$ have source u in $\mathcal{G}$, and let $f \in \mathcal{C}(u, v)$ be such that $y^f \in \Gamma_m$. By Lemma 2.3.23, we have $\Delta^{-m}(u) \preccurlyeq_{\mathcal{C}} y$ and $\Delta^{-m}(v) \preccurlyeq_{\mathcal{C}} y^f$. In other words, we have $\Delta^m(\phi^{-m}(u))y \in \mathcal{C}$ and

$$\begin{aligned} &\Delta^m(\phi^{-m}(v))y^f = \phi^{-m}(f^{-1})\Delta^m(\phi^{-m}(u))yf \in \mathcal{C} \\ \Leftrightarrow &\phi^{-m}(f) \preccurlyeq_{\mathcal{C}} \Delta^m(\phi^{-m}(u))yf. \end{aligned}$$

Let us write $s := \alpha(f)$ and $h := \omega(f)$. We have

$$\phi^{-m}(s)\phi^{-m}(h) \preccurlyeq_{\mathcal{C}} \Delta^m(\phi^{-m}(u))ysh.$$

Note that, since $\Delta^m(\phi^{-m}(u))y \in \mathcal{C}$, we actually have $\phi^{-m}(s) \preccurlyeq \Delta^m(\phi^{-m}(u))ysh$. Furthermore, since $\phi^{-m}(s)$ is a simple morphism, we have

$$\begin{aligned} \phi^{-m}(s) &\preccurlyeq \alpha(\Delta^m(\phi^{-m}(u))ysh) \\ &= \alpha(\Delta^m(\phi^{-m}(u))y\alpha(sh)) \\ &= \alpha(\Delta^m(\phi^{-m}(u))ys) \\ &\preccurlyeq \Delta^m(\phi^{-m}(u))ys. \end{aligned}$$

In other words $\phi^{-m}(s^{-1})\Delta^m(\phi^{-m}(u))ys = \Delta^m(\phi^{-m}(w))y^s \in \mathcal{C}$, where w is the target of s. We deduce that $\Delta^{-m}(w) \preccurlyeq_{\mathcal{C}} y^s$ and $-m \leqslant \inf(y^s)$.

We have $\inf(y^{-1}) = -\sup(y) \geqslant -m$ and $\inf((y^{-1})^f) \geqslant -m$ by Corollary 2.3.28 (inf and sup of inverse). Thus, applying the same reasoning as above to y^{-1} yields $\inf((y^{-1})^s) \geqslant -m$ and $\sup(y^s) \leqslant m$ again by Corollary 2.3.28. Since $y^s \in \Gamma$ by definition, we obtain $y^s \in \Gamma_m$ which terminates the proof. □

4.3.2 Garside conjugacy graphs

Our next goal is to endow conjugacy sets with a germ structure related to the quasi-Garside structure of the category $\mathrm{Conj}^+(\mathcal{G})$.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also consider a conjugacy set Γ for some endomorphism in $\mathcal{G}$.

Definition 4.3.4 (Conjugacy Graph). The *conjugacy graph* $\mathcal{S}_\Gamma$ associated to Γ is defined as follows

- The objects set of $\mathcal{S}_\Gamma$ is Γ.
- For $y, z \in \Gamma$, $\mathcal{S}_\Gamma(y, z) = \{s \in \mathcal{S} \mid y^s = z\}$. If need be, we will write s_y to emphasis that we see s in $\mathcal{S}_\Gamma(y, -)$.

We endow the conjugacy graph $\mathcal{S}_\Gamma$ with a partial product by saying that $s_y \cdot t_z$ is defined if and only if $st \in \mathcal{S}$ and $y^s = z$, in which case it is equal to $(st)_y$.

We first show that the partial product defined for Γ does induce a germ structure.

Lemma 4.3.5. *The partial product $\cdot$ on the conjugacy graph $\mathcal{S}_\Gamma$ endows is with a germ structure.*

Proof. Let $s_y, t_z \in \mathcal{S}_\Gamma$ such that $s_y \cdot t_z$ is defined. The target of s_y is y^s, which is the source of t_z. The source of $s_y \cdot t_z = (st)_y$ is y, and its target is $y^{st} = z^t$, which is the target of t_z. For $y \in \mathrm{Ob}(\mathcal{S}_\Gamma)$ with source u in $\mathcal{G}$, the element $(1_u)_y$ plays the role of an identity element. Lastly, let there be a diagram in $\mathcal{S}_\Gamma$

$$y \xrightarrow{s_y} z \xrightarrow{t_z} w \xrightarrow{r_w} v\ .$$

We have

$$\begin{aligned}(s_y \cdot t_z) \cdot r_w \text{ is defined} &\Leftrightarrow (st) \in \mathcal{S} \text{ and } ((st)r) = str \in \mathcal{S} \\ &\Leftrightarrow (tr) \in \mathcal{S} \text{ and } (s(tr)) \in \mathcal{S} \\ &\Leftrightarrow s_y \cdot (t_z \cdot r_w) \text{ is defined.}\end{aligned}$$

In this case, we have $(s_y \cdot t_z) \cdot r_w = (str)_y = s_y \cdot (t_z \cdot r_w)$. Thus $(\mathcal{S}_\Gamma, \cdot)$ is indeed a germ. □

Thanks to Lemma 4.3.5, we know have a germ $(\mathcal{S}_\Gamma, \cdot)$, and we can consider the associated category $\mathcal{C}_\Gamma := \mathcal{C}(\mathcal{S}_\Gamma)$, along with its enveloping groupoid $\mathcal{G}_\Gamma := \mathcal{G}(\mathcal{C}_\Gamma)$.

We now show that the germ attached to the conjugacy graph is actually a Garside germ. This is where we need to add a finiteness assumption on Γ.

Theorem 4.3.6 (Garside conjugacy germ). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let Γ be a finite conjugacy set for an endomorphism in $\mathcal{G}$.*
The germ structure on the conjugacy graph $\mathcal{S}_\Gamma$ is a Garside germ. Its Garside map is given by $\Delta_\Gamma(y) := \Delta(u)_y$, for $y \in \Gamma$ with source u in $\mathcal{G}$. The triple $(\mathcal{G}_\Gamma, \mathcal{C}_\Gamma, \Delta_\Gamma)$ is then a Garside groupoid.

Proof. First, we know by Lemma 4.3.5 that $(\mathcal{S}_\Gamma, \cdot)$ is indeed a germ. By assumptions, the set Γ is finite. For $y \in \Gamma$, we have $\mathcal{S}_\Gamma(y, -) \subset \mathcal{S}$, which is finite. Thus the set $\mathcal{S}_\Gamma$ is finite.

We then show that the germ $(\mathcal{S}_\Gamma, \cdot)$ is left-cancellative. Let there be an equality of the form $s'_y \cdot t_z = s_y \cdot t_z$ in $\mathcal{S}_\Gamma$. By definition, we have $y^s = y^{s'} = z$ and $s't = st$ in $\mathcal{S}$. As $\mathcal{C}$ is cancellative, we deduce that $s = s'$ and $s_y = s'_y$. Likewise, we obtain that $(\mathcal{S}_\Gamma, \cdot)$ is right-cancellative.

Let now $y \in \Gamma$ have source u in $\mathcal{G}$. By definition of a conjugacy set, we have $\Delta(u)_y \in \mathcal{S}_\Gamma(y, \phi(y))$. For any simple $s_y \in \mathcal{S}_\Gamma(y, -)$, we have $\overline{s}_{y^s} \in \mathcal{S}_\Gamma(y^s, \phi(y))$ and

$$s_y \cdot \overline{s}_{y^s} = \Delta(u)_y.$$

Thus $s_y \preccurlyeq \Delta(u)_y$ and $\Delta_\Gamma(y) := \Delta(u)_y$ is the maximum of $(\mathcal{S}_\Gamma(y, -), \preccurlyeq_{\mathcal{S}_\Gamma})$. Likewise, $\phi^{-1}(y) \in \Gamma$ is such that $\Delta_\Gamma(\phi^{-1}(y)) = \Delta(\phi^{-1}(u))_{\phi^{-1}(y)}$ is a maximum of $(\mathcal{S}_\Gamma(-, y), \succcurlyeq_{\mathcal{S}_\Gamma})$.

It remains to show that $(\mathcal{S}_\Gamma, \cdot)$ admits left-gcds. Let $y \in \Gamma$, and let $s_y, t_y \in \mathcal{S}_\Gamma(y, -)$. By assumption, we have $y^s, y^t \in \Gamma$. By definition of a conjugacy set, we have $y^{s \wedge t} \in \Gamma$ and thus $(s \wedge t) \in \mathcal{S}_\Gamma(y, -)$. By construction of $\mathcal{S}_\Gamma$, $(s \wedge t)_y$ is the left-gcd of s_y and t_y in $\mathcal{S}_\Gamma(y, -)$, which terminates the proof. □

The groupoid $\mathcal{G}_\Gamma$ comes equipped with two natural functors. First, the map $\mathcal{S}_\Gamma \to \mathrm{Conj}^+(\mathcal{G})$ sending Γ to itself and $s_x \in \mathcal{S}_\Gamma$ to $s_x \in \mathrm{Conj}^+_{\mathcal{G}}(x, -)$ induces a functor $I : \mathcal{C}_\Gamma \to \mathrm{Conj}^+(\mathcal{G})$. We also denote by I the induced functor $\mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$. Likewise, the map $\mathcal{S}_\Gamma \to \mathcal{C}$ sending $x \in \Gamma$ to the source of x in $\mathcal{G}$, and $s_x \in \mathcal{S}_\Gamma$ to $s \in \mathcal{S}$ induces a functor $\pi : \mathcal{C}_\Gamma \to \mathcal{C}$. We also denote by π the induced functor $\mathcal{G}_\Gamma \to \mathcal{C}$.

The functor $I : \mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$ provides an isomorphism between $\mathcal{G}_\Gamma$ and a full subcategory of $\mathrm{Conj}(\mathcal{G})$, implying that $\mathcal{G}_\Gamma$ is equivalent to a centralizer in $\mathcal{G}$ (without any finiteness assumption on Γ).

Corollary 4.3.7 (Conjugacy graph and conjugacy category). *The functor $I : \mathcal{C}_\Gamma \to \mathrm{Conj}^+(\mathcal{G})$ induces an isomorphism between $\mathcal{C}_\Gamma$ and the full subcategory of $\mathrm{Conj}^+(\mathcal{G})$ whose objects are the elements of Γ. In particular, for $x \in \Gamma$, the enveloping groupoid $\mathcal{G}(\mathcal{C}_\Gamma)$ is equivalent to the centralizer $C_{\mathcal{G}}(x)$ of x in $\mathcal{G}$.*

Proof. Let $\mathrm{Conj}^+(\Gamma)$ denote the full subcategory of $\mathrm{Conj}^+(\mathcal{G})$ whose objects are the elements of Γ. By construction, the functor I induces a functor $I : \mathcal{C}_\Gamma \to \mathrm{Conj}^+(\Gamma)$ which is bijective on objects.

We show that I is a full functor. Let $f_y \in \mathrm{Conj}^+_{\mathcal{G}}(y, z)$ with $y, z \in \Gamma$. Let $f = s_1 \cdots s_r$ be the greedy normal form of f in $\mathcal{C}$. By definition of a conjugacy set, we have

$$\forall i \in [\![1, r]\!],\ y^{s_1 \cdots s_i} \in \Gamma.$$

For $i \in [\![1, r]\!]$, we can then see s_i as a conjugating element from $y^{s_1 \cdots s_{i-1}} \in \Gamma$ to $y^{s_1 \cdots s_i} \in \Gamma$, thus it belongs to $\mathcal{S}_\Gamma$, and we can write

$$f_y = (s_1)_y \cdots (s_i)_{y^{s_1 \cdots s_{i-1}}} \cdots (s_r)_{y^{s_1 \cdots s_{r-1}}}.$$

The morphism f_y then lies in the image of I, which is then surjective.

It remains to show that I is injective. We begin by the case where Γ is finite. Let $f_y \in \mathcal{C}_\Gamma$. We saw in the proof of Theorem 4.3.6 that $\alpha(f_y) = \alpha(f)_y$. We then have $I(\alpha(f_y)) = \alpha(I(f_y))$. An easy induction then shows that the functor I preserves the greedy normal form. If $g, h \in \mathcal{C}_\Gamma$ share the same source and are such that $I(g) = I(h)$, then the greedy normal forms of $I(g)$ and $I(h)$ are equal. We then deduce that $g = h$ and that I is faithful.

Now, if Γ is infinite, consider $m > 0$ such that the finite conjugacy set Γ_m of Proposition 4.3.3 is nonempty. The natural functor $I_m : \mathcal{C}_{\Gamma_m} \to \mathrm{Conj}^+(\mathcal{G})$ is just the composition of I with the natural functor $\mathcal{C}_{\Gamma_m} \to \mathcal{C}_\Gamma$. Furthermore, the natural functor $\mathcal{C}_{\Gamma_m} \to \mathcal{C}_\Gamma$ is full. Indeed, let

$f_y \in \mathcal{C}_\Gamma$ with $y \in \Gamma_m$. We can consider the greedy normal form $f = s_1 \cdots s_r$ of f in $\mathcal{C}$. By definition of a conjugacy set, the product $s_{1y} \cdots s_{ry^{s_1\cdots s_{r-1}}}$ in $\mathcal{C}_{\Gamma_m}$ is sent to f_y by the functor $\mathcal{C}_{\Gamma_m} \to \mathcal{C}_\Gamma$. Let now $f, g \in \mathcal{C}_\Gamma$ such that $I(f) = I(g)$. We can choose $m > 0$ big enough so that both the source and target of f, g lie in Γ_m. By fullness of the functor $\mathcal{C}_{\Gamma_m} \to \mathcal{C}_\Gamma$, we deduce that $I_m(f) = I_m(g)$ since Γ_m is finite, and thus that $f = g$.

Lastly, by definition, the groupoid $\mathcal{G}(\mathrm{Conj}^+(\Gamma))$ is the full subgroupoid of $\mathrm{Conj}(\mathcal{G})$ whose objects are the elements of Γ. As a full connected groupoid, it is equivalent to the automorphism group of any of its objects, which in turn is isomorphic to the centralizer of x in $\mathcal{G}$. ☐

Remark 4.3.8. Notice that the functor $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$ is simply the composition of $I : \mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$ with the natural functor $\mathrm{Conj}(\mathcal{G}) \to \mathcal{G}$. Corollary 4.3.7 proves that, for $x \in \Gamma$, with source u in $\mathcal{G}$, the group morphism $\mathcal{G}_\Gamma(x, x) \to \mathcal{G}(u, u)$ induced by π identifies $\mathcal{G}_\Gamma(x, x)$ with the centralizer $C_\mathcal{G}(x)$.

The functor $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$ is useful to relate the Garside structure of $(\mathcal{G}_\Gamma, \mathcal{C}_\Gamma, \Delta_\Gamma)$ (when Γ is finite) with the Garside structure of $(\mathcal{G}, \mathcal{C}, \Delta)$, as in the two following results:

Proposition 4.3.9 (Preservation of lattice structures). *Let $y \in \Gamma$, and let u be the source of y in $\mathcal{G}$. The functor $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$ induces poset isomorphisms*

$$(\mathcal{G}_\Gamma(y, -), \preccurlyeq_{\mathcal{C}_\Gamma}) \simeq (\mathcal{G}(u, -), \preccurlyeq_\mathcal{C}) \text{ and } (\mathcal{G}_\Gamma(-, y), \succcurlyeq_{\mathcal{C}_\Gamma}) \simeq (\mathcal{G}(-, u), \succcurlyeq_\mathcal{C}).$$

Proof. First, the functor $I : \mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$ is a fully faithful functor, which is injective on objects by Corollary 4.3.7. Furthermore, $I(f) \in \mathrm{Conj}^+(\mathcal{G})$ if and only if $f \in \mathcal{C}_\Gamma$. Thus I induces isomorphisms of posets

$$(\mathcal{G}_\Gamma(y, -), \preccurlyeq_{\mathcal{C}_\Gamma}) \simeq (\mathrm{Conj}_\mathcal{G}(y, -), \preccurlyeq_{\mathrm{Conj}^+(\mathcal{G})}) \text{ and } (\mathcal{G}_\Gamma(-, y), \succcurlyeq_{\mathcal{C}_\Gamma}) \simeq (\mathrm{Conj}_\mathcal{G}(-, y), \succcurlyeq_{\mathrm{Conj}^+(\mathcal{G})}).$$

By Lemma 3.1.10, the natural functor $\mathrm{Conj}(\mathcal{G}) \to \mathcal{G}$ induces isomorphisms of posets

$$(\mathrm{Conj}_\mathcal{G}(y, -), \preccurlyeq_{\mathrm{Conj}^+(\mathcal{G})}) \simeq (\mathcal{G}(u, -), \preccurlyeq_\mathcal{C}) \text{ and } (\mathrm{Conj}_\mathcal{G}(-, y), \succcurlyeq_{\mathrm{Conj}^+(\mathcal{G})}) \simeq (\mathcal{G}(-, u), \succcurlyeq_\mathcal{C}).$$

The result then comes by noticing that π is the composition of I with the natural functor $\mathrm{Conj}(\mathcal{G}) \to \mathcal{G}$. ☐

Proposition 4.3.10 (Preservation of normal forms). *Assume that Γ is finite. The functor $\pi : \mathcal{C}_\Gamma \to \mathcal{C}$ preserves greedy normal forms. The functor $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$ preserves symmetric normal forms and left normal forms.*

Proof. Let $s_x, t_y \in \mathcal{S}_\Gamma$ be two composable simples, and assume that $s_x t_y$ is greedy. We have $\pi(s_x t_y) = st$, and we have to show that st is greedy. We have $y^{\overline{s}} = x^{s\overline{s}} = \phi(x) \in \Gamma$, and thus we have $\overline{s}_y \in \mathcal{S}_\Gamma(y, \phi(x))$ by definition of $\mathcal{S}_\Gamma$. Let $t' := \overline{s} \wedge t$. Since $y^{\overline{s}}, y^t \in \Gamma$, we have $y^{t'} \in \Gamma$ and $t'_y \in \mathcal{S}_\Gamma(y, -)$. By construction, we have $t'_y \preccurlyeq t_y$ and $s_x t'_y = (st')_x$ is a simple morphism, which divides $s_x t_y$. Since the latter is greedy, we deduce that $s_x t'_y \preccurlyeq s_x$ and $t'_y = 1_y$. We then have that $\overline{s} \wedge t$ is trivial, and st is greedy in $\mathcal{C}$.

Let now $f \in \mathcal{C}_\Gamma$ be written in greedy normal form $s_{1x_1} \cdots s_{rx_r}$. Since s_{rx_r} is nontrivial, $\pi(s_{rx_r}) = s_r$ is nontrivial. Since all the compositions $s_{ix_i} s_{i+1x_{i+1}}$ are greedy for $i \in [\![1, r-1]\!]$, the compositions $s_i s_{i+1}$ are greedy in $\mathcal{C}$ by the above argument. Thus $s_1 \cdots s_r = \pi(f)$ is the greedy normal form of $\pi(f)$ in $\mathcal{C}$, and π preserves greedy normal forms.

Let $f \in \mathcal{G}_\Gamma$ be written as a left-fraction $f = a^{-1}b$. We have $\pi(f) = \pi(a)^{-1}\pi(b)$, and $\pi(a) \wedge \pi(b) = \pi(a \wedge b)$ by Proposition 4.3.9. Thus, if $a^{-1}b$ is the reduced left-fraction decomposition of f, then $\pi(a)^{-1}\pi(b)$ is the reduced left-fraction decomposition of $\pi(f)$. Since $\pi : \mathcal{C}_\Gamma \to \mathcal{C}$ preserves greediness, we then deduce that $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$ preserves symmetric normal forms.

Since π also sends Δ_Γ to Δ, and since it preserves symmetric normal forms, it preserves left normal forms by Proposition 2.3.25 (symmetric to left normal form). ☐

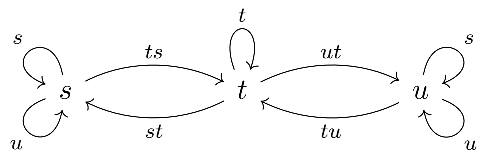
Let us also state the following easy corollary of Corollary 4.3.7, which gives a new stability result on the class of weak Garside groups.

Corollary 4.3.11. *Let G be a weak Garside group and let $x \in G$. The centralizer $C_G(x)$ is again a weak Garside group*

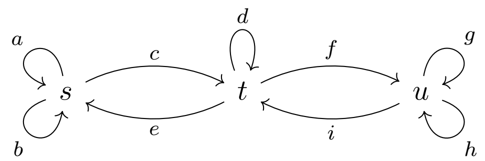
Proof. Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid such that G and $\mathcal{G}$ are equivalent. There is an object u in $\mathcal{G}$ and an isomorphism $G \simeq \mathcal{G}(u,u)$. Let y be the image of x in $\mathcal{G}(u,u)$ under this isomorphism. Let $\Gamma := \mathrm{SSS}(y)$ be the super-summit set of y in $\mathcal{G}$. The associated Garside groupoid $(\mathcal{G}_\Gamma, \mathcal{C}_\Gamma, \Delta_\Gamma)$ is equivalent to $C_{\mathcal{G}}(y)$, which itself is isomorphic to $C_G(x)$. ☐

Using Theorem 4.3.6 and Corollary 4.3.7, we can obtain a presentation for a groupoid which is equivalent to the centralizer of a given element in a weak Garside group. It was already shown in [FG03a] that classical conjugacy sets like the super-summit set can be used to obtain a generating set of the centralizer. However, Corollary 4.3.7, along with the Reidemeister-Schreier method for groupoids (see Section 1.5.1), can be used to obtain a presentation of the centralizer starting from the Garside category attached to a conjugacy graph. In this context [FG03a, Theorem 3.4] is a consequence of Lemma 1.5.3 (Schreier's Lemma). We detail below an example of computation of a centralizer.

Example 4.3.12. Consider the monoid $M = \langle s, t, u \mid sts = tst, tut = utu, su = us \rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. We want to compute the centralizer of s in the enveloping group $G(M)$ of M. The super-summit set of s is $\{s, t, u\}$. The atoms of the conjugacy category $\mathcal{C}$ attached to this super-summit set are given by



We relabel the elements of this graph by



The right-lcms of the atoms of $\mathcal{C}$ are easily computed using the relations in M. We have

$$\begin{cases} a \vee b = ab = ba, \\ a \vee c = ac = cd, \\ b \vee c = bcf = cfg, \end{cases} \qquad \begin{cases} d \vee e = de = ea, \\ d \vee f = df = fh, \\ e \vee f = ebc = fgi, \end{cases} \qquad \begin{cases} g \vee h = gh = hg, \\ h \vee i = hi = id, \\ g \vee i = gie = ieb. \end{cases}$$

Using Proposition 2.1.34, these relations give us a presentation of the category $\mathcal{C}$. A Schreier transversal for $\mathcal{C}$ rooted in s is given by $\{1_s, c, cf\}$. We obtain the following set of generators of $\mathcal{G}(\mathcal{C})(s,s) = C_{G(M)}(s)$:

$$\begin{cases}\gamma(a) = a,\\ \gamma(b) = b,\\ \gamma(c) = 1_s,\end{cases} \quad \begin{cases}\gamma(d) = cdc^{-1} = a,\\ \gamma(e) = ce,\\ \gamma(f) = 1_s,\end{cases} \quad \begin{cases}\gamma(g) = cfgf^{-1}c^{-1} = b,\\ \gamma(h) = cfhf^{-1}c^{1} = a,\\ \gamma(i) = cfic^{-1}.\end{cases}$$

By Proposition 1.5.4 (Reidemeister-Schreier method), we obtain that $\mathcal{G}(\mathcal{C})(s,s)$ is generated by $w := a, x := b, y := \gamma(e), z := \gamma(i)$, with the relations

$$\begin{cases}wx = xw,\\ w = w,\\ x = x,\end{cases} \quad \begin{cases}wy = yw,\\ w = w,\\ yx = xz,\end{cases} \quad \begin{cases}xw = wx,\\ wz = zw,\\ xzy = zyx.\end{cases}$$

After simplifications, we obtain that

$$\mathcal{G}(\mathcal{C})(s,s) = \langle w \rangle \times \langle x, y \mid xyxy = yxyx \rangle.$$

The isomorphism $\mathcal{G}(\mathcal{C})(s,s) \simeq C_{G(M)}(s)$ is given by

$$w \mapsto s,\ x \mapsto u,\ y \mapsto tsst.$$

Of course, if the centralizer of a given element in a weak Garside group is a finite index subgroup, then we already know by Section 4.2 that it is a weak Garside group. However, it is easy to find examples in which centralizers are not finite index subgroups.

Example 4.3.13. Consider the monoid $M = \langle s, t, u \mid sts = tst, tut = utu, su = us \rangle^+$ as in Example 3.2.14. It is a Garside monoid with Garside element $\Delta = stsuts$. The set of sliding circuits of the element $x = \Delta^{-1} tsuts \cdot stu \cdot u$ is given by $\mathrm{SC}(x) = \{sut, tus\}$. The atoms of the associated conjugacy graph are given by

$$\begin{tikzcd} sut \arrow[r, bend left, "su"] & tsu \arrow[l, bend left, "t"] \end{tikzcd},$$

and the centralizer of sut in $G(M)$ is infinite cyclic and generated by sut. We deduce that $C_{G(M)}(x) = \langle x \rangle$. We see that this subgroup is not a finite index subgroup of G, since it does not contain any nontrivial power of the atoms s, t, u.

We will see later (see Remark 8.4.4) that a centralizer in a complex braid group associated to an irreducible well-generated complex reflection group never has finite index, unless it is equal to the whole group (i.e. of index 1). Thus the construction of this section is the only one suitable for handling centralizers of arbitrary elements in this context.

4.4 Divided groupoids

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid. In [Bes07, Section 9], Bessis defines a Garside groupoid $(\mathcal{G}_m, \mathcal{C}_m, \Delta_m)$ (depending on an integer $m > 0$) attached to $(\mathcal{G}, \mathcal{C}, \Delta)$, called its m-divided groupoid. The intuition is that, while being larger (having more objects and more simple morphisms), the m-divided groupoid retains the main properties of the starting groupoid (see Theorem 4.4.21). The point of this construction is that, being larger, the m-divided groupoid admits "more symmetries", allowing us to consider interesting Garside groupoid of fixed points, which would not be accessible by only considering $(\mathcal{G}, \mathcal{C}, \Delta)$. Section 4.5 below gives an example of such an interesting Garside groupoid of fixed points.

The content of this section is mostly taken from the algebraic exposition of [DDGKM, Section XIV.1.1].

4.4.1 Tuples, decompositions of a Garside map

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also consider a positive integer m.

The construction of the m-divided groupoid of $(\mathcal{G}, \mathcal{C}, \Delta)$ needs us to consider several m-tuples of elements of $\mathcal{C}$. In this short section, we introduce the notation we will use when handling such tuples. We consider temporarily the category $\mathcal{C}^m = \mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C}$ (with m factors). One readily checks that $(\mathcal{G}^m, \mathcal{C}^m, \Delta^m)$ is a Garside groupoid (where Δ^m is the map $(\Delta, \Delta, \ldots, \Delta)$).

An element of $\mathcal{C}^m$, that is, an m-tuple of elements of $\mathcal{C}$, will be denoted by $f = (f_0, \cdots, f_{m-1})$. We will indifferently use the symbols $\preccurlyeq, \succcurlyeq, \wedge, \vee_L, \ldots$ to denote operations on $\mathcal{C}$ or on $\mathcal{C}^m$. For instance, if $f, g \in \mathcal{C}^p$, $f \wedge g$ will denote $(f_0 \wedge g_0, \cdots, f_{m-1} \wedge g_{m-1})$ (provided of course that f_i and g_i share the same source for all $i \in [\![0, m-1]\!]$).

The category $\mathcal{C}^m$ is endowed with an automorphism $\tau (= \tau_m)$ defined by

$$\forall f \in \mathcal{C}^m,\ \tau(f)(= \tau_m(f)) := (f_1, \ldots, f_{m-1}, \phi(f_0)).$$

Remark 4.4.1. One readily checks that $f \in \mathcal{C}^m$ is τ-invariant if and only if $f_0 = f_1 \cdots f_{m-1} = \phi(f_0)$. Thus the subcategory of $\mathcal{C}^m$ made of τ-invariant morphisms identifies with the category $\mathcal{C}^\phi$ of ϕ-invariant morphisms in $\mathcal{C}$.

Finally, for $f, g \in \mathcal{C}^m$, the notation (f, g) will denote the $2m$-tuple $(f_0, g_0, \ldots, f_{m-1}, g_{m-1})$, so that $\tau(f, g) = (g, \tau(f))$ by definition. Likewise, for $f, g, h \in \mathcal{C}^m$, the triple (f, g, h) of m-tuples denotes the $3m$-tuple $(f_0, g_0, h_0, \ldots, g_{m-1}, h_{m-1})$.

Now that we have the needed notation, we will no longer consider $\mathcal{C}^m$ as a category, but merely as a set equipped with some (partial) operations $\wedge, \tau, \ldots$

Definition 4.4.2 (Decompositions of a Garside map). [Bes07, Definition 9.1]
The set $D_m(\Delta)$ of *length m decompositions of* Δ is defined as

$$D_m(\Delta) = \{(s_0, \ldots, s_{m-1}) \in \mathcal{C}^m \mid s_0 \cdots s_{m-1} = \Delta(\eta) \text{ where } \eta \text{ is the source of } s_0\}.$$

Note that all if $(s_0, \ldots, s_{m-1}) \in D_m(\Delta)$, then all the s_i are simple morphisms. By construction, we see that $D_1(\Delta)$ is simply the set of morphisms $\Delta(u)$ for $u \in \mathrm{Ob}(\mathcal{G})$. In particular, it is in bijection with $\mathrm{Ob}(\mathcal{C})$. Likewise, $D_2(\Delta) = \{(s, \overline{s}) \mid s \in \mathcal{S}\}$ is in bijection with $\mathcal{S}$. The idea here is that, for $m \geqslant 1$, the sets $D_m(\Delta)$ and $D_{2m}(\Delta)$ are also the sets of objects and simple morphisms of some Garside groupoid. In fact, we are going to use the sets $D_m(\Delta)$, $D_{2m}(\Delta)$ and $D_{3m}(\Delta)$ to construct a germ structure. We first need the easy following lemma:

Lemma 4.4.3. *(a) For $(a, b) \in D_{2m}(\Delta)$, we have $ab, b\tau(a) \in D_m(\Delta)$.*

(b) For $(x, y, z) \in D_{3m}(\Delta)$, we have $(x, yz), (y, z\tau(x)), (xy, z) \in D_{2m}(\Delta)$.

Proof. This is an easy consequence of the definitions. Let $(a, b) \in D_{2m}(\Delta)$, and let u be the source of a_0. By definition, we have $a_0 b_0 a_1 \cdots a_{m-1} b_{m-1} = \Delta(u)$. We then have $(a_0 b_0, \ldots, a_{m-1} b_{m-1}) = ab \in D_m(\Delta)$. We also have $\overline{a_0} = b_0 \cdots a_{m-1} b_{m-1}$. Thus $\phi(a_0) = \overline{b_0 \cdots a_{m-1} b_{m-1}}$, and $b_0 \cdots a_{m-1} b_{m-1} \phi(a_0) = \Delta(v)$, where v is the target of a_0. We then have $(b_0 a_1, \ldots, b_{m-1} \phi(a_0)) = b\tau(a) \in D_m(\Delta)$. A similar reasoning deals with the case of $(x, y, z) \in D_{3m}(\Delta)$. □

Remark 4.4.4. Let W be a group, endowed with a positive generating set T and a balanced element c. Assume that the associated interval germ $(\mathcal{S}, \cdot)$ is a Garside germ. By construction, the set $\mathcal{S}$ is identified with the interval $I(c)_T$ in W. Under this identification, the set $D_m(\Delta)$ is identified with the set

$$D_m(c) := \{(x_0, \ldots, x_{m-1}) \in W^m \mid x_0 \cdots x_{m-1} = c \text{ and } \ell_T(x_0) + \cdots \ell_T(x_{m-1}) = \ell_T(c)\}$$

of *length-additive decompositions* of c in W. .

4.4.2 Divided germ of simples

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$, and Garside automorphism ϕ. We also fix a positive integer m.

Definition 4.4.5 (Divided groupoid). [DDGKM, Definition XIV.1.2]
The *m-divided graph of simples $\mathcal{S}_m$* is defined as follows.

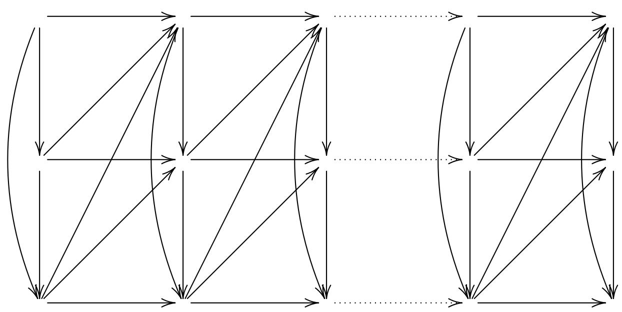
- The object set of $\mathcal{S}_m$ is $D_m(\Delta)$.
- The set of arrows is $D_{2m}(\Delta)$, where the respective source and target of $(a, b) \in D_{2m}(\Delta)$ are ab and $b\tau(a)$.

The set $D_{3m}(\Delta)$ endows $\mathcal{S}_m$ with a partial product, where (x, y, z) induces the equality $(x, yz) \cdot (y, z\tau(x)) = (xy, z)$. The germ $(\mathcal{S}_m, \cdot)$ is called the *m-divided germ of simples* attached to $(\mathcal{G}, \mathcal{C}, \Delta)$. The category $\mathcal{C}_m := \mathcal{C}(\mathcal{S}_m)$ is called the *m-divided category* associated to $(\mathcal{G}, \mathcal{C}, \Delta)$. The enveloping groupoid $\mathcal{G}_m$ of $\mathcal{C}_m$ is the *m-divided groupoid* associated to $(\mathcal{G}, \mathcal{C}, \Delta)$.

As usual, we will often amalgamate the graph $\mathcal{S}_m$ with its set of arrows $D_{2m}(\Delta)$. An arrow (a, b) in the m-divided graph can be represented by a commutative diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u_0"] \arrow[d, "a_0"'] & \cdot \arrow[r, "u_1"] \arrow[d, "a_1"] & \cdot \arrow[r, dotted] \arrow[d, "a_2"] & \cdot \arrow[r, "u_{m-1}"] \arrow[d, "a_{m-1}"'] & \cdot \arrow[d, "\phi(a_0)"] \\
\cdot \arrow[r, "v_0"'] \arrow[ur, "b_0"] & \cdot \arrow[r, "v_1"'] \arrow[ur, "b_1"] & \cdot \arrow[r, dotted] & \cdot \arrow[r, "v_{m-1}"'] \arrow[ur, "b_{m-1}"] & \cdot
\end{tikzcd}$$

where $u = ab$ and $v = b\tau(a)$ by commutativity. The product induced by a triple (x, y, z) may also be represented by a commutative diagram of the form



where all the arrows are simple morphisms in $\mathcal{C}$.

The first thing we need to show is that $(\mathcal{S}_m, \cdot)$ as defined above is indeed a germ. This is done in [DDGKM, Lemma XIV.1.11], in which the authors also show that $(\mathcal{S}_m, \cdot)$ is cancellative and has no nontrivial invertible elements.

By definition of $(\mathcal{S}_m, \cdot)$, it is easy to characterize the case where the product of two elements of $\mathcal{S}_m$ is defined, as in the following lemma:

Lemma 4.4.6. *Let $(a,b),(c,d) \in \mathcal{S}_m$ be such that $b\tau(a) = cd$. The product $(a,b)\cdot(c,d)$ is defined if and only if $c \preccurlyeq b$, in which case we have $(a,b)\cdot(c,d) = (ac,e)$, where $ce = b$.*

Proof. First, assume that $c \preccurlyeq b$, and let e be an m-tuple in $\mathcal{C}$ such that $ce = b$. We have $(a,c,e) \in D_{3m}(\Delta)$, which gives the product $(a,b)\cdot(c,d) = (ac,e)$ as claimed.
Conversely, assume that $(a,b)\cdot(c,d)$ is defined. By definition, there is some $(x,y,z) \in D_{3m}(\Delta)$ such that $(x,yz) = (a,b)$, and $(y, z\tau(x)) = (c,d)$. In particular, we have $x = a$, $y = c$ and $cz = b$. The product is given by (ac, z) as claimed. □

Remark 4.4.7. If $m = 1$, we already said that the set $D_{2m}(\Delta)$ identifies with $\mathcal{S}$. Now, if $s \in \mathcal{S}(u,v)$ for some $u, v \in \mathrm{Ob}(\mathcal{C})$, then $(s,\overline{s}) \in D_2(\Delta)$ has source $s\overline{s} = u$ and target $\overline{s}\phi(s) = v$, thus the graphs $\mathcal{S}_1$ and $\mathcal{S}$ are isomorphic. Furthermore, an element $(x,y,z) \in D_3(\Delta)$ induces the equation $(x,\overline{x})\cdot(y,\overline{y}) = (xy, \overline{xy})$, thus the germ $(\mathcal{S}_1, \cdot)$ is isomorphic with the germ of simples associated to $(\mathcal{C}, \Delta)$. From this we deduce that $\mathcal{C}_1 \simeq \mathcal{C}$, and this isomorphism preserves the set of simples.

We now proceed to show that $(\mathcal{S}_m, \cdot)$ is actually a Garside germ. This essentially comes from the following proposition, which characterizes divisibility in the germ $\mathcal{S}_m$ in terms of the divisibility in the category $\mathcal{C}$.

Proposition 4.4.8 (Characterization of divisibility in divided germ). *[DDGKM, Lemma XIV.1.12] Let $u \in \mathrm{Ob}(\mathcal{S}_m)$, and let $(a,b),(a',b') \in \mathcal{S}_m(u,-)$, we have*

$$(a,b) \preccurlyeq_{\mathcal{S}_m} (a',b') \Leftrightarrow a \preccurlyeq a' \Leftrightarrow b \succcurlyeq b'.$$

Likewise, for $(c,d),(c',d') \in \mathcal{S}_m(-,u)$, we have

$$(c',d') \succcurlyeq_{\mathcal{S}_m} (c,d) \Leftrightarrow c' \succcurlyeq c \Leftrightarrow d' \preccurlyeq d.$$

An immediate consequence of this proposition is a characterization of atoms in a divided germ.

Corollary 4.4.9 (Atoms of divided category). *Let $u \in \mathrm{Ob}(\mathcal{S}_m)$. A simple morphism $(a,b) \in \mathcal{S}_m(u,-)$ is an atom in $\mathcal{S}_m$ if and only if there is some $i \in [\![0, m-1]\!]$ such that a_k is trivial for $k \neq i$, and a_i is an atom in $\mathcal{C}$.*

Corollary 4.4.10. *Let $u \in \mathrm{Ob}(\mathcal{S}_m)$, and let θ be the target of u in $\mathcal{C}^m$. The element $\Delta_m(u) := (u, 1_\theta) \in \mathcal{S}_m(u, \tau(u))$ is the maximum of $(\mathcal{S}_m(u,-), \preccurlyeq_{\mathcal{S}})$, and the element $\Delta_m(\tau^{-1}(u))$ is the maximum of $(\mathcal{S}_m(-,u), \succcurlyeq_{\mathcal{S}})$.*

Proof. By definition, we have $1_\theta = (1_{\theta_0}, \dots, 1_{\theta_{m-1}})$, where θ_i is the target of u_i for $i \in [\![0, m-1]\!]$. Let $(a,b) \in \mathcal{S}_m(u,-)$, we have $ab = u$, thus $a \preccurlyeq u$ and $(a,b) \preccurlyeq \Delta_m(u)$ by Proposition 4.4.8. Likewise, for $(c,d) \in \mathcal{S}_m(-,u)$, we have $d\tau(c) = u$, thus $\tau^{-1}(d)c = \tau^{-1}(u)$ and $\Delta_m(\tau^{-1}(u)) \succcurlyeq (c,d)$. $\square$

Corollary 4.4.11 (Join and meet in divided germ). *Let $u \in \mathrm{Ob}(\mathcal{S}_m)$ and let $s := (a,b), s' := (a',b') \in \mathcal{S}_m(u,-)$. The join and meet of s and s' exist in $(\mathcal{S}_m(u,-), \preccurlyeq_{\mathcal{S}_m})$ and are respectively given by*

$$s \wedge s' = (a \wedge a', b \vee_L b'),\ \ s \vee s' = (a \vee a', b \wedge_R b').$$

Likewise, for $t = (c,d), t' := (c',d') \in \mathcal{S}_m(-,u)$, the join and meet of t and t' exist in $(\mathcal{S}_m(-,u), \succcurlyeq_{\mathcal{S}_m})$ and are respectively given by

$$t \wedge_R t' = (c \wedge_R c', d \vee d'),\ \ t \vee_L t' = (c \vee_L c', d \wedge d').$$

Proof. Let $d := a \wedge a'$, with $dx = a$ and $dx' = a'$. We have $dx'\beta' = a'b' = u = ab = dxb$ and $x'b' = xb$ by cancellativity. Let $p = b \vee_L b'$ with $yb = y'b' = p$. By definition of $b \vee_L b'$, there is some m-tuple g such that $xb = gm = x'b'$. We then have $x = gy$, $x' = gy'$ and

$$d = a \wedge a' = dgy \wedge dgy' = dg(y \wedge y').$$

In particular, we deduce that both g and $y \wedge y'$ are trivial, and that $u = ab = dxb = dp = (a \wedge a')(b \vee_L b')$. We then have $(a \wedge a', b \vee_L b') \in \mathcal{S}_m(u,-)$.

Likewise, let $q = a \vee a'$ with $ay = a'y'$. Since $u = ab = a'b'$ is a common right-multiple of a and a', there is some tuple δ such that $y\delta = b$ and $y'\delta = b'$. Let $d := b \wedge_R b'$, with $xd = b$ and $x'd = b'$. By definition, there is some m-tuple h such that $h\delta = d$. We then have $q = ay = a'y' = axh = ax'h$ and h is trivial. We deduce that $(a \vee a', b \wedge_R b') \in \mathcal{S}_m(u,-)$. The elements $(a \wedge a, b \vee_L b)$ and $(a \vee a', b \wedge_R b')$ are the respective meet and join of s and s' in $(\mathcal{S}_m(u,-), \preccurlyeq)$ by Proposition 4.4.8. A similar reasoning shows the second statement. $\square$

From this corollary, we are also able to deduce formulas for left- and right-complements in the divided germ. As we will soon see that divided germs are Garside germs, this formulas will give the complements in the divided category $\mathcal{C}_m$.

Corollary 4.4.12 (Left- and right-complement in divided germ). *Let $u \in \mathrm{Ob}(\mathcal{S}_m)$.*

(a) For $s := (a,b), s' := (a',b') \in \mathcal{S}_m(u,-)$, we have $s \backslash s' = (a \backslash a', (b \wedge_R b')\tau(a))$

(b) For $t := (c,d), t' := (c',d') \in \mathcal{S}_m(-,u)$, we have $t'/t = (c'/c, c(d \wedge d'))$.

Proof. a) By Corollary 4.4.11, we have $s \vee s' = (a \vee a', b \wedge_R b)$. The triple $(a, a\backslash a', b \wedge_R b') \in D_{3m}(\Delta)$ induces the desired product.
b) By Corollary 4.4.11, we have $t \vee_L t' = (c \vee_L c', d \wedge d')$. The triple $(c'/c, c, d \wedge d') \in D_{3m}(\Delta)$ induces the desired product. $\square$

Since we already know that $(\mathcal{S}_m, \cdot)$ is a finite cancellative germ with no nontrivial invertible element, combining Corollary 4.4.10 and Corollary 4.4.11 proves that $(\mathcal{S}_m, \cdot)$ is a Garside germ [DDGKM, Proposition XIV.1.13]. We then have

Theorem 4.4.13 (Garside divided groupoid). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let m be a positive integer.*
The triple $(\mathcal{G}_m, \mathcal{C}_m, \Delta_m)$ is a Garside groupoid with germ of simples $(\mathcal{S}_m, \cdot)$. The Garside automorphism ϕ_m is given on simple elements by $\phi_m(a, b) = (\tau(a), \tau(b))$ for $(a, b) \in \mathcal{S}_m$.

In particular, this theorem allows us to consider greedy paths in the category $\mathcal{C}_m$. The following corollary gives an easy characterization of greediness in $\mathcal{C}_m$. As usual it is sufficient to consider pats of length 2.

Corollary 4.4.14 (Greediness in divided category). *Let $s := (a, b), s' := (a', b') \in \mathcal{S}_m$ be two composable morphisms. Let also $d = a' \wedge b$ with $dx' = a'$ and $dy = b$. The greedy normal form of the product ss' in $\mathcal{C}_m$ is given by*

$$ss' = (ad, y)(x', b'\tau(d)).$$

In particular, ss' is already greedy if and only if $d = a' \wedge b$ is trivial.

Proof. We know that $\overline{s} = (b, \tau(a))$. By Corollary 4.4.11, we have $s' \wedge \overline{s} = (d, x'b)$. By Lemma 2.1.13 and Lemma 4.4.6, the greedy normal form of ss' is then given by

$$ss' = (a, b)(a', b') = (a, b)(dx', b) = (a, b)(d, x'b)(x', b\tau(d)) = (ad, y)(x', b\tau(d)).$$

□

4.4.3 Divided category and divided groupoid

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix a positive integer m, and we keep the notation of the last section.

We know by Theorem 4.4.13 that $(\mathcal{G}_m, \mathcal{C}_m, \Delta_m)$ is a Garside groupoid. We now study the relation between this groupoid and the starting groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$. This is achieved by considering collapse functors.

Definition 4.4.15 (Collapse functors). The map $\mathcal{S}_m \to \mathcal{S}$ sending (a, b) to a_i induces a functor $\pi_i : \mathcal{G}_m \to \mathcal{G}$, which is called the i-th collapse functor. We also denote by π_i the restricted functor $\mathcal{C}_m \to \mathcal{C}$.

The fact that collapse functors are functors is an immediate consequence of Lemma 4.4.6. For all $i \in [\![0, m-2]\!]$, we have $\pi_i \circ \phi_m = \pi_{i+1}$. Since ϕ_m is an automorphism of $\mathcal{C}_m$, all the collapse functors will share similar properties. A first property of collapse functors, which relies on Corollary 4.4.14, proves that the head of a morphism in $\mathcal{C}_m$ can be characterized in terms of gcds in $\mathcal{C}$.

Lemma 4.4.16. *[DDGKM, Lemma XIV.1.15] Let $u \in \mathrm{Ob}(\mathcal{C}_m)$, and let $f \in \mathcal{C}_m(u, -)$. We have $\alpha(f) = (a, b)$ where $a_i = \pi_i(f) \wedge u_i$.*

Note that in the above lemma, we do not have $\pi_i(\alpha(f)) = \alpha(\pi_i(f))$. In particular collapse functors do not need to preserve greediness. We give an explicit example below.

Example 4.4.17. Let (M, Δ) be a Garside monoid, and let $s \in \mathcal{S}$. Let also $t, x \in \mathcal{S}$ be such that

$tx = \overline{s}$. We have $x\phi(s) = \overline{t}$, and we have the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r, "s"] \arrow[d, "s"'] & {} \arrow[r, "\overline{s}"] \arrow[d, "\overline{s}"'] & {} \arrow[d, "\phi(s)"] \\
{} \arrow[ur, "1"'] \arrow[r, "\overline{s}"] \arrow[d, "t"'] & {} \arrow[ur, "1"] \arrow[r, "\phi(s)"] \arrow[d, "\phi(s)"'] & {} \arrow[d, "\phi(t)"] \\
{} \arrow[ur, "x"] \arrow[r, "\overline{t}"'] & {} \arrow[ur, "1"] \arrow[r, "\phi(t)"'] & {}
\end{tikzcd}$$

which denotes a greedy path in the 2-divided category attached to M. However, the image of this path under the 0-th collapse functor is st, which is not greedy in M.

Another remarkable property of collapse functors is that it characterizes $\mathcal{C}_m$-divisibility in $\mathcal{G}_m$ in terms of $\mathcal{C}$-divisibility in $\mathcal{G}$. This is summarized in the following proposition, which is a relatively easy consequence of [DDGKM, Proposition XIV.5.9], which states that $f \in \mathcal{G}_m$ lies in $\mathcal{C}_m$ if and only if $\pi_i(f) \in \mathcal{C}$ for all $i \in [\![0, m-1]\!]$.

Proposition 4.4.18 (Collapse functors and divisibility). *Let* $u \in \mathrm{Ob}(\mathcal{G}_m)$

(a) Let $f, g \in \mathcal{G}_m(u, -)$, *we have*

- *(i)* $f \preccurlyeq_{\mathcal{C}_m} g \Leftrightarrow \forall i \in [\![0, m-1]\!],\ \pi_i(f) \preccurlyeq_{\mathcal{C}} \pi_i(g)$,
- *(ii)* $\forall i \in [\![0, m-1]\!], \pi_i(f \wedge g) = \pi_i(f) \wedge \pi_i(g)$ *and* $\pi_i(f \vee g) = \pi_i(f) \vee \pi_i(g)$,
- *(iii)* $\forall i \in [\![0, m-1]\!], \pi_i(f \backslash g) = \pi_i(f) \backslash \pi_i(g)$.

(b) Let $f, g \in \mathcal{G}_m(-, u)$, *we have*

- *(i)* $g \succcurlyeq_{\mathcal{C}_m} f \Leftrightarrow \forall i \in [\![0, m-1]\!],\ \pi_i(g) \succcurlyeq_{\mathcal{C}} \pi_i(f)$,
- *(ii)* $\forall i \in [\![0, m-1]\!], \pi_i(f \wedge_R g) = \pi_i(f) \wedge_R \pi_i(g)$ *and* $\pi_i(f \vee_L g) = \pi_i(f) \vee_L \pi_i(g)$,
- *(iii)* $\forall i \in [\![0, m-1]\!], \pi_i(g/f) = \pi_i(g)/\pi_i(f)$.

Proof. We only prove point (a), point (b) being obtained by similar arguments.
(i) : By definition, we have $f \preccurlyeq_{\mathcal{C}_m} g$ if and only if $f^{-1}g \in \mathcal{C}_m$. By [DDGKM, Proposition XIV.1.19], this is equivalent to $\pi_i(f^{-1}g) = \pi_i(f)^{-1}\pi_i(g) \in \mathcal{C}$ for all $i \in [\![0, m-1]\!]$. By definition of $\mathcal{C}$-divisibility, this is equivalent to having $\pi_i(f) \preccurlyeq_{\mathcal{C}} \pi_i(g)$ for all $i \in [\![0, m-1]\!]$.
(ii) and (iii) : Let $h \in \mathcal{C}_m$ be such that $hf, hg \in \mathcal{C}_m$ (we can take for instance the lcm of the denominators of f and g in $\mathcal{G}_m$), we have

$$\pi(f \wedge g) = \pi_i(h^{-1})\pi_i(hf \wedge hg) = \pi_i(h^{-1})(\pi_i(hf) \wedge \pi_i(hg)) = \pi_i(f) \wedge \pi_i(g),$$
$$\pi(f \vee g) = \pi_i(h^{-1})\pi_i(hf \vee hg) = \pi_i(h^{-1})(\pi_i(hf) \vee \pi_i(hg)) = \pi_i(f) \vee \pi_i(g).$$

(iv) : This is an consequence of the functoriality of π_i, since $f \backslash g = f^{-1}(f \vee g)$. □

Of course, since π_i is a functor for $i \in [\![0, m-1]\!]$, [DDGKM, Proposition XIV.5.9] can be seen as a particular case of this proposition. Of course this is convenient only because we have omitted the proof of [DDGKM, Proposition XIV.5.9]. Note that, combining [DDGKM, Proposition

XIV.5.9] with Proposition 4.4.18 proves that collapse functors characterize divisibility in $\mathcal{C}_m$, along with gcds and lcms.

We now give two corollaries of Proposition 4.4.18.

Corollary 4.4.19 (Characterization of elements). *[DDGKM, Corollary XIV.1.18]* *Let $u \in \mathrm{Ob}(\mathcal{G}_m)$, and let $f, g \in \mathcal{G}_m(u, -)$, we have $f = g$ if and only if $\pi_i(f) = \pi_i(g)$ for all $i \in [\![0, m-1]\!]$.*

Corollary 4.4.20. *A left-fraction $f^{-1}g \in \mathcal{G}_m$ is reduced if and only if the fraction $\pi_i(f)^{-1}\pi_i(g)$ is reduced for all $i \in [\![0, m-1]\!]$.*

An even more remarkable property of collapse functors is that they provide equivalences of groupoids. In fact, we can actually exhibit the inverse equivalence.

Let $\eta \in \mathrm{Ob}(\mathcal{C})$. We denote by $\iota(\eta)$ the m-decomposition of $\Delta(\eta)$ given by $(1_\eta, \ldots, 1_\eta, \Delta(\eta))$. For $s \in \mathcal{S}(\eta, \theta)$, there is a unique morphism $\iota(s) \in \mathcal{C}_m(\iota(\eta), \iota(\theta))$, such that $\pi_i(\iota(s)) = s$ for all $i \in [\![0, m-1]\!]$. It is given by the following commutative diagram in $\mathcal{C}$, where an arrow with no label denotes an identity morphism (we exclude the diagonal arrows for readability)

$$
\begin{tikzcd}
\eta \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & \eta \arrow[r, dotted] \arrow[d] & \eta \arrow[r] \arrow[d] & \eta \arrow[r, "\Delta(\eta)"] \arrow[d, "s"] & \phi(\eta) \arrow[d] \\
\eta \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & \eta \arrow[r, dotted] \arrow[d] & \eta \arrow[r, "s"] \arrow[d, "s"] & \theta \arrow[r, "\overline{s}"] \arrow[d] & \phi(\eta) \arrow[d] \\
\eta \arrow[r] \arrow[d, dotted] & \eta \arrow[r] \arrow[d, dotted] & \eta \arrow[r, dotted] \arrow[d, dotted] & \theta \arrow[r] \arrow[d, dotted] & \theta \arrow[r, "\overline{s}"] \arrow[d, dotted] & \phi(\eta) \arrow[d, dotted] \\
\eta \arrow[r, "s"] \arrow[d, "s"'] & \theta \arrow[r] \arrow[d] & \theta \arrow[r, dotted] \arrow[d] & \theta \arrow[r] \arrow[d] & \theta \arrow[r, "\overline{s}"] \arrow[d] & \phi(\eta) \arrow[d, "\phi(s)"] \\
\theta \arrow[r] & \theta \arrow[r] & \theta \arrow[r, dotted] & \theta \arrow[r] & \theta \arrow[r, "\Delta(\theta)"'] & \phi(\theta)
\end{tikzcd}
$$

Theorem 4.4.21 (Equivalence of groupoids). *[DDGKM, Proposition XIV.1.6]* *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let m be a positive integer. The map $\iota : \mathcal{C} \to \mathcal{C}_m$ defines a well-defined functor. The functors $\pi_0 : \mathcal{G}_m \leftrightarrows \mathcal{G} : \iota$ provide quasi-inverse equivalences of groupoids.*

We will not detail the proof here. Let us simply mention that it proceeds by constructing a natural transformation $\mathrm{Dil} : \iota \circ \pi_0 \Rightarrow 1_{\mathcal{C}_m}$, which becomes a natural isomorphism in $\mathcal{G}_m$. Let $u \in \mathrm{Ob}(\mathcal{C}_m)$. There is a unique morphism $\mathrm{Dil}(u) : \iota(\pi_0(u)) \to u$ such that $\pi_i(\mathrm{Dil}(u)) = u_0 \cdots u_{i-1}$ for $i \in [\![1, m-1]\!]$ and with $\pi_0(\mathrm{Dil}(u)) = 1_{\eta_0}$, where η_i is the source of u_i for $i \in [\![0, m-1]\!]$. The morphism $\mathrm{Dil}(u)$ is given by the following commutative diagram in $\mathcal{C}$, where an arrow with no label denotes an identity morphism (we exclude the diagonal arrows for readability)

$$\begin{tikzcd}
\eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r, dotted] \arrow[d] & \eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r, "\Delta(\eta)"] \arrow[d, "u_0 \cdots u_{m-2}"] & \phi(\eta_0) \arrow[d] \\
\eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r, dotted] \arrow[d] & \eta_0 \arrow[r, "u_0 \cdots u_{m-2}"] \arrow[d, "u_0 \cdots u_{m-3}"] & \eta_{m-1} \arrow[r, "u_{m-1}"] \arrow[d] & \phi(\eta_0) \arrow[d] \\
\eta_0 \arrow[r] \arrow[d, dotted] & \eta_0 \arrow[r] \arrow[d, dotted] & \eta_0 \arrow[r, dotted] \arrow[d, dotted] & \eta_{m-2} \arrow[r, "u_{m-2}"] \arrow[d, dotted] & \eta_{m-1} \arrow[r, "u_{m-1}"] \arrow[d, dotted] & \phi(\eta_0) \arrow[d, dotted] \\
\eta_0 \arrow[r] \arrow[d] & \eta_0 \arrow[r, "u_0 u_1"] \arrow[d, "u_0"] & \eta_2 \arrow[r, dotted] \arrow[d] & \eta_{m-2} \arrow[r, "u_{m-2}"] \arrow[d] & \eta_{m-1} \arrow[r, "u_{m-1}"] \arrow[d] & \phi(\eta_0) \arrow[d] \\
\eta_0 \arrow[r, "u_0"'] & \eta_1 \arrow[r, "u_1"'] & \eta_2 \arrow[r, dotted] & \eta_{m-2} \arrow[r, "u_{m-2}"'] & \eta_{m-1} \arrow[r, "u_{m-1}"'] & \phi(\eta_0)
\end{tikzcd}$$

For an arbitrary morphism $f \in \mathcal{C}_m$, we then obtain a commutative square

$$\begin{tikzcd}
\iota(\pi_0(u)) \arrow[r, "\iota(\pi_0(f))"] \arrow[d, "\mathrm{Dil}(u)"'] & \iota(\pi_0(v)) \arrow[d, "\mathrm{Dil}(v)"] \\
u \arrow[r, "f"'] & v
\end{tikzcd}$$

which proves that Dil is a natural transformation.

Note that, since $\pi_i = \pi_0 \circ \phi_m^i$ for $i \in [\![0, m-1]\!]$, the pair $(\pi_i, \phi_m^{-i} \circ \iota)$ also provides inverse equivalences of groupoids between $\mathcal{G}_m$ and $\mathcal{G}$ for $i \in [\![0, m-1]\!]$.

According to the above theorem, it may seem like the construction of divided category achieves nothing: we obtained more or less the same groupoid that we started with, only with a larger (and thus less computationally efficient!) Garside structure. Nonetheless, this enlargement will prove useful later in Section 4.5. For now, we can already show that the Garside automorphism of a divided category has a higher order than the original Garside automorphism of the category $\mathcal{C}$.

Lemma 4.4.22. *Let n be the order of ϕ. The Garside automorphism ϕ_m of $(\mathcal{G}_m, \mathcal{C}_m, \Delta_m)$ has order nm.*

Proof. Let $(a, b) \in \mathcal{S}_m$. By construction, we have $\phi_m(a, b) = (\tau(a), \tau(b))$, thus $\phi_m^k(a, b) = (a, b)$ if and only if $\tau^k(a) = a$ and $\tau^k(b) = b$. Let $\eta, \theta \in \mathrm{Ob}(\mathcal{C})$, and let $s \in \mathcal{S}(\eta, \theta)$. We have $(s, \overline{s}, 1_{\phi(\eta)}, \ldots, 1_{\phi(\eta)}) =: (a, b) \in \mathcal{S}_m$, thus ϕ_m^k is trivial only if $\tau^k(a) = a$. If s is nontrivial, then $\tau^k(a) = a$ if and only if $k = k'm$ is a multiple of m, and if $\phi^{k'}(s) = s$. We then obtain that the order of ϕ_m is mK, where K is the lcm of the set

$$\{\min\{k \geqslant 1 \mid \phi^k(s) = s\} \mid s \in \mathcal{S} \setminus \{1_\eta\}_{\eta \in \mathrm{Ob}(\mathcal{C})}\}.$$

On the other hand, since $\mathcal{S}$ generates $\mathcal{C}$, ϕ^k is trivial if and only if $\phi^k(s) = s$ for all nontrivial simple morphism s of $\mathcal{C}$. Thus K is also the order of ϕ on $\mathcal{C}$ and the result follows. $\square$

We finish this section by briefly explaining what happens when one tries to iterate the divided category construction. Let $m, n \geqslant 1$ be integers. Consider a mn-decomposition of Δ of the form

$$U := (s_0^0, \ldots, s_{n-1}^0, s_0^1, \ldots, s_{n-1}^1, \ldots, s_{n-1}^{m-1}).$$

By regrouping the entries by blocks of size n, we obtain a m-decomposition of Δ $(s^0, s^1, \ldots, s^{m-1})$. We have the following commutative diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "s^0"] \arrow[d, "s_0^0"'] & \bullet \arrow[r, "s^1"] \arrow[d, "s_0^1"] & \bullet \arrow[r, dotted] \arrow[d, "s_0^2"] & \bullet \arrow[r, "s^{m-1}"] \arrow[d, "s_0^{m-1}"'] & \bullet \arrow[d, "\phi(s_0^0)"] \\
\bullet \arrow[d, "s_1^0"'] & \bullet \arrow[d, "s_1^1"'] & \bullet \arrow[d, "s_1^2"'] & \bullet \arrow[d, "s_1^{m-1}"'] & \bullet \arrow[d, "\phi(s_1^0)"] \\
\bullet \arrow[d, dotted] & \bullet \arrow[d, dotted] & \bullet \arrow[d, dotted] & \bullet \arrow[d, dotted] & \bullet \arrow[d, dotted] \\
\bullet \arrow[d, "s_{n-1}^0"'] & \bullet \arrow[d, "s_{n-1}^1"'] & \bullet \arrow[d, "s_{n-1}^2"'] & \bullet \arrow[d, "s_{n-1}^{m-1}"'] & \bullet \arrow[d, "\phi(s_{n-1}^0)"] \\
\bullet \arrow[r, "s^1"'] \arrow[uuuur, "1"] & \bullet \arrow[r, "s^2"'] \arrow[uuuur, "1"] & \bullet \arrow[r, dotted] & \bullet \arrow[r, "\phi(s^0)"'] \arrow[uuuur, "1"] & \bullet
\end{tikzcd}$$

Which gives that $U \in D_n(\Delta_m)$. Conversely, let $(s_0, \ldots s_{n-1})$ be a n-decomposition of Δ_m, then the sequence $(\pi_0(s_0), \pi_0(s_1), \ldots, \pi_0(s_{n-1}), \pi_1(s_0), \ldots, \pi_1(s_{n-1}), \pi_2(s_0), \ldots, \pi_{m-1}(s_{n-1}))$ is a mn-decomposition of Δ. This gives a bijection between $D_n(\Delta_m)$ and $D_{nm}(\Delta)$. Likewise, we have bijections between $D_{2n}(\Delta_m), D_{3n}(\Delta_m)$ and $D_{2nm}(\Delta), D_{3nm}(\Delta)$, respectively. This bijections are compatible with the germ structures on both sides, so that we obtain

Proposition 4.4.23 (Divided category of divided category). *[Bes07, Proposition 9.8] Let $n, m \geqslant 1$ be integers. The divided germs $((\mathcal{S}_m)_n, \cdot)$ and $(\mathcal{S}_{mn}, \cdot)$ are isomorphic. In particular, we have $((\mathcal{G}_m)_n, (\mathcal{C}_m)_n, (\Delta_m)_n) \simeq (\mathcal{G}_{mn}, \mathcal{C}_{mn}, \Delta_{mn})$.*

4.5 Groupoids of periodic elements

In Section 3.4, we studied the conjugacy of periodic elements in Garside groupoids. We gave a complete description in Theorem 3.4.4 (conjugacy of periodic elements). In this section, we want to study centralizers of periodic elements. Theoretically, this is possible using the tools of Section 4.3 for instance on the super-summit set. However, in the case of periodic elements, we saw (Corollary 3.4.9) that computation of the super-summit set was actually an easy consequence of Theorem 3.4.4, thus we would like to construct a Garside category using Theorem 3.4.4, in the hopes of getting stronger results.

Such a construction was given by Bessis in [Bes07, Section 10] (see also [DDGKM, Section XIV.1.1]). It is actually built on constructions we have already encountered: groupoids of fixed points and divided groupoids.

Definition 4.5.1 (Groupoid of periodic elements). Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. The *groupoid of (p, q)-periodic elements* attached to $\mathcal{G}$ is defined as $\mathcal{G}_p^q := (\mathcal{G}_p)^{\phi_p^q}$. The *category of (p, q)-periodic elements* attached to $\mathcal{C}$ is defined similarly as $\mathcal{C}_p^q := (\mathcal{C}_p)^{\phi_p^q}$.

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. By combining Theorem 4.4.13 (Garside divided groupoid) and Theorem 4.1.11 (Garside groupoid of fixed points), we obtain that $(\mathcal{G}_p^q, \mathcal{C}_p^q, \Delta_p)$ is a Garside groupoid (provided that $\mathcal{G}_p^q$ is nonempty). Theorem 4.5.2 shows that $\mathcal{G}_p^q$ can be used to understand the conjugacy of (p,q)-periodic elements.

In Theorem 3.4.4, we saw that, in the study of (p,q)-periodic elements up to conjugacy, we can restrict our attention to the case where p and q are coprime. This phenomenon has an analogue in the context of categories of periodic elements: if $p = dp'$ and $q = dq'$ where d is the gcd of p and q, then the isomorphism $\mathcal{G}_p \simeq (\mathcal{G}_{p'})_d$ of Proposition 4.4.23 restricts to an isomorphism between $\mathcal{G}_p^q$ and the divided groupoid $(\mathcal{G}_{p'}^{q'})_d$. The latter is equivalent to the groupoid $\mathcal{G}_{p'}^{q'}$ of (p',q')-periodic elements by Theorem 4.4.21 (equivalence of groupoids).

We saw that the divided groupoid $\mathcal{G}_p$ comes equipped with a family of collapse functors π_i for $i \in [\![0, p-1]\!]$ (see Definition 4.4.15). Among several useful properties, we saw in Theorem 4.4.21 that π_0 provides an equivalence of groupoids between $\mathcal{G}_p$ and $\mathcal{G}$, with quasi-inverse $\iota : \mathcal{G} \to \mathcal{G}_p$. Restricting the functor π_0 to the category $\mathcal{G}_p^q$ allows us to completely understand the conjugacy of (p,q)-periodic elements.

Theorem 4.5.2 (Periodic elements and groupoid of periodic elements). *[DDGKM, Proposition XIV.1.8] Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers.*

(a) Let $u \in \mathrm{Ob}(\mathcal{G}_p^q)$, then $\rho(u) := \pi_0(\Delta_p^q(u))$ is (p,q)-periodic in $\mathcal{G}$.

(b) Let ρ be an endomorphism in $\mathcal{G}$ conjugate to a (p,q)-periodic element of $\mathcal{G}$. There exists an object u of $\mathcal{G}_p^q$ such that $\iota(\rho)$ is conjugate to $\Delta_p^q(u)$ in $\mathcal{G}_m$.

(c) Let $u \in \mathrm{Ob}(\mathcal{G}_p^q)$. The collapse functor $\pi_0 : \mathcal{G}_p \to \mathcal{G}$ sends $\mathcal{G}_p^q(u,u)$ to the centralizer in $\mathcal{G}(\pi_0(u), \pi_0(u))$ of the (p,q)-regular element $\rho(u)$.

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. A first consequence of Theorem 4.5.2 is that the groupoid $\mathcal{G}_p^q$ is nonempty if and only if there are (p,q)-periodic elements in $\mathcal{G}$.

Remark 4.5.3. Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. Note that the map $u \mapsto \rho(u) := \pi_0(\Delta_p^q(u))$ is actually a bijection. Indeed, if λ, μ are positive integers such that $p\lambda - q\mu = 1$, then we saw in Theorem 3.4.4 that $u = \rho(u)^{-\mu}\Delta^{\lambda}(\eta)$, where $\eta = \pi_0(u)$ is the source of $\rho(u)$.

Moreover, we can actually classify conjugacy classes of periodic elements using groupoids of periodic elements.

Corollary 4.5.4. *[Bes07, Corollary 10.4] Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. The map $u \mapsto \rho(u)$ induces a bijection between the connected components of $\mathcal{G}_p^q$ and the conjugacy classes of $\mathcal{G}$ containing (p,q)-periodic elements.*

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $p, q > 0$ be integers. Since $\mathcal{G}_p^q$ has a finite number of objects (as a subgroupoid of $\mathcal{G}_p$), Corollary 4.5.4 gives a new proof of Corollary 3.4.11, since $\mathcal{G}_p^q$ must then have finitely many connected components.

4.5.1 Germs of periodic elements

Theorem 4.5.2 gives a strong motivation for computing groupoids of periodic elements. However in practice, it is very long to compute the whole divided groupoid before computing the groupoid of fixed points. Thus we are interested in computing groupoids of periodic elements, and their germs of simples, more directly.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We otherwise keep the notation from the last section. We also fix coprime integers $p, q > 0$ be integers, along with positive integers λ, μ such that $p\lambda - q\mu = 1$.

In Section 4.4, we considered decompositions of the Garside map Δ. In the conventions of Section 4.4, the set $D_m(\Delta)$ of length m decompositions of Δ is endowed with an automorphism τ. For $m, n > 0$ two integers, we introduce the set

$$D_m^n(\Delta) = \{u \in D_m(\Delta) \mid \tau^n(u) = u\}.$$

Recall from Definition 4.4.5 (divided groupoid) that the germ of simples of the category $\mathcal{C}_p$ is given by the set $\mathcal{S}_p = D_{2p}(\Delta)$ of length $2p$ decompositions of Δ. The action of the Garside automorphism ϕ_p is given by $\phi_p(a,b) = (\tau(a), \tau(b)) = \tau^2(a,b)$ in the conventions of Section 4.4.1. We then have

$$\forall (a,b) \in \mathcal{S}_p, (\phi_p)^q(a,b) = (a,b) \Leftrightarrow \tau^{2q}(a,b) = (a,b) \Leftrightarrow (a,b) \in D_{2p}^{2q}(\Delta).$$

Likewise, we obtain that the objects of $\mathcal{C}_p^q$ are given by $D_p^q(\Delta)$, and that the germ structure on $(\mathcal{S}_p)^{\phi^q}$ is given by $D_{3p}^{3q}(\Delta)$. More precisely, for $(a,b), (c,d), (e,f) \in D_{2p}^{2q}(\Delta)$, we have $(a,b) \cdot (c,d) = (e,f)$ if and only if $(a,c,f) \in D_{3p}^{3q}(\Delta)$ and if $b = cf$, $d = f\tau(a)$, $e = ac$.

A priori, an element of $D_{np}^{nq}(\Delta)$ depends on np parameters in $\mathcal{S}$. However, we will show that an element of $D_{np}^{nq}(\Delta)$ actually depends only on n parameters in $\mathcal{S}$. We begin with a technical lemma, which is a general result about fixed points under some cyclic group actions.

Lemma 4.5.5. *Let A be a set, and let $\sigma \in \mathfrak{S}(A)$. We define an bijection of A^p by $\tau(a_0, \ldots, a_{p-1}) := (a_1, \ldots, a_{p-1}, \sigma(a_0))$. For $a \in A^p$, we have*

$$\tau^q(a) = a \Leftrightarrow \sigma^q(a_0) = a_0 \text{ and } \forall i \in [\![1, p-1]\!],\ a_i = \sigma^{i\lambda}(a_0).$$

In particular, a then depends only on a_0.

Proof. Let $q = pm + r$ be the euclidean division of q by p. For $a \in A^p$, we have

$$\tau^p(a) = (\sigma(a_0), \ldots, \sigma(a_{p-1})),$$

and thus,

$$\tau^q(a) = (\sigma^m(a_r), \ldots, \sigma^m(a_{p-1}), \sigma^{m+1}(a_0), \ldots, \sigma^{m+1}(a_{r-1})).$$

($\Leftarrow$) Assume that $\sigma^q(a_0) = a_0$, and that $\forall i \in [\![1, p-1]\!]$, we have $a_i = \sigma^{i\lambda}(a_0)$. We then have

$$\begin{aligned} a &= \left(a_0, \sigma^{\lambda}(a_0), \ldots, \sigma^{(p-1)\lambda}(a_0)\right), \\ \tau^q(a) &= \left(\sigma^{r\lambda+m}(a_0), \ldots, \sigma^{(p-1)\lambda+m}(a_0), \sigma^{m+1}(a_0), \ldots, \sigma^{m+(r-1)\lambda+1}(a_0)\right). \end{aligned}$$

Since we have $pm + r = q$, we have $pm + r \equiv 0[q]$ and $\lambda pm + r\lambda = 0[q]$. However, since $p\lambda - q\mu = 1$ implies $p\lambda \equiv 1[q]$, we deduce that $m + r\lambda \equiv 0[q]$. Since a_0 is invariant under σ^q, we have

$$\begin{cases} \forall j \in [\![0, p-r-1]\!],\ \sigma^{(r+j)\lambda+m}(a_0) = \sigma^{j\lambda}(a_0), \\ \forall j \in [\![p-r, p-1]\!],\ \sigma^{m+(j-p+r)\lambda+1}(a_0) = \sigma^{j\lambda - p\lambda + 1}(a_0) = \sigma^{j\lambda}(a_0). \end{cases}$$

and thus $a = \tau^q(a)$.
$(\Rightarrow)$ Conversely, assume that $\tau^q(a) = a$. We have by definition

$$\forall i \in [\![0, p-1]\!], a_i = \begin{cases} \sigma^m(a_{i+r}) & \text{if } i < p - r, \\ \sigma^{m+1}(a_{i+r-p}) & \text{if } i \geqslant p - r. \end{cases}$$

By an immediate induction, we obtain that

$$a_0 = \sigma^m(a_r) = \ldots = \sigma^{pm+r}(a_{pr}) = \sigma^q(a_0),$$

and a_0 is σ^q-invariant. Now, we show that, for all $i \in [\![0, p-1]\!]$, we have $a_i = \sigma^{i\lambda}(a_0)$. The result is obvious for $i = 0$. Now, assume that $a_i = \sigma^{i\lambda}(a_0)$ for some $i \in [\![0, p-1]\!]$. If $i > r$, then we have

$$a_{i-r} = \sigma^m(a_i) = \sigma^{-r\lambda}(a_i) = \sigma^{(i-r)\lambda}(a_0)$$

since $m + r\lambda \equiv 0[q]$, and since the value of $\sigma^n(a_0)$ depends only on the value of n modulo q. Likewise, if $i < r$, then we have

$$a_{i-r} = a_{i-r+p} = \sigma^{m+1}(a_i) = \sigma^{-r\lambda+1+i\lambda}(a_0) = \sigma^{-r\lambda+p\lambda+i\lambda}(a_0) = \sigma^{(p-r+i)\lambda}(a_0).$$

Since r and p are coprime, every i in $[\![0, p-1]\!]$ can be reached from 0 by successively subtracting r (mod p). This terminates the proof.

□

By applying this lemma, we obtain characterizations of the sets $D^{nq}_{np}(\Delta)$ for $n = 1, 2, 3$.

Proposition 4.5.6. *(a) The maps $u \mapsto u_0$ induces a bijection between $D^q_p(\Delta)$ and the set*

$$\mathcal{O} := \left\{ \upsilon \in \mathcal{S}^{\phi^q} \mid (\upsilon, \phi^\lambda(\upsilon), \ldots, \phi^{(p-1)\lambda}(\upsilon)) \in D_p(\Delta) \right\}.$$

(b) The map $(a, b) \mapsto (a_0, b_0)$ induces a bijection between $D^{2q}_{2p}(\Delta)$ and the set

$$\left\{ (\alpha, \beta) \in (\mathcal{S}^{\phi^q})^2 \mid \alpha\beta \in \mathcal{O} \right\}.$$

(c) The map $(x, y, z) \mapsto (x_0, y_0, z_0)$ induces a bijection between $D^{3q}_{3p}(\Delta)$ and the set

$$\left\{ (\chi, \gamma, \zeta) \in (\mathcal{S}^{\phi^q})^3 \mid \chi\gamma\zeta \in \mathcal{O} \right\}.$$

Proof. We apply Lemma 4.5.5 to $A = \mathcal{S}$, $\sigma = \phi$. We obtain that, for $u \in \mathrm{Ob}(\mathcal{G}_m)$,

$$\tau^q(u) = u \Leftrightarrow \phi^q(u_0) = u_0 \text{ and } \forall i \in [\![1, p-1]\!], u_i = \phi^{i\lambda}(u_0).$$

Thus $D^q_p(\Delta)$ is exactly made of all the tuples $(\upsilon, \phi^\lambda(\upsilon), \cdots, \phi^{(p-1)\lambda}(\upsilon))$ with $\upsilon \in \mathcal{O}$, which proves (a).

(b) Let $(a, b) \in D_{2p}(\Delta)$. By definition, we have $(a, b) \in D^{2q}_{2p}(\Delta)$ if and only if $\tau^q(a) = a$ and $\tau^q(b) = b$. We then have $a = (a_0, \phi^\lambda(a_0), \ldots, \phi^{(p-1)\lambda}(a_0))$ and $b = (b_0, \phi^\lambda(b_0), \ldots, \phi^{(p-1)\lambda}(b_0))$. In particular the map $(a, b) \mapsto (a_0, b_0)$ is injective. Furthermore, the source of (a, b) is ab, which is sent to a_0b_0 by the map $D^q_p(\Delta) \to \mathcal{O}$ of point (a). Thus the map $(a, b) \mapsto (a_0, b_0)$ takes its values in the considered set. Conversely, let $(\alpha, \beta) \in (\mathcal{S}^{\phi^q})^2$ be such that $\alpha\beta \in \mathcal{O}$. By definition, we have $(\alpha\beta, \phi^\lambda(\alpha\beta), \ldots, \phi^{(p-1)\lambda}(\alpha\beta)) \in D^q_p(\Delta)$, and thus the $2p$-tuple

$$(\alpha, \beta, \phi^\lambda(\alpha), \phi^\lambda(\beta), \ldots, \phi^{(p-1)\lambda}(\alpha), \phi^{(p-1)\lambda}(\beta)) \in D^{2q}_{2p}(\Delta)$$

is a preimage of (α, β). A similar reasoning applies to point (c). □

Using Proposition 4.5.6, we can give an alternative definition of the germ $((\mathcal{S}_p)^{\phi^q}, \cdot)$.

Definition 4.5.7 (Germ of periodic elements). The *germ of (p,q)-periodic elements* is defined in the following way:

- The object set is $\mathcal{O} := \{u \in \mathcal{S}^{\phi^q} \mid (u, \phi^\lambda(u), \ldots, \phi^{(p-1)\lambda}(u)) \in D_p(\Delta)\}$.
- An arrow is a couple $(a,b) \in (\mathcal{S}^{\phi^q})^2$ such that $ab \in \mathcal{O}$.
- The partial product is given by $(x, yz) \cdot (y, z\phi^\lambda(x)) = (xy, z)$ for $(x,y,z) \in (\mathcal{S}^{\phi^q})^3$ such that $xyz \in \mathcal{O}$.

By Proposition 4.5.6, the germ of (p,q)-periodic elements defined above is isomorphic to the germ $(\mathcal{S}_p)^{\phi^q}$ attached to $\mathcal{G}$. In particular it is a Garside germ, and its associated Garside groupoid is $(\mathcal{G}_p^q, \mathcal{C}_p^q, \Delta_p)$. The interesting thing here is that we are now dealing with simple elements of $\mathcal{S}$, instead of p-tuples of simple elements.

Under this new definition, we can easily rephrase results of Section 4.4.3. First, we can characterize divisibility of simple morphisms in $\mathcal{S}_p^q$ in terms in divisibility in $\mathcal{C}$ (and not only in $\mathcal{C}^p$ as in Proposition 4.4.8 (characterization of divisibility in divided germ)).

Proposition 4.5.8 (Divisibility between simples). *Let $u \in \mathrm{Ob}(\mathcal{G}_p^q)$, and let $(a,b), (a',b') \in \mathcal{S}_p^q(u, -)$. We have $(a,b) \preccurlyeq (a',b') \Leftrightarrow a \preccurlyeq a \Leftrightarrow b \succcurlyeq b'$.*
Likewise, for $(c,d), (c',d') \in \mathcal{S}_p^q(-,u)$, we have $(c',d') \succcurlyeq (c,d) \Leftrightarrow c' \succcurlyeq c \Leftrightarrow d' \preccurlyeq d'$

Then we can also use this proposition to show a characterization of greediness, adapting Corollary 4.4.14 (greediness in divided category).

Corollary 4.5.9 (Greediness in category of periodic elements). *Let $s := (a,b), s' := (a',b') \in \mathcal{S}_p^q$ be two composable morphisms in $\mathcal{C}_p^q$. Let also $d := a' \wedge b$ with $dw = a'$ and $dy = b$. The greedy normal form of the product ss' in $\mathcal{C}_p^q$ is given by*

$$ss' = (ad, y)(x', b\phi^\lambda(d)).$$

In particular, ss' is already greedy if and only if $d = a' \wedge b$ is trivial, and ss' is a simple morphism if and only if $a' \preccurlyeq b$ in $\mathcal{C}$.

Our characterization of divisibility between simples also allows us to characterize atoms of $\mathcal{C}_p^q$.

Corollary 4.5.10 (Atoms in category of periodic elements). *Let $u \in \mathrm{Ob}(\mathcal{G}_p^q)$. A simple morphism $(a,b) \in \mathcal{S}_p^q(u,-)$ is an atom in $\mathcal{C}_p^q$ if and only if a is an atom in $\mathcal{C}^{\phi^q}$.*

Note that atoms of the category of fixed points $\mathcal{C}^{\phi^q}$ can be computed using Lemma 4.1.10 (atoms in category of fixed points). Lastly, we were able to characterize several operations like gcds and lcms in divided groupoids using the collapse functors π_i for $i \in [\![0, p-1]\!]$. However, for $s = (a,b) \in \mathcal{S}_p^q$, we have $\pi_i(s) = \phi^{i\lambda}(a)$. Thus $\pi_i = \phi^{i\lambda} \circ \pi_0$. Since ϕ is an automorphism of $\mathcal{C}$, we can then deduce all collapse functors (restricted to $\mathcal{G}_p^q$) from π_0. Applying this to Proposition 4.4.18 (collapse functors and divisibility) yields the following proposition:

Corollary 4.5.11 (Characterization of divisibility). *Let $u \in \mathrm{Ob}(\mathcal{G}_p^q)$.*

(a) For $f, g \in \mathcal{G}_p^q(u,-)$, we have

(i) $f \preccurlyeq_{\mathcal{C}_p^q} g \Leftrightarrow \pi_0(f) \preccurlyeq \pi_0(g)$.

(ii) $\pi_0(f \wedge g) = \pi_0(f) \wedge \pi_0(g)$ and $\pi_0(f \vee g) = \pi_0(f) \vee \pi_0(g)$.

(iii) $\pi_0(f\backslash g) = \pi_0(f)\backslash\pi_0(g)$.

(b) For $f, g \in \mathcal{G}_p^q(-, u)$*, we have*

(i) $g \succcurlyeq_{\mathcal{C}_p^q} f \Leftrightarrow \pi_0(g) \succcurlyeq \pi_0(f)$.

(ii) $\pi_0(f \wedge_R g) = \pi_0(f) \wedge_R \pi_0(g)$ *and* $\pi_0(f \vee_L g) = \pi_0(f) \vee_L \pi_0(g)$.

(iii) $\pi_0(g/f) = \pi_0(g)/\pi_0(f)$.

In particular, we have $f = g$ *if and only if* $\pi_0(f) = \pi_0(g)$.

Now, let ρ be conjugate to a (p, q)-periodic element in $\mathcal{G}$. By Theorem 4.5.2, there is an object u of $\mathcal{G}_p^q$ such that $\iota(\rho)$ is conjugate to $\Delta_p^q(u)$ in $\mathcal{G}_p^q$. Let then $\mathcal{G}(\rho)$ be the connected component of u in $\mathcal{G}_p^q$. We know by Theorem 4.5.2 that $\mathcal{G}(\rho)$ is equivalent to the centralizer $C_{\mathcal{G}}(\rho)$. Moreover, we saw in Section 4.3 that we could construct Garside groupoids that are also equivalent to $C_{\mathcal{G}}(\rho)$ using finite conjugacy sets. We show in the following proposition that these two constructions overlap in the sense that the groupoid of periodic elements provides a conjugacy set in the sense of Lemma 4.3.1.

Proposition 4.5.12. *Let* ρ *be conjugate to a* (p, q)*-periodic element in* $\mathcal{G}$*. The set*

$$\Gamma := \{\rho(u) \mid u \in \mathrm{Ob}(\mathcal{G}_p^q)\} \cap \mathrm{Cl}_{\mathcal{G}}(\rho)$$

is a finite conjugacy set for ρ*. Furthermore,* $\mathcal{G}(\rho)$ *is isomorphic to* $\mathcal{G}_\Gamma$*.*

Proof. First, Γ is nonempty by Theorem 4.5.2 (b). Then, we have

$$\phi(\rho(u)) = \phi(\pi_0(\Delta_p^q(u)) = \pi_0(\phi_p^p(\Delta_p^q(u)) = \pi_0(\Delta_p^q(\phi_p^p(u))) \in \Gamma,$$

and Γ is stable under ϕ. It remains to show that, for $\rho(u) \in \Gamma$ and $f \in \mathcal{C}$ such that $\rho(u)^f \in \Gamma$, we have $\rho(u)^{\alpha(f)} \in \Gamma$.

Let $f \in \mathcal{C}$ conjugate $\rho(u)$ to $\rho(v)$ for some $u, v \in \mathrm{Ob}(\mathcal{G}_p^q)$. By Corollary 4.5.4, there is some $g \in \mathcal{G}_p^q(u, v)$. Applying π_0 yields a conjugating element $\pi_0(g)$ from $\rho(u)$ to $\rho(v)$. The element $f(\pi_0(g))^{-1}$ lies in the centralizer of $\rho(u)$. By Theorem 4.5.2 (c), there is a unique $h \in \mathcal{G}_p^q(u, u)$ such that $\pi_0(h) = f(\pi_0(g))^{-1}$. The element $hg \in \mathcal{G}_p^q(u, v)$ is then the unique element such that $\pi_0(hg) = f$. Furthermore, since $\pi_0(hg) = f \in \mathcal{C}$, we have $hg \in \mathcal{C}_p^q(u, v)$ by Corollary 4.5.11. We can then compute $s = hg \wedge \Delta_p^p(u) \in \mathcal{C}_p^q(u, -)$. By Corollary 4.5.11, we have $\pi_0(s) = f \wedge \pi_0(\Delta_p^p(u)) = f \wedge \Delta(\eta)$, where η is the source of f in $\mathcal{C}$. If u' denotes the target of s in $\mathcal{C}_p^q$, then $\alpha(f) = f \wedge \Delta(\eta)$ conjugates $\rho(u)$ to $\rho(u')$. In particular $\rho(u)^{\alpha(f)} \in \Gamma$, which is then a conjugacy set for ρ.

Let now $s := (a, b) \in \mathcal{S}_p^q(u, v)$. We have that $a = \pi_0(s)$ is a conjugating element from $\rho(u)$ to $\rho(v)$ and a simple morphism. Thus we can consider $\pi_0(s)_{\rho(u)} \in \mathcal{S}_\Gamma$. By Corollary 4.5.9, this map is compatible with the germ structure on $\mathcal{S}_p^q$, and induces a functor $\mathcal{G}(\rho) \to \mathcal{G}_\Gamma$. This functor is a bijection on objects by Remark 4.5.3, and it is an equivalence by Theorem 4.5.2 and Corollary 4.3.7 (conjugacy graph and conjugacy category) (both groupoids are equivalent to $C_{\mathcal{G}}(\rho)$). It is thus an isomorphism of groupoids. □

In other words, $(\mathcal{G}_p^q, \mathcal{C}_p^q, \Delta_p)$ and $(\mathcal{G}_\Gamma, \mathcal{C}_\Gamma, \Delta_\Gamma)$ give two Garside structures on the same groupoid, which is the full subgroupoid of $\mathrm{Conj}(\mathcal{G})$ whose objects are the elements of Γ. However, the germ of periodic elements contains fewer elements than the germ $\mathcal{S}_\Gamma$: we have $\pi_0(\mathcal{S}_p^q) \subset \mathcal{S}_\Gamma$, and $\Delta \in \mathcal{S}_\Gamma$, whereas $\Delta \notin \mathcal{S}_p^q$.

4.5.2 Conjugacy classes of periodic elements

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

We want to use groupoids of periodic elements to strengthen Corollary 3.4.11, which states that, for any fixed couple of positive integers (p, q), there are finitely many conjugacy classes of (p, q)-periodic elements in $\mathcal{G}$. We actually wish for a finiteness result on <u>all</u> conjugacy classes of periodic elements. Of course, there are infinitely many conjugacy classes of periodic elements in $\mathcal{C}$ (if ϕ has order p, then $\Delta^{pn}(u)$ is a $(1, pn)$-periodic element for all $n > 1$), so we need to restrict our attention to a particular kind of periodic elements.

An endomorphism $g \in \mathcal{G}(\eta, \eta)$ will be called *rootless* if it has no n-th roots in the group $\mathcal{G}(\eta, \eta)$ for $n > 1$. Note that this is a group theoretic property, which does not depend on the choice of a Garside structure. Note also that the existence of an n-th root of an endomorphism g is equivalent to the existence of an n-th root of any conjugate of g in $\mathcal{G}$.

Of course, if $\rho \in \mathcal{G}$ is a rootless periodic elements, then all the conjugates of ρ in $\mathcal{G}$ are also rootless, even though they may only be periodic for parameters different than those of ρ.

Let $\eta \in \mathrm{Ob}(\mathcal{G})$. Recall that, by Proposition 2.1.23, the category $\mathcal{C}$ is strongly Noetherian, and we can consider $\Lambda(\Delta(\eta)) \in \mathbb{N}^*$ the maximal length of a decomposition of $\Delta(\eta)$ as a product of atoms in $\mathcal{C}$. Since $\mathcal{C}$ has a finite number of objects, we can then define

$$L := \max_{u \in \mathrm{Ob}(\mathcal{G})} \Lambda(\Delta(\eta)).$$

Proposition 4.5.13. *Let $p, q > 0$ be coprime integers.*

(a) If $\mathcal{G}$ admits (p, q)-periodic elements, then $p \leqslant L$.

(b) If $\mathcal{G}$ admits rootless (p, q)-periodic elements, then q divides the order of ϕ.

Proof. (a) Let ρ be a (p, q)-periodic element, and let λ, μ be positive integers such that $p\lambda - q\mu = 1$. By Theorem 3.4.4, we can, up to conjugacy, write ρ as $s\phi^{\lambda}(s) \cdots \phi^{(q-1)\lambda}(s)$ for some simple s such that

$$s\phi^{\lambda}(s) \cdots \phi^{(p-1)\lambda}(s) = \Delta(\eta),$$

where η is the source of ρ. In particular, we have

$$p \leqslant p\Lambda(s) \leqslant \Lambda(\Delta(\eta)) \leqslant L.$$

(b) Let ρ be a (p, q)-rootless periodic element. By Theorem 4.5.2 (b), there is an object $u \in \mathrm{Ob}(\mathcal{G}_p)$ such that $\iota(\rho)$ is conjugate to $\Delta_p^q(u)$. In particular we have $\phi_p^q(u) = u$, and $\Delta_p^q(u)$ is rootless in $\mathcal{G}_p(u, u)$. Let $q' > 0$ be the smallest integer such that $\phi_p^{q'}(u) = u$. We have $q'|q$, and $\Delta_p^{q'}(u)$ is a $\frac{q'}{q}$-th root of $\Delta_p^q(u)$. By assumption, we have $q' = q$. Furthermore, we saw in Lemma 4.4.22 that, if n is the order of the Garside automorphism ϕ of $\mathcal{G}$, then ϕ_p has order np. Since $\phi_p^{np}(u) = u$ by definition, we have $q' = q|np$, and $q|n$ since q and p are coprime. ☐

Let $p, q > 0$ be coprime integers. The fact that $\mathcal{G}$ admits rootless (p, q)-periodic elements does not mean that all (p, q)-periodic elements are rootless. For instance one can show that in the monoid $M = \langle a, b \mid a^2 = b^4 \rangle$ (which is Garside for $\Delta = a^2$), a and b^2 are both $(2, 1)$-periodic elements, with a rootless, while b^2 admits b as a square root.

Remark 4.5.14. If $\mathcal{C}$ is a homogeneous category, endowed with a length function ℓ, which is preserved by ϕ, then the equality $s\phi^{\lambda}(s)\cdots\phi^{(p-1)\lambda}(s) = \Delta(\eta)$ implies that $p\ell(s) = \ell(\Delta(\eta))$. In particular, if $\mathcal{C} = M$ is a monoid, then we get that p divides the length of the Garside element Δ.

As a first corollary, we can show that it is possible to express all periodic elements using only rootless periodic elements.

Corollary 4.5.15. *Periodic endomorphisms in $\mathcal{G}$ are exactly the powers of rootless periodic elements in $\mathcal{G}$.*

Proof. Of course, powers of rootless periodic elements are periodic elements. Conversely, we have to show that any periodic element can be written as some power of a rootless periodic element.

In general, let ρ be a (p,q)-periodic element, with p,q positive and coprime. If x is a n-th root of ρ with $n \geqslant 1$, then x is (np,q)-periodic. Since $p \wedge q = 1$, we have $(np) \wedge q = n \wedge q$, and x is conjugate to a $(p\frac{n}{n\wedge q}, \frac{q}{n\wedge q})$-periodic element by Theorem 3.4.4. Furthermore, we have

$$\left(p\frac{n}{n\wedge q}, \frac{q}{n\wedge q}\right) = (p,q) \Leftrightarrow \frac{n}{n\wedge q} = 1 \text{ and } n\wedge q = 1 \Leftrightarrow n = 1.$$

Conversely, we have $n > 1$ if and only if either $p\frac{n}{n\wedge q} > p$ or $\frac{q}{n\wedge q} < q$.

Let now $x = x_0$ be a $(p,q) = (p_0,q_0)$-periodic element, with $p_0,q_0 > 0$ coprime integers. If x is not a power of some rootless periodic elements, then we can consider a sequence of integers $n_1, n_2, \ldots$ all of them bigger than 1, and a sequence $x_1, x_2, \cdots$ of endomorphisms in $\mathcal{G}$ such that $(x_{i+1})^{n_{i+1}} = x_i$ for all $i \geqslant 1$. For each $i \geqslant 1$, we can consider coprime integers $p_i, q_i > 0$ such that x_i is conjugate to a (p_i,q_i)-periodic element.

Let $i \geqslant 1$, and let $c \in \mathcal{G}$ be such that $y := (x_{i-1})^c$ is (p_{i-1},q_{i-1})-periodic. The element $(x_i)^c$ is a n_i-th root of y with $n_i > 1$, which is conjugate to a (p_i,q_i)-periodic element. We then have either $p_i > p_{i-1}$ or $q_i < q_{i-1}$ by the first part of the proof.

The sequence $(q_i)_{i\geqslant 1}$ is a nonincreasing sequence of positive integers, thus there is some $i_0 \geqslant 1$ such that $q_i = q_{i_0}$ for all $i \geqslant i_0$. We then have $p_{i+1} > p_i$ for all $i \geqslant i_0$. For i big enough however, we obtain $p_i > L$, which contradicts Proposition 4.5.13. This contradiction shows that x must be a power of some rootless periodic elements. □

Since L is a finite integer, as well as the order of the Garside automorphism of $\mathcal{G}$, we see that p and q can take only finitely many values. Whence the following result:

Corollary 4.5.16. *There are finitely many conjugacy classes of rootless periodic elements in $\mathcal{G}$.*

Proof. Let E be the set of couples (p,q) of coprime integers such that rootless (p,q)-periodic elements exist in $\mathcal{G}$. The set E is finite by Proposition 4.5.13. For each $(p,q) \in E$, there is a finite number of conjugacy classes in $\mathcal{G}$ which contains (p,q)-periodic elements by Corollary 3.4.11.

By Theorem 3.4.4, any conjugacy class of rootless periodic elements in $\mathcal{G}$ contains elements which are (p,q)-periodic with p,q coprime. Thus the set of conjugacy class of rootless periodic elements in $\mathcal{G}$ is contained in the finite set of conjugacy classes in $\mathcal{G}$ which contains (p,q)-periodic elements for $(p,q) \in E$. □

Example 4.5.17. Corollary 4.5.16 fails when one allows an infinite number of simple morphisms. For instance the monoid

$$M := \langle a_i, i \in \mathbb{N}^* \mid a_1 = (a_i)^i \ \forall i \in \mathbb{N}^* \rangle$$

is a quasi-Garside monoid, with $(n,1)$-periodic elements for all $n \in \mathbb{N}$.

Example 4.5.18. Consider the monoid $M = \langle s,t,u \mid sts = tst, su = us, tut = utu\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. It is also a homogeneous monoid with $\ell(s) = \ell(t) = \ell(u) = 1$. Since ϕ permutes the atoms of M, this length function is ϕ-invariant.

If $G(M)$ admit rootless (p,q)-periodic elements, with $p, q > 0$ coprime integers, then by Proposition 4.5.13 and Remark 4.5.14, we have $p|\ell(\Delta) = 6$, and $q|2$. We then have

$$(p,q) \in \{(1,1),(2,1),(3,1),(6,1),(1,2),(3,2)\}.$$

For $n \geqslant 1$, Δ^n is the unique $(1,n)$-periodic element in $G(M)$. It is always a power of the unique $(1,1)$-periodic element Δ.

For $n \geqslant 1$, an $(n,1)$-periodic element in $G(M)$ is an n-th root of Δ. Such an element commutes with its power Δ, and thus must be ϕ-invariant. For $n = 6$, a 6-th root of Δ would have length 1, the only ϕ-invariant element of length 1 is t, and $t^6 \neq \Delta$. For $n = 3$, a 3-rd root of Δ would have length 2, the only ϕ-invariant element of length 2 is su, and $sususu \neq \Delta$. For $n = 2$, tsu and sut are the only square roots of Δ in $\mathcal{S}$, and they are conjugate (by t for instance).

It remains to check for $(3,2)$-periodic elements. We have $3x - 2y$ with $x = 1 = y$. By Proposition 4.5.6, we can compute the set of simple elements a such that $a\phi(a)\phi^2(a) = \Delta$. This set is given by $\{ts, ut, tu, st\}$, and all of its elements are conjugate. For a in this set, a $(3,2)$-periodic element is given by $a\phi(a)$. We obtain the set of $(3,2)$-periodic elements $\{tstu, utst, tuts, stut\}$, and all of its elements are conjugate.

By Corollary 4.5.15, we obtain that any periodic element in $G(M)$ is conjugate to a power of either $stut$ or sut. Since $(stut)^3 = \Delta^2 = (sut)^4$ is central, we deduce that an element of $G(M)$ is periodic only if either its third or fourth power is central. Since it is possible to show that $Z(G(M)) = \langle \Delta^2 \rangle$, we obtain that an element of $G(M)$ is periodic if and only if either its third or fourth power is central.

Chapter 5

Parabolicity in Garside groupoids

In this chapter we propose an in-depth study of standard parabolic subgroupoids in Garside groupoids (the concept itself is not new and was introduced by Godelle).

In particular we introduce the concept of shoal of standard parabolic subgroupoids, as a way to manage intersections of standard parabolic subgroupoids in a general Garside groupoid. This concept provides us with a framework suitable for adapting the results of González-Meneses and Marin on the intersection of arbitrary parabolic subgroups in a Garside group.

We then provide various constructions of shoals adapted to the various constructions of Garside groupoids detailed in Chapter 4.

Contents

The general concept of standard parabolic subgroup of a Garside group was introduced by Godelle in [God07] in order to generalize standard parabolic subgroups of an Artin group (i.e. a subgroup of an Artin group generated by a subset of the Artin generators).

In [God10], this notion was generalized to (quasi-)Garside categories. However, as the goal of [God10] was to study ribbons associated to Garside groups rather than general Garside categories, this notion was not investigated further. In [DDGKM, Section VII.1.4], a general concept of parabolic subcategory of a Garside category is defined. However, this concept is too broad for our purpose: In general, parabolic subcategories in the sense of [DDGKM] are not Garside categories in the sense of Definition 2.1.5 (see Example 5.1.6 below). Thus we will follow the definition of [God10] and show the basic properties of standard parabolic subgroupoids in the first section.

In a Garside group, parabolic subgroups are naturally defined as the conjugates of the standard parabolic subgroups. In [GM22], González-Meneses and Marin provide general arguments allowing to prove (under suitable assumptions) that arbitrary elements in Garside groups are contained in a smallest parabolic subgroup, and even that parabolic subgroups are stable under intersection. The starting point of their proof is to consider the smallest standard parabolic subgroup containing a given element. Such a group always exists since standard parabolic subgroups of Garside groups are always stable under intersection.

The natural extension of the arguments of González-Meneses and Marin to groupoids is, for once, not immediate. Indeed, contrary to the case of Garside groups, the intersection of standard parabolic subgroupoids of a Garside groupoid may not be again a Garside groupoid. This leads us to introduce in the second section the notion of shoals of parabolic subgroupoids. We then use this notion to generalize the arguments of González-Meneses and Marin to the categorical context, in order to obtain similar results, now on parabolic subgroups (associated to a shoal) of weak Garside groups.

Since a shoal of parabolic subgroups is an additional structure on a Garside groupoid, it is not obvious how to obtain an “interesting” shoal on a given Garside groupoid. In the third section, given a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, we provide a way to construct a shoal on $(\mathcal{G}', \mathcal{C}', \Delta')$, where $(\mathcal{G}', \mathcal{C}', \Delta')$ is a Garside groupoid obtained from $(\mathcal{G}, \mathcal{C}, \Delta)$ through one of the constructions of Chapter 4, provided of course that $(\mathcal{G}, \mathcal{C}, \Delta)$ is itself endowed with a shoal. In each case, we also show how properties of the shoal on $(\mathcal{G}, \mathcal{C}, \Delta)$ carry on to the shoal on $(\mathcal{G}', \mathcal{C}', \Delta')$.

5.1 Standard parabolic subgroupoids

We begin by introducing the definition of standard parabolic subgroupoid of a Garside groupoid we are going to work with. Just like we did with Garside groupoids in Chapter 2, we will begin with standard parabolic subcategories of a Garside category, before moving on to groupoids.

The guiding intuition is that a standard parabolic subcategory of a Garside category should be a Garside category whose germ of simples is a subgerm of the ambient germ of simples. This condition alone is too broad, as it encompasses for instance the categories of fixed points considered in Section 4.1. We thus add that a standard parabolic subcategory should be stable under (left- or right-) divisor. This is the idea behind [DDGKM, Definition VII.1.30]. However, this last definition does not require a standard parabolic subcategory to be a Garside category in the sense of Definition 2.1.5, which is why we will consider parabolic Garside maps instead of following [DDGKM, Definition VII.1.30].

5.1.1 Parabolic Garside maps

Our starting definition is that of a parabolic Garside map, which will allow us to define standard parabolic subcategories. This definition already appears in [God10, Definition 3.1], although with additional assumptions.

In this section, we fix a Garside category $(\mathcal{C}, \Delta)$ with set of simples $\mathcal{S}$ and Garside automorphism ϕ. In the case where $\mathcal{C}$ is a monoid, we will rather denote it by (M, Δ).

Let $\delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ be a partial map (i.e. defined on a subset of $\mathrm{Ob}(\mathcal{C})$). We can consider the sets $\mathrm{Div}(\delta)$ and $\mathrm{Div}_R(\delta)$ defined as in Section 2.1.1.

$$\mathrm{Div}(\delta) := \bigsqcup_{\substack{u \in \mathrm{Ob}(\mathcal{C}) \\ \delta(u) \text{ is defined}}} \mathrm{Div}(\delta(u)) \text{ and } \mathrm{Div}_R(\delta) := \bigsqcup_{\substack{u \in \mathrm{Ob}(\mathcal{C}) \\ \delta(u) \text{ is defined}}} \mathrm{Div}_R(\delta(u)).$$

Definition 5.1.1 (Parabolic Garside map). [God10, Definition 3.1]
Let $\delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ be a partial map and let $\mathcal{C}_\delta$ be the subcategory of $\mathcal{C}$ generated by $\mathrm{Div}(\delta)$. We say that δ is a *parabolic Garside map* if the following two conditions are met:

- For all object $u \in \mathrm{Ob}(\mathcal{C})$ such that $\delta(u)$ is defined, we have $\delta(u) \in \mathcal{S}(u, -)$.
- $\mathcal{S}_\delta := \mathrm{Div}(\delta) = \mathrm{Div}_R(\delta) = \mathcal{C}_\delta \cap \mathcal{S}$.

We then say that $\mathcal{C}_\delta$ is a *standard parabolic subcategory* of $\mathcal{C}$

This definition seems weaker than [God10, Definition 3.1], which also requires $\mathcal{C}_\delta(u, -)$ and $\mathcal{C}_\delta(-, u)$ to be lattices for all $u \in \mathrm{Ob}(\mathcal{C}_\delta)$. We will show later that these two definitions are equivalent (see Proposition 5.1.10).

In the case of a Garside monoid, a parabolic Garside map is simply a balanced element $\delta \in \mathcal{S}$ such that

$$M_\delta \cap \mathcal{S} = \mathrm{Div}(\delta)(= \mathrm{Div}_R(\delta)).$$

We will call such an element a parabolic Garside element, and M_δ will be called a standard parabolic submonoid.

The second condition in Definition 5.1.1 both states that δ is a balanced map, and that all products of simples dividing δ still divides δ. This last condition can be rephrased as in the following lemma:

Lemma 5.1.2. *Let $\delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ be a partial map such that $\delta(u) \in \mathcal{C}(u, -)$ whenever it is defined, and let $\mathcal{C}_\delta$ be the subcategory of $\mathcal{C}$ generated by $\mathrm{Div}(\delta)$. The following conditions are equivalent:*

(i) $\mathrm{Div}(\delta) = \mathcal{C}_\delta \cap \mathcal{S}$.

(ii) $\forall s, t \in \mathrm{Div}(\delta),\ st \in \mathcal{S} \Rightarrow st \in \mathrm{Div}(\delta)$.

(iii) $(\mathrm{Div}(\delta), \cdot) \subset (\mathcal{S}, \cdot)$ *is a subgerm of the germ of simples of* $(\mathcal{C}, \Delta)$.

Proof. $(i) \Rightarrow (ii)$ Let $s, t \in \mathrm{Div}(\delta)$ be such that $st \in \mathcal{S}$. By definition, we have $st \in \mathcal{C}_\delta$ as a product of elements of $\mathrm{Div}(\delta)$, thus $st \in \mathcal{C}_\delta \cap \mathcal{S} = \mathrm{Div}(\delta)$.

$(ii) \Leftarrow (i)$ Conversely, let $g = s_1 \cdots s_r \in \mathcal{S} \cap \mathcal{C}_\delta$, with $s_i \in \mathrm{Div}(\delta)$ for all $i \in [\![1, r]\!]$. We proceed by induction on r (the cases $r = 1, 2$ are obvious). By assumption, we have $s_1 s_2 \preccurlyeq g \preccurlyeq \Delta(u)$, where u is the source of g. Thus, we have $t := s_1 s_2 \in \mathcal{S}$ and $t \in \mathrm{Div}(\delta)$ by assumption. We then obtain $g = t s_3 \cdots s_r \in \mathcal{C}_\delta$ by induction hypothesis.

By definition, saying that $(\mathrm{Div}(\delta), \cdot) \subset (\mathcal{S}, \cdot)$ is a subgerm amounts to say that, if $s, t \in \mathrm{Div}(\delta)$ are such that $s \cdot t$ is defined in $(\mathcal{S}, \cdot)$ (i.e. $st \in \mathcal{S}$), then $s \cdot t$ is defined in $\mathrm{Div}(\Delta)$ and equal to its value in $\mathcal{S}$, which is just a rephrasing of (ii). $\square$

Using this lemma, we obtain an equivalent definition of a parabolic Garside map which is checkable in finite time using a computer.

Example 5.1.3.

- Let M be a Garside monoid, and let $a \in M$ be an atom such that $a^2 \notin \mathcal{S}$. Then, a is a parabolic Garside element with $\mathrm{Div}(a) = \{1, a\}$, and $\langle a \rangle^+$ is a parabolic submonoid.
- Consider the monoid $M = \langle a, b \mid a^2 = b^2 \rangle^+$. It is a Garside monoid with $\Delta = a^2$ and $\mathcal{S} = \{1, a, b, \Delta\}$. The element a is balanced, but it is not a parabolic Garside element since $a^2 \in \mathcal{S} \cap \langle a \rangle^+$ without $a^2 \preccurlyeq a$. Thus $\langle a \rangle^+$ is not a parabolic submonoid of M. Actually, the only parabolic submonoids of M are $\{1\}$ and M.

In Definition 5.1.1, we defined a standard parabolic subcategory using a parabolic Garside map. This raises the question of whether or not the map $\delta \mapsto \mathcal{C}_\delta$ is injective. In other words, can a standard parabolic subcategory come from distinct parabolic Garside maps? The answer is negative by the following lemma:

Lemma 5.1.4. *Let δ, δ' be two parabolic Garside maps in $\mathcal{C}$. If $\mathcal{C}_\delta = \mathcal{C}_{\delta'}$, then $\delta = \delta'$.*

Proof. Let $u \in \mathrm{Ob}(\mathcal{C}_\delta)$. We have $\delta(u) \in \mathcal{C}_{\delta'} \cap \mathcal{S} = \mathrm{Div}(\delta')$ and thus $\delta(u) \preccurlyeq \delta'(u)$. Likewise, we have $\delta'(u) \preccurlyeq \delta(u)$ and $\delta(u) = \delta'(u)$. As this holds for every object u of $\mathrm{Ob}(\mathcal{C}_\delta)$, we have $\delta = \delta'$. □

An important feature of parabolic Garside maps is that they are actually Garside maps in the associated standard parabolic subcategories.

Proposition 5.1.5 (Standard parabolic subcategories are Garside categories). *Let δ be a parabolic Garside map in $\mathcal{C}$. The couple $(\mathcal{C}_\delta, \delta)$ is a Garside category. Its set of simple morphisms is $\mathcal{S}_\delta := \mathrm{Div}(\delta)$.*

Proof. First, as a subcategory of $\mathcal{C}$, the category $\mathcal{C}_\delta$ is a cancellative category with no nontrivial invertible morphisms. By definition, the notion of a divisor of some $\delta(u)$ in $\mathcal{C}_\delta$ and in $\mathcal{C}_\delta$ coincide, thus $\mathrm{Div}(\delta) = \mathrm{Div}_R(\delta) \subset \mathcal{S}$ is a finite family which generates $\mathcal{C}_\delta$ by definition.

It remains to show that, for $u \in \mathrm{Ob}(\mathcal{C}_\delta)$ and $g \in \mathcal{C}_\delta(u, -)$, there is a left-gcd of g and $\delta(u)$ in $\mathcal{C}_\delta$. Let s be the left-gcd of g and $\delta(u)$ in $\mathcal{C}$. It is a left-divisor of $\delta(u)$ and thus $s \in \mathcal{C}_\delta$. Let now $h \in \mathcal{C}_\delta(u, -)$ be a common left-divisor of g and $\delta(u)$. By definition of s, there is some $t \in \mathcal{C}$ such that $ht = s$. We have $s \in \mathrm{Div}(\delta) = \mathrm{Div}_R(\delta)$, thus $t \in \mathrm{Div}_R(\delta) \subset \mathcal{C}_\delta$. In particular $ht = s$ holds in $\mathcal{C}_\delta$ and $h \preccurlyeq s$ in $\mathcal{C}_\delta$, whence the result. □

As stated above, there is another definition of parabolic subcategory in [DDGKM, Definition VII.1.30], which relies on the weaker notion of head subcategory. The definition we use is more restrictive than that of [DDGKM] by [DDGKM, Proposition VII.1.33]. The example below show that our definition is strictly more restrictive than [DDGKM, Definition VII.1.30].

Example 5.1.6. Consider an atom $a : u \to v$ of $\mathcal{C}$ with $u \neq v$. Let also $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ whose morphisms are $\{1_u, a, 1_v\}$. It is a parabolic subcategory of $\mathcal{C}$ in the sense of [DDGKM, Definition VII.1.30]. However, it is not a standard parabolic subcategory according to Definition 5.1.1. Actually, it is not a Garside category at all. Indeed, the only possibilities for a Garside map are $(\Delta(u), \Delta(v)) \in \{(1_u, 1_v), (a, 1_v)\}$. In the first case, $\mathrm{Div}(\Delta)$ does not generate $\mathcal{C}'$. In the second case, $\mathrm{Div}_R(\Delta) = \{1_v, a\}$ does not contain $1_u \in \mathrm{Div}(\Delta)$. It is interesting to compare this situation with that of a monoid (see Example 5.1.3).

From now on, we fix a standard parabolic subcategory $(\mathcal{C}_\delta, \delta)$ of $(\mathcal{C}, \delta)$. We write $\mathcal{S}_\delta := \mathrm{Div}(\delta)$ for its simple elements and φ_δ for the Garside automorphism of $(\mathcal{C}_\delta, \delta)$.

Proposition 5.1.5 is only a preliminary result: the main point of considering standard parabolic subcategories is not only to obtain a Garside structure, but rather a Garside structure which is related to that of the ambient category. Our next goal is to show that the embedding $\mathcal{C}_\delta \to \mathcal{C}$ preserves the Garside structure of the former. First, we show that the head functions on $(\mathcal{C}_\delta, \delta)$ is simply the restriction to $\mathcal{C}_\delta$ of the head function on $(\mathcal{C}, \Delta)$.

Lemma 5.1.7. *Let $u \in \mathrm{Ob}(\mathcal{C}_\delta)$ and $g \in \mathcal{C}_\delta(u, -)$. We have $g \wedge \Delta(u) = g \wedge \delta(u)$ in $\mathcal{C}$.*

Proof. We obviously always have $g \wedge \delta(u) \preccurlyeq g \wedge \Delta(u)$ since $\delta(u) \preccurlyeq \Delta(u)$ by definition. We just have to show that $g \wedge \Delta(u) \preccurlyeq g \wedge \delta(u)$. Since $g \in \mathcal{C}_\delta$, we can decompose g as a product $g = s_1 \cdots s_r$ of elements of $\mathcal{S}_\delta$. We proceed by induction on r.

By definition of $\mathcal{C}_\delta$, if g is a simple morphism, then

$$g \in \mathcal{C}_\delta \Leftrightarrow g \in \mathcal{S}_\delta \Leftrightarrow g \preccurlyeq \delta(u).$$

In this case, we have $g = g \wedge \Delta(u) = g \wedge \delta(u)$.

Now, if g is a product of two simple morphisms $g = st$ with $s, t \in \mathcal{S}_\delta$, then we have

$$(st) \wedge \Delta(u) = (st) \wedge (s\overline{s}) = s(t \wedge \overline{s}).$$

This last element is a product of two elements of $\mathcal{S}_\delta$, thus it belongs to $\mathcal{C}_\delta \cap \mathcal{S} = \mathcal{S}_\delta$, and $s(t \wedge \overline{s}) \preccurlyeq \delta(u)$. We then have $g \wedge \Delta(u) \preccurlyeq \delta(u)$ and $g \wedge \Delta(u) \preccurlyeq g \wedge \delta(u)$.

In general, we can write $g = s_1 \cdots s_r$ as a product of elements of $\mathcal{S}_\delta$. We write $g' := s_2 \cdots s_r$. By Lemma 2.1.11, we have

$$g \wedge \Delta(u) = \alpha(g) = \alpha(s_1 \alpha(g')) = (s_1(g' \wedge \Delta(u)) \wedge \Delta(u).$$

By induction hypothesis, we have $\sigma := g' \wedge \Delta(u) = g' \wedge \delta(u)$. Thus, $s_1\sigma$ is a product of two simple morphisms that both belong to $\mathcal{S}_\delta$. Thus $s_1\sigma \wedge \Delta(u) = s_1\sigma \wedge \delta(u)$, which finishes the proof. $\square$

This lemma allows us to show the following compatibility result:

Proposition 5.1.8 (Compatibility of standard parabolic subcategories).
The inclusion functor $\mathcal{C}_\delta \to \mathcal{C}$ preserves greedy normal forms. Furthermore, an element $f \in \mathcal{C}$ belongs to $\mathcal{C}_\delta$ if and only if every term of its greedy normal form does.

Proof. Let $u \in \mathrm{Ob}(\mathcal{C}_\delta)$, and let $f \in \mathcal{C}_\delta(u, -)$. We denote by $\alpha_\delta(f)$ the left-gcd of f and $\delta(u)$ in $\mathcal{C}_\delta$, we saw in the proof of Proposition 5.1.5 that $\alpha_\delta(f)$ is also the left-gcd of f and $\delta(u)$ in $\mathcal{C}$. By Lemma 5.1.7, we deduce that $\alpha_\delta(f) = \alpha(f)$. By an immediate induction, we deduce that the greedy normal forms of f seen in $\mathcal{C}_\delta$ or in $\mathcal{C}$ are equal. In particular, all the terms of the greedy normal form of f in $\mathcal{C}$ belong to $\mathcal{C}_\delta$.

Conversely, if $f \in \mathcal{C}$ has all the terms of its greedy normal form in $\mathcal{S}_\delta$, then $f \in \mathcal{C}_\delta$ by definition, and its greedy normal forms in $\mathcal{C}_\delta$ or in $\mathcal{C}$ are equal. $\square$

Corollary 5.1.9. *Let $f \in \mathcal{C}_\delta$, the values of* $\sup(f)$ *defined for $\mathcal{C}_\delta$ and for $\mathcal{C}$ are equal.*

Proposition 5.1.8 is also a direct application of [DDGKM, Corollary VII.2.22], since our notion of a standard parabolic subcategory is a particular case of the notion of parabolic subcategory of [DDGKM]. However, we are able to give a direct argument.

Proposition 5.1.10. *Let u be an object of $\mathcal{C}_\delta$.*

(a) If $f, g \in \mathcal{C}_\delta(u, -)$ are such that $f \preccurlyeq g$, then $g \in \mathcal{C}_\delta \Rightarrow f \in \mathcal{C}_\delta$. The inclusion map $(\mathcal{C}_\delta(u,-), \preccurlyeq) \to (\mathcal{C}(u,-), \preccurlyeq)$ is a morphism of posets which preserves $\wedge$ and $\vee$.

(b) If $f, g \in \mathcal{C}_\delta(-, u)$ are such that $g \succcurlyeq f$, then $g \in \mathcal{C}_\delta \Rightarrow f \in \mathcal{C}_\delta$. The inclusion map $(\mathcal{C}_\delta(-,u), \succcurlyeq) \to (\mathcal{C}(-,u), \succcurlyeq)$ is a morphism of posets which preserves $\wedge_R$ and $\vee_L$.

Proof. Since the definition of a parabolic Garside map is self-dual, δ^{op} is a parabolic Garside map in the category $(\mathcal{C}^{\mathrm{op}}, \Delta^{\mathrm{op}})$. Statement (b) is then just statement (a) applied in $\mathcal{C}^{\mathrm{op}}$.

Let $f, g \in \mathcal{C}_\delta(u,-)$ be such that $f \preccurlyeq g$ and $g \in \mathcal{C}_\delta$. We show by induction on $\sup(f)$ that $f \in \mathcal{C}_\delta$. If $\sup(f) = 1$, i.e. if $f \in \mathcal{S}$, then $f \preccurlyeq \alpha(g) \in \mathcal{S}_\delta$ and $f \in \mathcal{S}_\delta$. In general, by Proposition 5.1.8, we have $\alpha(f) \preccurlyeq \alpha(g) \in \mathcal{S}_\delta$ and $\alpha(f) \in \mathcal{S}_\delta$. If we write $\alpha(g) = \alpha(f)s$ for some $s \in \mathcal{S}_\delta$, we have $\omega(f) \preccurlyeq s\omega(g) \in \mathcal{C}_\delta$. Since $\sup(\omega(f)) = \sup(f) - 1$, we obtain that $\omega(f) \in \mathcal{C}_\delta$ by induction hypothesis, and $f \in \mathcal{C}_\delta$.

In particular, we see that $\mathcal{C}_\delta$ is also stable under right-divisor by working in $\mathcal{C}^{\mathrm{op}}$.

The inclusion map $(\mathcal{C}_\delta(u,-), \preccurlyeq) \to (\mathcal{C}(u,-), \preccurlyeq)$ is a morphism of posets by definition of a subcategory. Let $f, g \in \mathcal{C}_\delta(u,-)$. By the first part of the proof, the set of common divisors of f and g in $\mathcal{C}_\delta$ and in $\mathcal{C}$ are equal. In particular, $f \wedge g \in \mathcal{C}_\delta(u,-)$ is the maximum of this set in $\mathcal{C}_\delta(u,-)$ and in $\mathcal{C}(u,-)$.

We show that $f \vee g \in \mathcal{C}_\delta$ by induction on $\sup(f)$ and $\sup(g)$. If $\sup(f) = 0$ (resp. $\sup(g) = 0$), then f (resp. g) is trivial and $f \vee g = g \in \mathcal{C}_\delta$ (resp $f \vee g = f \in \mathcal{C}_\delta$). If $\sup(f) = \sup(g) = 1$, then $f, g \in \mathcal{S}_\delta = \mathrm{Div}(\delta)$ and $f \vee g \in \mathrm{Div}(\delta) = \mathcal{S}_\delta$ by definition. In general, let us write $s := \alpha(f)$, $f' := \omega(f)$, $t := \alpha(g)$ and $t' := \omega(g)$. We can compute the lcm of f and g with the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "t"'] & \cdot \arrow[r, "f'"] \arrow[d, "s\backslash t"'] & \cdot \arrow[d] \\
\cdot \arrow[r, "t\backslash s"] \arrow[d, "g'"'] & \cdot \arrow[r, "(s\backslash t)\backslash f'"] \arrow[d, "(t\backslash s)\backslash g'"'] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

which gives that

$$f \vee g = (s \vee t)\left((s\backslash t)\backslash f') \vee ((t\backslash s)\backslash g')\right).$$

We have

$$s((s\backslash t) \wedge f') = (s \vee t) \wedge f \preccurlyeq \Delta(u) \wedge f = s.$$

Thus $(s\backslash t) \wedge f'$ is trivial and $\sup((s\backslash t)\backslash f') = \sup(f') = \sup(f) - 1$ by Corollary 2.3.31. Likewise, we have $\sup((t\backslash s)\backslash g') = \sup(g') = \sup(g) - 1$.

Now, we have $s\backslash t, t\backslash s \in \mathcal{S}_\delta$ by definition. By induction hypothesis, we have $(s\backslash t) \vee f', (t\backslash s) \vee g' \in \mathcal{C}_\delta$, and $(s\backslash t)\backslash f', (t\backslash s)\backslash g' \in \mathcal{C}_\delta$ as right-divisors of elements of $\mathcal{C}_\delta$. By induction hypothesis, we have $((s\backslash t)\backslash f') \vee ((t\backslash s)\backslash g') \in \mathcal{C}_\delta$ and

$$f \vee g = (s \vee t)\left((s\backslash t)\backslash f') \vee ((t\backslash s)\backslash g')\right) \in \mathcal{C}_\delta,$$

which finishes the proof. □

In particular, Proposition 5.1.10 gives that both $\mathcal{C}_\delta(u,-)$ and $\mathcal{C}_\delta(-,u)$ are lattices. Thus Definition 5.1.1 is indeed equivalent to [God10, Definition 3.1].

5.1.2 Standard parabolic subgroupoids

In this section, we fix a Garside groupoid $(\mathcal{G},\mathcal{C},\Delta)$ with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix $(\mathcal{C}_\delta,\delta)$ a standard parabolic subcategory of $(\mathcal{C},\Delta)$, with set of simples $\mathcal{S}_\delta$ and Garside automorphism φ_δ. In the case where $\mathcal{G}$ is a group, we will rather denote the associated Garside group by (G,M,Δ).

Let $\mathcal{G}_\delta$ be the subgroupoid of $\mathcal{G}$ generated by $\mathcal{S}_\delta$. The embedding $\mathcal{C}_\delta \to \mathcal{C}$ induces a functor $\iota : \mathcal{G}(\mathcal{C}_\delta) \to \mathcal{G}$. We first show that this functor identifies $\mathcal{G}(\mathcal{C}_\delta)$ with $\mathcal{G}_\delta$.

Lemma 5.1.11. *The functor $\iota : \mathcal{G}(\mathcal{C}_\delta) \to \mathcal{G}$ preserves reduced left-fraction decompositions and symmetric normal forms. It induces an isomorphism of groupoids $\mathcal{G}(\mathcal{C}_\delta) \simeq \mathcal{G}_\delta$.*

Proof. Let $f \in \mathcal{G}(\mathcal{C}_\delta)$, and let $f = a^{-1}b$ be the reduced left-fraction decomposition of f in $\mathcal{G}(\mathcal{C}_\delta)$. We have $\iota(f) = a^{-1}b$ seen in $\mathcal{G}$. The functor $\mathcal{C}_\delta$ sends $a \wedge b$ in $\mathcal{C}_\delta$ to $a \wedge b$ in $\mathcal{C}$ by Proposition 5.1.10. Since, by definition, $a \wedge b$ is trivial in $\mathcal{C}_\delta$, it is also trivial in $\mathcal{C}$, and $a^{-1}b = \iota(f)$ is the reduced left-fraction decomposition of $\iota(f)$ in $\mathcal{G}$. Since the embedding $\mathcal{C}_\delta \to \mathcal{C}$ also preserves greedy normal forms by Proposition 5.1.8 (compatibility of standard parabolic subcategories), we deduce that $\iota : \mathcal{G}(\mathcal{C}_\delta) \to \mathcal{G}$ preserves symmetric normal forms.

By construction we have $\iota(\mathcal{G}(\mathcal{C}_\delta)) \subset \mathcal{G}_\delta$, furthermore, as every element of $\mathcal{S}_\delta$ is the image under ι of some element of $\mathcal{C}_\delta$, we have that ι is a full functor, which is bijective on objects. It remains to show that ι is faithful, which is an easy consequence of the fact that it preserves symmetric normal forms and that the embedding $\mathcal{C}_\delta \to \mathcal{C}$ is faithful. □

From now on, we will identify $\mathcal{G}(\mathcal{C}_\delta)$ with $\mathcal{G}_\delta$ using the functor ι. A first consequence of Lemma 5.1.11 is that studying recurrent endomorphisms in $\mathcal{G}_\delta$ amounts to studying recurrent endomorphisms in $\mathcal{G}$ which belong to $\mathcal{G}_\delta$.

Corollary 5.1.12. *The swap operation on $\mathcal{G}_\delta$ is the restriction to $\mathcal{G}_\delta$ of the swap operation on $\mathcal{G}$. In particular, the recurrent conjugates of an endomorphism x in $\mathcal{G}_\delta$ are exactly the recurrent conjugates of x in $\mathcal{G}$ which belong to $\mathcal{G}_\delta$.*

Using Lemma 5.1.11, we obtain the following natural extension of Definition 5.1.1 (parabolic Garside map) to Garside groupoids.

Definition 5.1.13 (Standard parabolic subgroupoid). A Garside groupoid of the form $(\mathcal{G}_\delta,\mathcal{C}_\delta,\delta)$, where $\delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ is a parabolic Garside map, is called a *standard parabolic subgroupoid* of $(\mathcal{G},\mathcal{C},\Delta)$.

The natural embedding $\mathcal{C}_\delta \to \mathcal{C}$ preserves greediness by 5.1.8 (compatibility of standard parabolic subcategories). We easily deduce an analogue result at the level of groupoids.

Proposition 5.1.14 (Compatibility of standard parabolic subgroupoids). *The embedding $\mathcal{G}_\delta \to \mathcal{G}$ preserves symmetric normal forms. Furthermore, an element $f \in \mathcal{G}$ belongs to $\mathcal{G}_\delta$ if and only if every term of its symmetric normal form does.*

Proof. The fact that the embedding $\mathcal{G}_\delta \to \mathcal{G}$ preserves symmetric normal forms is Lemma 5.1.11. In particular, all the terms of the symmetric normal form of $f \in \mathcal{G}_\delta$ lie in $\mathcal{G}_\delta$.

Conversely, if $f \in \mathcal{G}$ has all the terms of its symmetric normal form in $\mathcal{G}_\delta$, then $f \in \mathcal{G}_\delta$ by definition, and its symmetric normal forms in $\mathcal{G}_\delta$ and in $\mathcal{G}$ are equal. □

Again, Proposition 5.1.14 is also a direct application of [DDGKM, Corollary VII.2.27]. But we are again able to provide a direct argument. The above proposition has several useful corollaries.

Corollary 5.1.15. *We have $\mathcal{C}_\delta = \mathcal{G}_\delta \cap \mathcal{C}$. In particular an element of $\mathcal{G}_\delta$ is positive (resp. negative) in $\mathcal{G}_\delta$ if and only if it is positive (resp. negative) in $\mathcal{G}$.*

Proof. First, the inclusion $\mathcal{C}_\delta \subset \mathcal{G}_\delta \cap \mathcal{C}$ is immediate. Conversely, let $f \in \mathcal{G}_\delta \cap \mathcal{C}$. Since $f \in \mathcal{C}$, the symmetric normal form of f in $\mathcal{G}$ is actually its greedy normal form in $\mathcal{C}$. By Proposition 5.1.14, all the terms of the greedy normal form of f lie in $\mathcal{G}_\delta \cap \mathcal{S} = \mathcal{S}_\delta$. In particular, we have that all the terms of the greedy normal form of f lie in $\mathcal{C}_\delta$, and that f lies in $\mathcal{C}_\delta$. □

Corollary 5.1.16. *Let δ, δ' be two parabolic Garside maps in $(\mathcal{G}, \mathcal{C}, \Delta)$. If $\mathcal{G}_\delta = \mathcal{G}_{\delta'}$, then $\delta = \delta'$.*

Proof. By Corollary 5.1.15, we have $\mathcal{G}_\delta \cap \mathcal{C} = \mathcal{C}_\delta = \mathcal{C}_{\delta'} = \mathcal{G}_{\delta'} \cap \mathcal{C}$, and thus $\delta = \delta'$ by Lemma 5.1.4. □

This corollary shows that the data of a standard parabolic subgroupoid is equivalent to that of its associated standard parabolic subcategory, and to that of its associated parabolic Garside map. In particular, we will often write that $\mathcal{G}_\delta$ is a standard parabolic subgroupoid of $\mathcal{G}$, without explicit reference to either $\mathcal{C}_\delta$ or to δ.

Corollary 5.1.17. *Let f in $\mathcal{G}_\delta$.*

(a) If f is positive, then the values of $\sup(f)$ *defined for $\mathcal{G}_\delta$ and for $\mathcal{G}$ are equal.*

(b) If f is negative, then the values of $\inf(f)$ *defined for $\mathcal{G}_\delta$ and for $\mathcal{G}$ are equal.*

(c) If f is neither positive nor negative, then the values of $\inf(f)$ *and of* $\sup(f)$ *defined for $\mathcal{G}_\delta$ and for $\mathcal{G}$ are equal.*

Proof. Point (a) is simply Corollary 5.1.9. If f is negative, then $\inf(f) = -\sup(f^{-1})$, point (b) is then an immediate consequence of point (a). Lastly, if f is neither negative nor positive, then let us write $f = a^{-1}b$ as a reduced left-fraction in $\mathcal{G}_\delta$. By Proposition 5.1.14, $a^{-1}b$ is also the reduced left fraction decomposition of f in $\mathcal{G}$. We deduce point (c) from point (a) along with Corollary 2.3.27 (inf and sup from symmetric normal form). □

Notice that the result of Proposition 5.1.14 does not extend to left normal forms: this normal form is not preserved by the embedding $\mathcal{G}_\delta \to \mathcal{G}$, as the following example shows. This makes Corollary 5.1.17 even more interesting: even though the left normal form is not preserved, the infimum and supremum are preserved.

Example 5.1.18. Consider the monoid $M := \langle s, t \mid sts = tst \rangle$ (Artin-Tits monoid of type A_2). It is a Garside monoid with Garside element $\Delta = sts$. The element $\delta := s$ is a parabolic Garside element since we have

$$\mathrm{Div}(s) = \{1, s\} = \mathrm{Div}_R(s) = \langle s \rangle^+ \cap \mathcal{S}.$$

The associated parabolic submonoid is $\langle s\rangle^+$, and the associated parabolic subgroup is $\langle s\rangle$. However, one easily checks that the left normal form of s^{-1} in $G(M)_s$ is δ^{-1}, and that the left normal form of s^{-1} in $G(M)$ is $\Delta^{-1}st$.

We now show that, as in the case of categories, Proposition 5.1.14 implies that the embedding $\mathcal{G}_\delta \to \mathcal{G}$ preserves the lattice structures on $\mathcal{G}_\delta$.

Proposition 5.1.19 (Preservation of lattice structures). *Let u be an object of $\mathcal{G}_\delta$.*

(a) The inclusion map $(\mathcal{G}_\delta(u,-), \preccurlyeq_{\mathcal{C}_\delta}) \to (\mathcal{G}(u,-), \preccurlyeq_{\mathcal{C}})$ is a morphism of posets which preserves $\wedge$ and $\vee$.

(b) The inclusion map $(\mathcal{G}_\delta(-,u), \succcurlyeq_{\mathcal{C}_\delta}) \to (\mathcal{G}(-,u), \succcurlyeq_{\mathcal{C}})$ is a morphism of posets which preserves $\wedge_R$ and $\vee_L$.

Proof. Again, since $\mathcal{G}_\delta^{\mathrm{op}}$ is a standard parabolic subgroupoid of $\mathcal{G}^{\mathrm{op}}$, it is sufficient to prove point (a). First, let $f, g \in \mathcal{G}_\delta(u,-)$. We have

$$f \preccurlyeq_{\mathcal{C}_\delta} g \Leftrightarrow f^{-1}g \in \mathcal{C}_\delta \Leftrightarrow f^{-1}g \in \mathcal{C} \Leftrightarrow f \preccurlyeq_{\mathcal{C}} g,$$

and the embedding $\mathcal{G}_\delta(u,-) \to \mathcal{G}(u,-)$ is a poset morphism. Then, if $f, g \in \mathcal{C}_\delta$, then the result is already known by Proposition 5.1.8 (compatibility of standard parabolic subcategories). Otherwise, let $m \geqslant -\inf(f), -\inf(g)$, we have $\delta^m(\phi^{-m}(u))f, \delta^m(\phi^{-m}(u))g \in \mathcal{C}_\delta$. By Proposition 5.1.8, the inclusion $\mathcal{G}_\delta \to \mathcal{G}$ preserves $\delta^m(\phi^{-m}(u))f \wedge \delta^m(\phi^{-m}(u))g$ and $\delta^m(\phi^{-m}(u))f \vee \delta^m(\phi^{-m}(u))g$, by composing by $\delta^m(u)$, we obtain the result. $\square$

In the case of a Garside groups (G, M, Δ), standard parabolic subgroup(oids) and their conjugates are naturally subgroups of G. However, in the case of a Garside groupoid, standard parabolic subgroupoids are not subgroups of an associated weak Garside group. This leads us to the definition of (standard) parabolic subgroups in a Garside groupoid.

Definition 5.1.20 (Parabolic subgroup). Let $u \in \mathrm{Ob}(\mathcal{G})$. A subgroup $H \subset \mathcal{G}(u,u)$ is called a *standard parabolic subgroup* if there is a standard parabolic subgroupoid $\mathcal{G}_\delta$ of $\mathcal{G}$ such that $H = \mathcal{G}_\delta(u,u)$. It is called a *parabolic subgroup* if there is some $f \in \mathcal{G}(u,-)$ such that H^f is a standard parabolic subgroup.

Since standard parabolic subgroupoids are in particular Garside groupoids, parabolic subgroups in $\mathcal{G}$ are obvious examples of weak Garside groups.

Thanks to this definition, we can now talk about the parabolic subgroups of a weak Garside group. Note however that the notion of (standard) parabolic subgroup depends on the Garside structure, as in the following example:

Example 5.1.21. Consider the monoid $M_1 := \langle s,t \mid sts = tst\rangle^+$ (Artin-Tits monoid of type A_2). It is a Garside monoid with Garside element $\Delta_1 = sts$. Let $G := G(M)$ be the enveloping group of M_1. The balanced simple elements of M_1 are $\{1, s, t, \Delta\}$, they all are parabolic Garside elements. The parabolic subgroups of (G, M_1, Δ_1) up to conjugacy are then given by

$\{1\}, \langle s\rangle, \langle t\rangle, G$. However, we have

$$\begin{aligned} G &= \langle s,t \mid sts = tst\rangle \\ &= \langle s,t,x,y \mid x = sts, y = st, sts = tst\rangle \\ &= \langle s,t,x,y \mid y^{-1}x = s, xy^{-1} = t, x = sts = tst\rangle \\ &= \langle x,y \mid x = xy^{-1}y^{-1}xxy^{-1}\rangle \\ &= \langle x,y \mid y^3 = x^2\rangle. \end{aligned}$$

The monoid $M_2 = \langle x, y \mid x^2 = y^3\rangle^+$ is also a Garside monoid, with Garside element $\Delta_2 = x^2 (= \Delta_1^2)$. The only parabolic Garside elements of M_2 are 1 and Δ_2. The parabolic subgroups of (G, M_2, Δ_2) up to conjugacy are then given by $\{1\}$ and G.

In the sequel, we will be interested in the question of constructing smallest standard parabolic subgroups containing a given element. This is a hard problem in general, and it will lead us to introduce the notion of shoal of standard parabolic subgroupoids. However, in the case of powers of parabolic Garside maps, it is easy to describe such a smallest standard parabolic subgroup.

Lemma 5.1.22. *Let $u \in \mathrm{Ob}(\mathcal{G})$, and let $\mathcal{G}_\delta(u,u)$ be a standard parabolic subgroup of $\mathcal{G}(u,u)$. Let also $k > 0$ be the smallest integer such that $x := \delta^k(u) \in \mathcal{G}_\delta(u,u)$. A standard parabolic subgroup $H \subset \mathcal{G}(u,u)$ contains x if and only if $\mathcal{G}_\delta(u,u) \subset H$.*

Proof. Let $H := \mathcal{G}_{\delta'}(u,u)$ be a standard parabolic subgroup. If $\mathcal{G}_\delta(u,u) \subset \mathcal{G}_{\delta'}(u,u)$, then x obviously belongs to $\mathcal{G}_{\delta'}(u,u)$. Conversely, assume that $x \in \mathcal{G}_{\delta'}(u,u)$. Let $y \in \mathcal{C}_\delta(u,u)$ and let $r = \sup(y)$. We have $y \preccurlyeq \delta^r(u)$ by Proposition 2.1.25 (powers of a Garside map). Let $n \geqslant 1$ be such that $kn \geqslant r$, we have

$$y \preccurlyeq \delta^r(u) \preccurlyeq \delta^{kn}(u) = x^n \in \mathcal{C}_{\delta'}(u,u),$$

and thus $y \in \mathcal{G}_{\delta'}(u,u)$ by Proposition 5.1.10. Since $\mathcal{G}_\delta(u,u)$ is generated by $\mathcal{C}_\delta(u,u)$ along with x^{-1}, we obtain that $\mathcal{G}_\delta(u,u) \subset \mathcal{G}_{\delta'}(u,u)$. □

In the case of a Garside group, we need not worry about the sources and targets. As a direct application of Lemma 5.1.22, we obtain the following result:

Corollary 5.1.23. *Let (G, M, Δ) be a Garside group, and let δ, δ' be two parabolic Garside elements. We have $G_\delta \subset G_{\delta'}$ if and only if $\delta \preccurlyeq \delta'$.*

Lastly, another consequence of Lemma 5.1.22 is a general group theoretic result on parabolic subgroups.

Corollary 5.1.24. *Let $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u,u)$ be a parabolic subgroup. If $H \neq \mathcal{G}(u,u)$, then the index $[\mathcal{G}(u,u) : H]$ is infinite.*

Proof. It is sufficient to check the case where $H = \mathcal{G}_\delta(u,u)$ is standard. Let $k > 0$ be the smallest integer such that $x := \Delta^k(u) \in \mathcal{G}(u,u)$. Since $\mathcal{G}_\delta \neq \mathcal{G}$ by assumption, we have $x \notin H$ by Lemma 5.1.22. Likewise, we see that no positive power of x lies in H, and H has infinite index. □

In particular, a (nontrivial) parabolic subgroup H of some $\mathcal{G}(u,u)$ cannot be made into a weak Garside group by using the methods of Section 4.2. The associated parabolic subgroupoid of $\mathcal{G}$ is then in general the only Garside groupoid to which H is naturally equivalent.

Before we close this section, we give some other results on standard parabolic subgroupoids which will be useful later.

First, since the definition of a standard parabolic subgroupoid depends mostly on divisibility conditions and on the germ of simples $\mathcal{S}$, we expect that the image under the Garside automorphism ϕ of a standard parabolic subgroupoid of $(\mathcal{G}, \mathcal{C}, \Delta)$ is again a standard parabolic subgroupoid. This is true by the following lemma:

Lemma 5.1.25. *The map $\phi(\delta)$ defined by $\phi(\delta)(u) = \phi(\delta(\phi^{-1}(u)))$ whenever $\delta(\phi^{-1}(u))$ is defined, is a parabolic Garside map for $(\mathcal{G}, \mathcal{C}, \Delta)$. Furthermore, we have $\mathcal{G}_{\phi(\delta)} = \phi(\mathcal{G}_\delta)$ and $\mathcal{C}_{\phi(\delta)} = \phi(\mathcal{C}_\delta)$.*

Proof. First, we have $\phi(\delta)(u) \in \mathcal{S}(\phi(u), -)$ whenever $\phi(\delta)(u)$ is defined since ϕ preserves $\mathcal{S}$ globally. Then, let $s \in \mathcal{S}(u, v)$, we have

$$\begin{aligned} s \preccurlyeq \phi(\delta)(u) &\Leftrightarrow \phi^{-1}(s) \preccurlyeq \delta(\phi^{-1}(u)) \\ &\Leftrightarrow \delta(\varphi^{-1}\phi^{-1}(v)) \succcurlyeq \phi^{-1}(s) \\ &\Leftrightarrow \phi(\delta)(\phi\varphi^{-1}\phi^{-1}(v)) \succcurlyeq s, \end{aligned}$$

and thus $\phi(\delta)$ is a balanced map. If $s, t \in \mathrm{Div}(\phi(\delta))$ are such that $st \in \mathcal{S}$, then we have $\phi^{-1}(s), \phi^{-1}(t) \in \mathrm{Div}(\delta)$, and $\phi^{-1}(s)\phi^{-1}(t) = \phi^{-1}(st) \in \mathcal{S}$. Thus $\phi^{-1}(st) \in \mathrm{Div}(\delta)$ since δ is a parabolic Garside map and $st \in \mathrm{Div}(\phi(\delta))$. We obtain that $\phi(\delta)$ is a parabolic Garside map by Lemma 5.1.2. By definition, $\mathcal{G}_{\phi(\delta)}$ is generated (as a subgroupoid of $\mathcal{G}$) by $\mathrm{Div}(\phi(\delta))$. We obtain that $\mathcal{G}_{\phi(\delta)} = \mathcal{G}_\delta$ since $\mathrm{Div}(\phi(\delta)) = \phi(\mathrm{Div}(\delta))$. We obtain similarly that $\mathcal{C}_{\phi(\delta)} = \phi(\mathcal{C}_\delta)$. ☐

Recall that a Garside groupoid needs not be connected. Later on, we will need to consider decompositions of standard parabolic subgroupoids as disjoint unions, along with disjoint unions of standard parabolic subgroupoids. The situation is easily understood in the following lemma:

Lemma 5.1.26 (Standard parabolic subgroupoids and disjoint unions).

(a) Let $\mathcal{G}_{\delta_1}$ and $\mathcal{G}_{\delta_2}$ be two standard parabolic subgroupoids of $\mathcal{G}$ such that $\mathcal{G}_{\delta_1} \cap \mathcal{G}_{\delta_2} = \varnothing$. Then $\mathcal{G}_{\delta_1} \sqcup \mathcal{G}_{\delta_2}$ is a standard parabolic subgroupoid of $\mathcal{G}$.

(b) Let $\mathcal{G}_1, \mathcal{G}_2$ be subgroupoids of $\mathcal{G}$ such that $\mathcal{G}_1 \cap \mathcal{G}_2 = \varnothing$ and $\mathcal{G}_1 \sqcup \mathcal{G}_2 = \mathcal{G}_\delta$. Then $\delta_1 := \delta_{|\,\mathrm{Ob}(\mathcal{G}_1)}$ (resp. $\delta_2 := \delta_{|\,\mathrm{Ob}(\mathcal{G}_2)}$) is a parabolic Garside map, and $\mathcal{G}_1 = \mathcal{G}_{\delta_1}$ (resp. $\mathcal{G}_2 = \mathcal{G}_{\delta_2}$).

Proof. (a) Let δ be the partial map on $\mathrm{Ob}(\mathcal{G})$ defined by

$$\delta(u) = \begin{cases} \delta_1(u) & \text{if } u \in \mathrm{Ob}(\mathcal{G}_{\delta_1}), \\ \delta_2(u) & \text{if } u \in \mathrm{Ob}(\mathcal{G}_{\delta_2}). \end{cases}$$

Since $\mathrm{Ob}(\mathcal{G}_{\delta_1})$ and $\mathrm{Ob}(\mathcal{G}_{\delta_2})$ are disjoint, these cases are mutually exclusive, and we have $\mathrm{Div}(\delta) = \mathcal{S}_{\delta_1} \sqcup \mathcal{S}_{\delta_2}$, which shows in particular that δ is a parabolic Garside map, and that $\mathcal{G}_\delta = \mathcal{G}_{\delta_1} \sqcup \mathcal{G}_{\delta_2}$.

(b) It is sufficient to prove the result for $\mathcal{G}_1$ since $\mathcal{G}_1$ and $\mathcal{G}_2$ play symmetric roles. First, since $\mathcal{G}_1 \cap \mathcal{G}_2 = \varnothing$, and since $\mathcal{G}_1, \mathcal{G}_2$ are groupoids, we have $\mathrm{Ob}(\mathcal{G}_1) \cap \mathrm{Ob}(\mathcal{G}_2) = \varnothing$. For $f \in \mathcal{G}_\delta(u, v)$, we then have

$$f \in \mathcal{G}_1 \Leftrightarrow u \in \mathrm{Ob}(\mathcal{G}_1) \Leftrightarrow v \in \mathrm{Ob}(\mathcal{G}_1).$$

For instance if $u \in \mathrm{Ob}(\mathcal{G}_1)$, then $f \notin \mathcal{G}_2$, and thus $f \in \mathcal{G}_1$ since $f \in \mathcal{G}_\delta = \mathcal{G}_1 \sqcup \mathcal{G}_2$.

We then show that the Garside automorphism φ attached with δ induces a bijection of $\mathrm{Ob}(\mathcal{G}_1)$ to itself. Let $u \in \mathrm{Ob}(\mathcal{G}_1)$, we have $\delta(u) = \delta_1(u) \in \mathcal{G}_\delta$, and thus the target $\varphi(u)$ of $\delta_1(u)$ also lies in $\mathrm{Ob}(\mathcal{G}_1)$. The Garside automorphism φ then induces an injective map from $\mathrm{Ob}(\mathcal{G}_1)$ to itself. Since $\mathrm{Ob}(\mathcal{G}_1)$ is finite, this map is also bijective.

Now, by definition, δ_1 is a partial map defined on $\mathrm{Ob}(\mathcal{G}_1) \subset \mathrm{Ob}(\mathcal{G}_\delta)$ and it takes its values in $\mathcal{S}$. Let now $s \in \mathcal{S}(u,v)$. If $u \in \mathrm{Ob}(\mathcal{G}_1)$ and if $s \preccurlyeq \delta_1(u) = \delta(u)$, then $s \in \mathcal{G}_1$ and $v \in \mathrm{Ob}(\mathcal{G}_1)$. We also have $\delta_1(\varphi^{-1}(v)) = \delta(\varphi^{-1}(v)) \succcurlyeq s$ as δ is a parabolic Garside map. Likewise, if $v \in \mathrm{Ob}(\mathcal{G}_1)$, and if $\delta_1(\varphi^{-1}(v)) \succcurlyeq s$, then $u \in \mathrm{Ob}(\mathcal{G}_1)$ and $s \preccurlyeq \delta_1(u)$. Thus δ_1 is a balanced map.

Lastly, let $s, t \in \mathcal{S} \cap \mathrm{Div}(\delta_1)$ be such that $st \in \mathcal{S}$. Since δ is a parabolic Garside map, we have $st \in \mathrm{Div}(\delta)$ and $st \in \mathrm{Div}(\delta_1)$ since the source of st (which is the source of s) lies in $\mathrm{Ob}(\mathcal{G}_1)$. □

In particular, the above lemma proves that connected components of standard parabolic subgroupoids are also standard parabolic subgroupoids.

5.1.3 Intersection of standard parabolic subgroupoids

In [God07], Godelle shows that the intersection of two standard parabolic subgroups of a Garside group is again a Garside group. Unfortunately, this situation does not extend to standard parabolic subgroupoids of Garside groupoids. However, we give in this short section a characterization of the cases where the intersection of two standard parabolic subgroupoids is again a standard parabolic subgroupoid.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

Let $\mathcal{G}_\delta, \mathcal{G}_{\delta'}$ be two standard parabolic categories of $\mathcal{G}$, and let φ, φ' be the respective associated Garside automorphisms. For $u \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'}) = \mathrm{Ob}(\mathcal{G}_\delta) \cap \mathrm{Ob}(\mathcal{G}_{\delta'})$, we set

$$\rho(u) := \delta(u) \wedge \delta'(u) \text{ and } \lambda(v) := \delta(\varphi^{-1}(v)) \wedge_R \delta'(\varphi'^{-1}(v)).$$

Note that v refers to the target of $\lambda(v)$, whereas u refers to the source of $\rho(u)$. We denote by $R(u)$ the target of $\rho(u)$, and by $L(u)$ the source of $\lambda(u)$.

Proposition 5.1.27 (Intersection of standard parabolic subgroupoids).
Let $\mathcal{G}_\delta$ and $\mathcal{G}_{\delta'}$ be two standard parabolic subgroupoids of $\mathcal{G}$. Let also ρ, λ, R, L as above. The following assertions are equivalent:

(i) The maps $u \mapsto R(u)$ and $u \mapsto L(u)$ are both injective.

(ii) The maps $u \mapsto R(u)$ and $v \mapsto L(v)$ are bijections of $\mathrm{Ob}(\mathcal{C}_\delta \cap \mathcal{C}_{\delta'})$ to itself.

(iii) For $u, v \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$, we have $\lambda(R(u)) = \rho(u)$ and $\rho(L(v)) = \lambda(v)$. In particular R and L are inverse bijections of one another.

(iv) $\mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$ is a standard parabolic subgroupoid of $\mathcal{G}$.

If these conditions are met, then ρ is a parabolic Garside map in $(\mathcal{G}, \mathcal{C}, \Delta)$, and $(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'}, \mathcal{C}_\delta \cap \mathcal{C}_{\delta'}, \rho)$ is a standard parabolic subgroupoid of $\mathcal{G}$.

Proof. $(i) \Leftrightarrow (ii)$. Let $u \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$. By definition, we have $\rho(u) \preccurlyeq \delta(u), \delta'(u)$, thus its target $R(u)$ belongs to $\mathrm{Ob}(\mathcal{G}_\delta) \cap \mathrm{Ob}(\mathcal{G}_{\delta'}) = \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$. Likewise, we have $L(u) \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$. Since $\mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$ is finite, we deduce that R and L are bijective if and only if they are injective.

(iii) $\Rightarrow$ *(ii)* is obvious. *(ii)* $\Rightarrow$ *(iii)* Let $u, v \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$. By definition, we have $\lambda(R(u)) \succcurlyeq \rho(u)$ and $\lambda(v) \preccurlyeq \rho(L(v))$. By iterating this, we obtain diagrams of the following form in $\mathcal{C}$:

$$\begin{tikzcd}
\vdots \arrow[d, dotted] & \vdots \\
(LR)^2(u) \arrow[r, dotted, dash, "\rho((LR)^2(u))"] \arrow[d, dashed] \arrow[dr, "\lambda(RLR(u))"] & R(LR)^2(u) \arrow[u, dotted] \\
LR(u) \arrow[r, "\rho(LR(u))"] \arrow[d, dashed] \arrow[dr, "\lambda(R(u))"] & RLR(u) \arrow[u, dashed] \\
u \arrow[r, "\rho(u)"] & R(u) \arrow[u, dashed]
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, dotted] & \vdots \\
L(RL)^2(v) \arrow[r, dotted, dash, "\lambda((RL)^2(v))"] \arrow[d, dashed] & (RL)^2(v) \arrow[u, dotted] \\
LRL(v) \arrow[r, "\lambda(RL(v))"] \arrow[d, dashed] \arrow[ur, "\rho(LRL(v))"] & RL(v) \arrow[u, dashed] \\
L(v) \arrow[r, "\lambda(v)"'] \arrow[ur, "\rho(L(v))"] & v \arrow[u, dashed]
\end{tikzcd}$$

Since L and R are both bijective, the compositions LR and RL are also bijective. Let $n \geqslant 1$ be the smallest integer such that $(LR)^n(u) = u$. Using the above diagrams, we have morphisms $f, g \in \mathcal{C}$ such that $f\rho(u)g = \rho(u)$. Since $\mathcal{C}$ is Noetherian, we must have $f = 1_u$ and $g = 1_{R(u)}$. In particular the morphism $LR(u) \to u$ is trivial and $\lambda(R(u)) = \rho(u)$. Likewise, we obtain that $\rho(L(v)) = \lambda(v)$.

(iii) $\Rightarrow$ *(iv)* We show that ρ is a parabolic Garside map in $\mathcal{G}$, and that $\mathcal{G}_\delta \cap \mathcal{G}_{\delta'} = \mathcal{G}_\rho$. First, we obviously have $\rho(u) \in \mathcal{S}(u, -)$ for $u \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$. Then for $u \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$, and $f \in \mathcal{C}(u, -)$, we have

$$f \preccurlyeq \rho(u) \Leftrightarrow f \preccurlyeq \delta(u), \delta'(u).$$

Thus, $\mathrm{Div}(\rho) = \mathcal{S}_\delta \cap \mathcal{S}_{\delta'}$. Likewise, we get

$$\mathrm{Div}_R(\lambda) = \mathrm{Div}_R(\delta) \cap \mathrm{Div}_R(\delta') = \mathcal{S}_\delta \cap \mathcal{S}_{\delta'} = \mathrm{Div}(\rho).$$

However, by assumption, we have $\mathrm{Div}_R(\lambda) = \mathrm{Div}_R(\rho)$, thus ρ is a balanced map. Let then $s, t \in \mathrm{Div}(\rho)$ be such that $st \in \mathcal{S}$. Since δ and δ' are parabolic Garside map, we obtain $st \in \mathcal{S}_\delta \cap \mathcal{S}_{\delta'} = \mathrm{Div}(\rho)$. Thus ρ is a parabolic Garside map in $\mathcal{G}$. It remains to show that $\mathcal{G}_\rho = \mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$. We have $\mathrm{Div}(\rho) \subset \mathcal{S}_\delta \cap \mathcal{S}_{\delta'}$ and $\mathcal{G}_\rho \subset \mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$ by definition of ρ. Conversely, if $f \in \mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$, then, by Proposition 5.1.14 (compatibility of standard parabolic subgroupoids), all the terms of the symmetric normal form of f lie in $\mathcal{S}_\delta \cap \mathcal{S}_{\delta'} = \mathrm{Div}(\rho)$. Thus $f \in \mathcal{G}_\rho$ and $\mathcal{G}_\rho = \mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$ as claimed.

(iv) $\Rightarrow$ *(iii)* Lastly, assume that $\mathcal{G}_\delta \cap \mathcal{G}_{\delta'} = \mathcal{G}_\gamma$ is a standard parabolic subcategory of $\mathcal{G}$. In this case $\mathcal{G}_\gamma \cap \mathcal{C}$ is a standard parabolic subcategory of $\mathcal{C}$, and we have $\mathcal{G}_\gamma \cap \mathcal{C} = \mathcal{G}_\delta \cap \mathcal{G}_{\delta'} \cap \mathcal{C} = \mathcal{C}_\delta \cap \mathcal{C}_{\delta'}$ by Corollary 5.1.15. We denote by ψ the Garside automorphism of $\mathcal{G}_\gamma$. We have

$$\begin{aligned}
\mathrm{Div}(\gamma) &= \mathcal{C}_\gamma \cap \mathcal{S} \\
&= \mathcal{C}_\delta \cap \mathcal{S} \cap \mathcal{C}_{\delta'} \cap \mathcal{S} \\
&= \mathrm{Div}(\delta) \cap \mathrm{Div}(\delta') = \mathrm{Div}(\rho) \\
&= \mathrm{Div}_R(\delta) \cap \mathrm{Div}_R(\delta') = \mathrm{Div}_R(\lambda).
\end{aligned}$$

For $u \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$, we deduce that $\rho(u) \preccurlyeq \gamma(u)$ and $\gamma(u) \preccurlyeq \rho(u)$, thus $\rho(u) = \gamma(u)$ and $R(u) = \psi(u)$. Likewise, for $v \in \mathrm{Ob}(\mathcal{G}_\delta \cap \mathcal{G}_{\delta'})$, we have $\gamma(\psi^{-1}(v)) = \lambda(v)$ and $L(v) = \psi^{-1}(v)$. We obtain that (ii) holds, and thus that (iii) holds. Furthermore, the implication $(iii) \Rightarrow (iv)$ proves that ρ is a parabolic Garside map with $\mathcal{G}_\rho = \mathcal{G}_\gamma$. We then have $\rho = \gamma$ by Lemma 5.1.4. $\square$

Corollary 5.1.28. *Let $\mathcal{G}_\delta$ and $\mathcal{G}_{\delta'}$ be two standard parabolic subcategories of $\mathcal{G}$ such that $\mathcal{G}_{\delta''} := \mathcal{G}_\delta \cap \mathcal{G}_{\delta'}$ is also a standard parabolic subcategory. For $u \in \mathrm{Ob}(\mathcal{G}_{\delta''})$ and $k > 0$ an integer, we have $\delta''^k(u) = \delta^k(u) \wedge \delta'^k(u)$.*

Proof. Let $f \in \mathcal{C}$ have source u. If $f \preccurlyeq \delta^k(u)$, then $\sup(f) \leqslant \sup(\delta^k(u)) = k$, and $f \in \mathcal{C}_\delta$ since f divides $\delta^k(u) \in \mathcal{C}_\delta$ (Proposition 5.1.10). Conversely, if $\sup(f) \leqslant k$ and if $f \in \mathcal{C}_\delta$, then the supremum of f in $\mathcal{C}_\delta$ is also inferior or equal to k by Corollary 5.1.9. By Proposition 2.1.25 (powers of a Garside map), we then have $f \preccurlyeq \delta^k(u)$. From this we deduce that

$$\begin{aligned} f \preccurlyeq \delta^k(u) \wedge \delta'^k(u) &\Leftrightarrow \sup(f) \leqslant k \text{ and } f \in \mathcal{C}_\delta \cap \mathcal{C}_{\delta'} \\ &\Leftrightarrow \sup(f) \leqslant k \text{ and } f \in \mathcal{C}_\delta \cap \mathcal{C}_{\delta'} = \mathcal{C}_{\delta''} \\ &\Leftrightarrow f \preccurlyeq \delta''^k(u), \end{aligned}$$

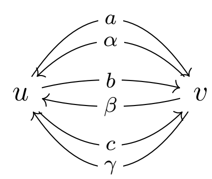
and the result follows. $\square$

In the case where $\mathcal{G} = G$ is a group, condition (i) in Proposition 5.1.27 is always satisfied, and we obtain the following result:

Corollary 5.1.29. *Let (G, M, Δ) be a Garside group. Standard parabolic subgroups of G are stable under intersection. More precisely, if δ, δ' are parabolic Garside elements in G, then $\delta \wedge \delta'$ is also a parabolic Garside element, and $G_\delta \cap G_{\delta'} = G_{\delta \wedge \delta'}$.*

Unfortunately, Proposition 5.1.27 doesn't always apply when $\mathcal{G}$ is a category with more than one object, as seen in the example below.

Example 5.1.30. Consider the following graph $\mathcal{S}$:



endowed with the relations $a\alpha a = b\beta b = c\gamma c$ and $\alpha a\alpha = \beta b\beta = \gamma c\gamma$. The category $\mathcal{C}$ presented by this data is a Garside category, with Garside map $\Delta(u) = a\alpha a$ and $\Delta(v) = \beta b\beta$. We denote by $\mathcal{G}$ the enveloping groupoid of $\mathcal{C}$. Consider the map δ defined by $\delta(u) = a$ and $\delta(v) = \beta$. We claim that δ is a parabolic Garside map. Indeed, we have

$$\mathrm{Div}(\delta) = \mathrm{Div}(a) \sqcup \mathrm{Div}(\beta) = \{1_u, a, \beta, 1_v\} = \mathrm{Div}_R(a) \sqcup \mathrm{Div}_R(\beta) = \mathrm{Div}_R(\delta),$$

and δ is balanced. The only pairs of composable simples in $\mathrm{Div}(\delta)$ which do not contain an identity morphism are a, β and β, a. Since neither $a\beta$ nor βa are simple morphisms in $\mathcal{C}$, we deduce that δ is indeed a Garside map by Lemma 5.1.2 (vacuously so). With a similar reasoning, we obtain that the map δ' defined by $\delta'(u) = c$ and $\delta'(v) = \beta$ is also a parabolic Garside map. However, the map $\delta \wedge \delta'$ is given by

$$(\delta \wedge \delta')(u) = a \wedge c = 1_u \text{ and } (\delta \wedge \delta')(v) = \beta \wedge \beta = \beta.$$

In particular, both $(\delta \wedge \delta')(u)$ and $(\delta \wedge \delta')(v)$ share the same target, and point (i) in Proposition 5.1.27 is not satisfied.

Note however, that $\mathcal{C}_\delta \cap \mathcal{C}_{\delta'} = \{1_u, \beta, 1_v\}$ is a parabolic subcategory of $\mathcal{C}$ in the sense of [DDGKM, Definition VII.1.30], as in Example 5.1.6.

This is the main difference between Garside groups and Garside groupoids when considering parabolic subcategories, which leads us to introducing the notion of shoal in the next section.

5.2 Shoals of standard parabolic subgroupoids

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ.

We saw in Example 5.1.30 that the intersection of standard parabolic subgroupoids of $\mathcal{G}$ is not always a standard parabolic subgroupoid. To compensate this issue, we will not consider the family of all standard parabolic subgroupoids of $\mathcal{G}$, but we will restrict our attention to particular subsets of this family, that we call shoals[1].

Definition 5.2.1 (Shoal of standard parabolic subgroupoids). A *shoal* of standard parabolic subgroupoids for $(\mathcal{G}, \mathcal{C}, \Delta)$ (or simply, a *shoal*) is a subset $\mathcal{T}$ of the set of standard parabolic subgroupoids of $(\mathcal{G}, \mathcal{C}, \Delta)$ such that

- $\mathcal{G} \in \mathcal{T}$ and $\{1_u\}_{u \in \mathrm{Ob}(\mathcal{G})} \in \mathcal{T}$.
- If $\mathcal{G}_\delta \in \mathcal{T}$, then $\phi(\mathcal{G}_\delta) \in \mathcal{T}$.
- The intersection of two elements of $\mathcal{T}$, if nonempty, lies in $\mathcal{T}$.

Note that the set of all standard parabolic subgroupoids of $\mathcal{G}$ is stable under ϕ by Lemma 5.1.25.

Alternatively, we can define a *shoal* of standard parabolic subcategories for $(\mathcal{C}, \Delta)$ (or simply, a *shoal*) as a subset $\mathcal{T}$ of the set of standard parabolic subcategories of $\mathcal{C}$ with the same conditions (except $\mathcal{C} \in \mathcal{T}$ instead of $\mathcal{G} \in \mathcal{T}$). If $\mathcal{T}$ is a shoal for $\mathcal{G}$, then $\{\mathcal{C}_\delta = \mathcal{G}_\delta \cap \mathcal{C} \mid \mathcal{G}_\delta \in \mathcal{T}\}$ is a shoal for $\mathcal{C}$. Conversely, if $\mathcal{T}$ is a shoal for $\mathcal{C}$, then $\{\mathcal{G}_\delta \mid \mathcal{C}_\delta \in \mathcal{T}\}$ is a shoal for $\mathcal{G}$. Thus we will often consider shoals for $(\mathcal{G}, \mathcal{C}, \Delta)$ without distinguishing between $\mathcal{G}$ and $\mathcal{C}$.

Notice that, as the family of standard parabolic subgroupoids of $\mathcal{G}$ is finite, the third condition in the definition of a shoal is equivalent to the stability under all intersections (provided that said intersections are nonempty).

Fixing a shoal $\mathcal{T}$ allows us to consider the particular parabolic subgroups which come from a standard parabolic subgroup belonging to $\mathcal{T}$, as in the following definition:

Definition 5.2.2 ($\mathcal{T}$-parabolic subgroup). Let $\mathcal{T}$ be a shoal for $\mathcal{G}$, and let $u \in \mathrm{Ob}(\mathcal{G})$. A subgroup $H \subset \mathcal{G}(u, u)$ is called a *$\mathcal{T}$-standard parabolic subgroup* if there is some $\mathcal{G}_\delta \in \mathcal{T}$ such that $H = \mathcal{G}_\delta(u, u)$. It is called a *$\mathcal{T}$-parabolic subgroup* if there exists $f \in \mathcal{G}(u, -)$ such that H^f is a $\mathcal{T}$-standard parabolic subgroup. The set of $\mathcal{T}$-parabolic subgroups of $\mathcal{G}(u, u)$ is denoted by $\mathcal{P}_\mathcal{T}(\mathcal{G}(u, u))$.

[1]We thank Ramla Abdellatif and Arthur Garnier for fruitful discussions which gave rise to the following concise definition.

Note that, a priori, a standard parabolic subgroup in $\mathcal{G}$ which is a $\mathcal{T}$-parabolic subgroup may not be a $\mathcal{T}$-standard parabolic subgroup in $\mathcal{G}$, although the converse is always true. In Section 5.2.1, we will see that in the case of a support-preserving shoal, the $\mathcal{T}$-standard parabolic subgroups are exactly the $\mathcal{T}$-parabolic subgroups which are standard parabolic subgroups (see Corollary 5.2.15).

In the sequel, we will construct shoals adapted to deal with the various constructions that we detailed in Chapter 4. Our basic building block for this is the fact that the family of all standard parabolic subgroups of a Garside groups is a shoal.

Lemma 5.2.3. *Let (G, M, Δ) be a Garside group. The family of all standard parabolic subgroups of G is a shoal for G.*

Proof. Let $\mathcal{T}$ be the family of standard parabolic subgroups of G. First, we have $G = G_\Delta \in \mathcal{T}$, and $\{1\} = G_1 \in \mathcal{T}$. The set $\mathcal{T}$ is stable under ϕ by Lemma 5.1.25, and stable under intersection by Corollary 5.1.29. □

The definition of a shoal is made so that we can consider standard parabolic closure of elements of $\mathcal{G}$. This definition is the first step towards building parabolic closures in Section 5.2.1.

Definition 5.2.4 (Standard parabolic closure). Let $\mathcal{T}$ be a shoal for $\mathcal{G}$, and let $f \in \mathcal{G}$. The *$\mathcal{T}$-standard categorical parabolic closure* $\mathrm{SCPC}(f)$ of f is defined as the intersection of all the elements of $\mathcal{T}$ which contain f.
If x is an endomorphism of some object $u \in \mathrm{Ob}(\mathcal{G})$, the *$\mathcal{T}$-standard parabolic closure* of x is $\mathrm{SPC}(x) := \mathrm{SCPC}(x)(u, u)$.

Note that, since $\mathcal{G} \in \mathcal{T}$, the intersection which defines $\mathrm{SCPC}(f)$ is nonempty. If the shoal $\mathcal{T}$ is clear by context, we will simply talk about the standard (categorical) parabolic closure of a morphism $f \in \mathcal{G}$. Note also that the $\mathcal{T}$-standard parabolic closure of an endomorphism $x \in \mathcal{G}(u, u)$ can also be defined as the intersection of all the $\mathcal{T}$-standard parabolic subgroups of $\mathcal{G}(u, u)$ which contain x.

Example 5.2.5. We always have a (somewhat trivial) shoal for $\mathcal{G}$ given by

$$\mathcal{T} := \{\{1_u\}_{u \in \mathrm{Ob}(\mathcal{C})}, \mathcal{G}\}.$$

For $f \in \mathcal{G}$, we either have $\mathrm{SCPC}(f) = \mathcal{G}$ if f is nontrivial, and $\mathrm{SCPC}(f) = \{1_u\}_{u \in \mathrm{Ob}(\mathcal{C})}$ if $f = 1_v$ for some $v \in \mathrm{Ob}(\mathcal{G})$, in which case $\mathrm{SPC}(f) = \{1_v\}$.

The second condition in the definition of a shoal is that the Garside automorphism ϕ preserves $\mathcal{T}$-standard parabolic closures.

Lemma 5.2.6. *Let $\mathcal{T}$ be a shoal for $\mathcal{G}$, and let $x \in \mathcal{G}$ be an endomorphism. We have* $\mathrm{SPC}(\phi(x)) = \phi(\mathrm{SPC}(x))$.

Proof. Let $u \in \mathrm{Ob}(\mathcal{G})$ be the source of x, and let $\mathcal{G}_\delta \in \mathcal{T}$ be such that $\mathcal{G}_\delta(u, u) = \mathrm{SPC}(x)$. By definition, $\phi(\mathrm{SPC}(x)) = \mathcal{G}_{\phi(\delta)}(\phi(u), \phi(u))$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(\phi(u), \phi(u))$, and it contains $\phi(x)$. We then have $\phi(\mathrm{SPC}(x)) \subset \mathrm{SPC}(\phi(x))$. Applying the same reasoning, replacing x by $\phi(x)$ and ϕ by ϕ^{-1} yields $\mathrm{SPC}(\phi(x)) \subset \phi(\mathrm{SPC}(x))$. □

By definition of a shoal, every element of $\mathcal{G}$ admits a $\mathcal{T}$-standard parabolic closure. This in itself does not prove that every element of the shoal $\mathcal{T}$ is the $\mathcal{T}$-standard parabolic closure of some element of $\mathcal{G}$. The following easy lemma proves that this is the case.

Corollary 5.2.7. *Let $\mathcal{T}$ be a shoal for $\mathcal{G}$, and let $\mathcal{G}_\delta \in \mathcal{T}$. Let also $u \in \mathrm{Ob}(\mathcal{G}_\delta)$, and let $k > 0$ be the smallest integer such that $x := \delta^k(u) \in \mathcal{G}_\delta(u,u)$. We have $\mathcal{G}_\delta(u,u) = \mathrm{SPC}(x)$.*

Proof. Since $\mathcal{G}_\delta \in \mathcal{T}$, we have that $\mathcal{G}_\delta(u,u)$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$. Conversely, if $\mathcal{G}_{\delta'} \in \mathcal{T}$ is such that $x \in \mathcal{G}_{\delta'}(u,u)$, then $\mathcal{G}_\delta(u,u) \subset \mathcal{G}_{\delta'}(u,u)$ by Lemma 5.1.22. We obtained that $\mathcal{G}_\delta(u,u)$ is the smallest $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$ containing x, which is the desired result. □

5.2.1 The support-preservingness property

We now introduce a key notion for studying the behavior of parabolic subgroups in weak Garside groups. It essentially states that standard parabolic closure behaves well in regards to conjugacy, and it will then allow us to construct parabolic closure of endomorphisms, starting from standard parabolic closure of endomorphisms which are recurrent for swap.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix a shoal $\mathcal{T}$ for $\mathcal{G}$.

Definition 5.2.8 (Support-preserving shoal). The shoal $\mathcal{T}$ is *support-preserving* if, for every pair of conjugate positive endomorphisms $x, y \in \mathcal{C}$, and every $\alpha \in \mathcal{G}$ such that $x^\alpha = y$, one has

$$\mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y).$$

This definition is directly inspired by [GM22, Definition 4.27]. In the case of a Garside group (G, M, Δ), the authors of *loc. cit.* define the notion of support of an element $x \in G$, which corresponds to the set of atoms of M lying in $\mathrm{SPC}(x)$, whence the terminology.

Note that, with the notation of Definition 5.2.8, we only require that conjugation by α induces an isomorphism between the groups $\mathrm{SPC}(x)$ and $\mathrm{SPC}(y)$. We make no assumptions regarding the underlying monoids and categories. See below an example where an isomorphism only happens at the level of parabolic subgroups.

Example 5.2.9. Consider the monoid $M = \langle s,t,u \mid sts = tst, tut = utu, su = us\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with $\Delta = stsuts$. The parabolic Garside elements of M are $\{1, s, t, u, su, sts, tut, \Delta\}$. In particular the standard parabolic closure of st in $G(M)$ is $\langle s, t\rangle = G(M)_{sts}$. The element $s \in M$ conjugates st to ts, however, it does not conjugate M_{sts} to itself since $t^s = s^{-1}ts \notin M_{sts}$. It only conjugates $G(M)_{sts}$ to itself.

Thanks to our definition of shoal, we can replace the conjugating element α in Definition 5.2.8 by a **positive** conjugating element, as in the following lemma:

Lemma 5.2.10. *The shoal $\mathcal{T}$ is support-preserving if and only if, for every pair of conjugate positive endomorphisms $x, y \in \mathcal{C}$, and every positive α such that $x^\alpha = y$, one has*

$$\mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y).$$

Proof. The only if statement is immediate, so let us assume that conjugation by positive elements preserves the standard parabolic closure. Let u (resp. v) be the source of x (resp. of y), and let

$\beta \in \mathcal{G}(u,v)$ be such that $x^\beta = y$. Since ϕ has finite order, we can consider $k > 0$ a big enough integer so that $\Delta^k(u) \in Z(\mathcal{G}(u,u))$ and $k + \inf(\beta) \geqslant 0$. We set $\beta' := \Delta^k(u)\beta$. We have

$$x^{\beta'} = x^{\Delta^k(u)\beta} = (\phi^k(x))^\beta = x^\beta = y.$$

Furthermore, $\beta' \in \mathcal{C}$ by construction, thus $\mathrm{SPC}(x)^{\beta'} = \mathrm{SPC}(y)$ by assumption. The result then comes from Lemma 5.2.6 since

$$\mathrm{SPC}(x)^{\beta'} = \mathrm{SPC}(x)^{\Delta^k(u)\beta} = (\phi^k(\mathrm{SPC}(x)))^\beta = \mathrm{SPC}(\phi^k(x))^\beta = \mathrm{SPC}(x)^\beta.$$

□

The support-preservingness condition is quite strong, and it is relatively easy to construct a shoal that is not support-preserving, as in the example below:

Example 5.2.11. Consider the monoid $M = \langle s,t,u \mid sts = tst, su = us, tut = utu\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. The set $\mathcal{T} := \{\{1\}, \langle s\rangle^+, \langle u\rangle^+, M\}$ is a shoal for M. Indeed the atoms are parabolic Garside elements as seen in Example 5.1.3, and $\phi(s) = u$, $\phi(u) = s$. The $\mathcal{T}$-standard parabolic closure of s is $\langle s\rangle$. However, we have $s^{ts} = t$, and the parabolic closure of t is $M \neq \langle s\rangle^\Delta$. Thus, the shoal $\mathcal{T}$ is not support-preserving.

Of course, this example is built with the sole purpose of not being support-preserving. Natural examples of shoals we will encounter later are support-preserving. It is not yet known whether or not the shoal of all parabolic subgroups of a Garside group is always support-preserving. However, we can already state a first strong consequence of the support-preservingness property in the case of a Garside group. The following proposition is directly inspired by the proofs of [DMM11, Proposition 2.2 and Corollary 2.5].

Proposition 5.2.12. *Let (G, M, Δ) be a Garside group, endowed with the shoal $\mathcal{T}$ of all its parabolic subgroups. Assume that, for a an atom, $a^2 \notin \mathcal{S}$, and that $\mathcal{T}$ is support-preserving. Then, if $U \subset G$ is a finite index subgroup, we have $Z(U) \subset Z(G)$.*

Proof. Let $z \in Z(U)$, and let a be an atom of M. Since U has finite index, we have $a^n \in U$ for some positive integer n. By assumption, we have $(a^n)^z = a^n$. Since $a^2 \notin \mathcal{S}$, $\langle a\rangle$ is a standard parabolic subgroup of G as in Example 5.1.3. We then have $\mathrm{SPC}(a^n) = \langle a\rangle$, and $\langle a\rangle^z = \langle a\rangle$ by support-preservingness. In particular, we have $a^z \in \langle a\rangle$ and $a^z = a^k$ for some integer $k \neq 0$. We have $(a^n)^z = a^{kn} = a^n$ and $a^{(k-1)n} = 1$. Since Garside groups are torsion-free (Corollary 2.3.14), we deduce that $k = 1$ and that $a^z = a$. Since the atoms of M generate G, we deduce that $z \in Z(G)$ as claimed. □

The authors of [DMM11] do not use the support-preservingness property. Rather they require several assumptions on the monoid M, and they essentially prove that the shoal made of subgroups of G generated by atoms of M is support-preserving, which is sufficient for the above proof to work.

Let us recall that the $\mathcal{T}$-parabolic closure of an endomorphism in $\mathcal{G}$ is the unique minimal (relative to inclusion) $\mathcal{T}$-parabolic subgroup which contains it. We aim to show that such a parabolic closure exists whenever the shoal $\mathcal{T}$ is support-preserving. We will follow the same pattern of proof as in [GM22, Section 4.6], which studies the shoal of all standard parabolic subgroups of a Garside group.

Our starting point will be the $\mathcal{T}$-standard parabolic closure of recurrent endomorphisms of $\mathcal{G}$. In order to show that the $\mathcal{T}$-standard parabolic closure of a recurrent endomorphism is actually a $\mathcal{T}$-parabolic closure, we first show the following proposition, which is a strengthening of Definition 5.2.8 (to which it is actually equivalent).

Proposition 5.2.13. *Assume that $\mathcal{T}$ is support-preserving. For every pair of conjugate recurrent endomorphisms $y, z \in \mathcal{G}$, and every $\alpha \in \mathcal{G}$ such that $x^\alpha = y$, one has*

$$\mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y).$$

Proof. The proof is similar to that of [GM22, Proposition 4.28]. First, assume that x is positive. In this case, y is also positive, as a recurrent conjugate of the positive endomorphism x (see Proposition 3.3.8). The result is then immediate by support-preservingness of $\mathcal{T}$. Likewise, if x is negative, then x^{-1} is positive, as well as its conjugate y^{-1}. We then have $\mathrm{SPC}(x^{-1})^\alpha = \mathrm{SPC}(y^{-1})$ by support-preservingness, and we have the result since $\mathrm{SPC}(z) = \mathrm{SPC}(z^{-1})$ for all endomorphisms z in $\mathcal{G}$.

Now, suppose that x is conjugate to neither a positive nor a negative element. For every $i \geqslant 0$, let $\mathrm{sw}^i(x) = a_i^{-1} b_i$ (resp. $\mathrm{sw}^i(y) = c_i^{-1} d_i$) be the reduced left-fraction decomposition of $\mathrm{sw}^i(x)$ (resp. of $\mathrm{sw}^i(y)$). We have $x^\alpha = y$. Let $u \in \mathrm{Ob}(\mathcal{G})$ be the source of x. Up to replacing α by $\Delta^k(u)\alpha$, where $\Delta^k(u) \in \mathcal{G}(u,u)$ is central, we can assume that α is positive. We have

$$(a_0\alpha)^{-1}(b_0\alpha) = \alpha^{-1}x\alpha = y = c_0^{-1}d_0.$$

Consider the swap functor (see Proposition 3.3.16) applied to the conjugating element $\alpha = \alpha_x$. We have

$$\mathrm{sw}(\alpha) = a_0\alpha \wedge b_0\alpha = a_0\alpha c_0^{-1} = b_0\alpha d_0^{-1}.$$

Notice that we have the commutative diagrams of conjugations

$$\begin{tikzcd} x \arrow[r, "a_0^{-1}"] \arrow[d, "\alpha"'] & \mathrm{sw}(x) \arrow[d, "\mathrm{sw}(\alpha)"] \\ y \arrow[r, "c_0^{-1}"'] & \mathrm{sw}(y) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "b_0^{-1}"] \arrow[d, "\alpha"'] & \mathrm{sw}(x) \arrow[d, "\mathrm{sw}(\alpha)"] \\ y \arrow[r, "d_0^{-1}"'] & \mathrm{sw}(y) \end{tikzcd}$$

Now, recall that both x and y are recurrent, so there exists some $k > 0$ such that $\mathrm{sw}^k(x) = x$, $\mathrm{sw}^k(y) = y$ and $\mathrm{sw}^k(\alpha_x) = \alpha_x$ by Proposition 3.3.17. We obtain the following commutative diagram of conjugations:

$$\begin{tikzcd} x \arrow[r, "a_0^{-1}"] \arrow[d, "\alpha"'] & \mathrm{sw}(x) \arrow[r, "a_1^{-1}"] \arrow[d, "\mathrm{sw}(\alpha)"] & \mathrm{sw}^2(x) \arrow[r, dotted] \arrow[d, "\mathrm{sw}^2(\alpha)"] & \mathrm{sw}^{k-1}(x) \arrow[r, "a_{k-1}^{-1}"] \arrow[d, "\mathrm{sw}^{k-1}(\alpha)"] & x \arrow[d, "\mathrm{sw}^k(\alpha)=\alpha"] \\ y \arrow[r, "c_0^{-1}"'] & \mathrm{sw}(y) \arrow[r, "c_1^{-1}"'] & \mathrm{sw}^2(y) \arrow[r, dotted] & \mathrm{sw}^{k-1}(y) \arrow[r, "c_{k-1}^{-1}"'] & y \end{tikzcd}$$

Simplifying the diagram, we have

$$\begin{tikzcd} x \arrow[r, "(a_{k-1}\cdots a_0)^{-1}"] \arrow[d, "\alpha"'] & x \arrow[d, "\alpha"] \\ y \arrow[r, "(c_{k-1}\cdots c_0)^{-1}"'] & y \end{tikzcd}$$

Let us denote $g_1 = a_{k-1} \cdots a_0$ and $h_1 = c_{k-1} \cdots c_0$. Both elements are positive, and we have $\alpha = g_1 \alpha h_1^{-1}$.

Now notice that we also have $\mathrm{sw}(\alpha) = b_0 \alpha d_0^{-1}$. Repeating the above arguments, if we define $g_2 = b_{k-1} \cdots b_0$ and $h_2 = d_{k-1} \cdots d_0$, we have $\alpha = \mathrm{sw}^k(\alpha) = g_2 \alpha h_2^{-1}$. Therefore $\alpha = g_1 g_2 \alpha h_2^{-1} h_1^{-1}$. That is

$$(g_1 g_2)^\alpha = h_1 h_2.$$

Since $g_1 g_2$ and $h_1 h_2$ are positive, step 2 gives that $\mathrm{SPC}(g_1 g_2)^\alpha = \mathrm{SPC}(h_1 h_2)$. The proof will then finish by showing that $\mathrm{SPC}(g_1 g_2) = \mathrm{SPC}(x)$ and that $\mathrm{SPC}(h_1 h_2) = \mathrm{SPC}(y)$. Recall that $g_1 g_2 = a_{k-1} \cdots a_0 b_{k-1} \cdots b_0$ where all factors in this expression are positive elements. Hence $a_0, b_0 \in \mathrm{SCPC}(g_1 g_2)$ and $\mathrm{SPC}(x) \subset \mathrm{SPC}(g_1 g_2)$. On the other hand, since $x \in \mathrm{SPC}(x)$, all elements $\{\mathrm{sw}^i(y)\}_{i \geqslant 0}$ belong to $\mathrm{SCPC}(x)$. Hence, all positive elements $a_{k-1}, \ldots, a_0, b_{k-1}, \ldots, b_0$ belong in $\mathrm{SCPC}(c)$. Therefore $\mathrm{SPC}(g_1 g_2) \subset \mathrm{SPC}(x)$, and hence $\mathrm{SPC}(g_1 g_2) = \mathrm{SPC}(x)$.

The same argument shows that $\mathrm{SPC}(h_1 h_2) = \mathrm{SPC}(y)$, and this finally implies that $\mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y)$, as we wanted to show. □

Using the above proposition, we show that the $\mathcal{T}$-parabolic closure of a recurrent element exists and is actually equal to its $\mathcal{T}$-standard parabolic closure.

Theorem 5.2.14 ($\mathcal{T}$-parabolic closure of recurrent endomorphisms). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $\mathcal{T}$ be a support-preserving shoal for $\mathcal{G}$. Let also $x \in \mathcal{G}$ be an endomorphism which is recurrent for swap. The group $\mathrm{PC}(x) := \mathrm{SPC}(x)$ is the $\mathcal{T}$-parabolic closure of x.*

Proof. The proof is similar to that of [GM22, Theorem 4.29]. Let $u \in \mathrm{Ob}(\mathcal{G})$ be the source of x, and let $H \subset \mathcal{G}(u,u)$ be a $\mathcal{T}$-parabolic subgroup containing x. We can assume that $H \neq \mathcal{G}(u,u)$, otherwise $\mathrm{SPC}(x) \subset H$ is immediate. By definition of a $\mathcal{T}$-parabolic subgroup, we can consider $f \in \mathcal{G}(u,v)$ such that $H^f = \mathcal{G}_\delta(v,v)$ is a $\mathcal{T}$-standard parabolic subgroup. By assumption, we have $x^f \in H^f = \mathcal{G}_\delta(v,v)$. By applying iterated swaps to x, we can consider a conjugating element $g \in \mathcal{G}_\delta(v,w)$ such that $(x^f)^g \in \mathcal{G}_\delta$ is a recurrent endomorphism. By definition of $\mathcal{T}$-standard parabolic closure, we have $\mathrm{SPC}(x^{fg}) \subset \mathcal{G}_\delta(w,w)$. Since both x and x^{fg} are recurrent, and since $\mathcal{T}$ is support-preserving, we have $\mathrm{SPC}(x)^{fg} = \mathrm{SPC}(x^{fg}) \subset \mathcal{G}_\delta(w,w)$. Therefore, we have

$$\mathrm{SPC}(x) = \mathrm{SPC}(x^{fg})^{(fg)^{-1}} \subset (\mathcal{G}_\delta(w,w))^{g^{-1}f^{-1}} = (\mathcal{G}_\delta(v,v))^{f^{-1}} = H.$$

This shows that $\mathrm{SPC}(x)$ is the smallest $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$ which contains x. □

An easy corollary of Theorem 5.2.14 is that if $\mathcal{T}$ is a support-preserving shoal, then the $\mathcal{T}$-standard parabolic subgroups in $\mathcal{G}$ are exactly the $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$ which are standard parabolic subgroups.

Corollary 5.2.15. *Assume that $\mathcal{T}$ is support-preserving, and let $u \in \mathrm{Ob}(\mathcal{G})$. A $\mathcal{T}$-parabolic subgroup $H \subset \mathcal{G}(u,u)$ is a $\mathcal{T}$-standard parabolic subgroup if and only if $H = \mathcal{G}_\delta(u,u)$ for some standard parabolic subgroupoid $\mathcal{G}_\delta$ (not necessarily in $\mathcal{T}$).*

Proof. First, if H is a $\mathcal{T}$-standard parabolic subgroup, then $H = \mathcal{G}_\delta(u,u)$ with $\mathcal{G}_\delta \in \mathcal{T}$. Conversely, let $\mathcal{G}_\delta$ be a standard parabolic subgroupoid of $\mathcal{G}$ such that $H = \mathcal{G}_\delta(u,u)$, and let $k > 0$ be the smallest integer such that $x := \delta^k(u) \in \mathcal{G}_\delta(u,u)$. We have $H = \mathrm{PC}(x)$ by Lemma 5.1.22, and, since x is positive, $\mathrm{PC}(x) = \mathrm{SPC}(x)$ by Theorem 5.2.14. Thus $H = \mathrm{SPC}(x)$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$. □

Thanks to this corollary, when considering a support-preserving shoal $\mathcal{T}$, will sometimes say standard parabolic subgroup instead of $\mathcal{T}$-standard parabolic subgroup when talking about $\mathcal{T}$-parabolic subgroups.

Now that we know that $\mathcal{T}$-parabolic closures of recurrent elements exist, we can easily deduce the existence of $\mathcal{T}$-parabolic subgroups of arbitrary endomorphisms using the following lemma:

Lemma 5.2.16. *Let $x \in \mathcal{G}$ be an endomorphism, and let $c \in \mathcal{G}$ share the same source as x. If x admits a $\mathcal{T}$-parabolic closure* $\mathrm{PC}(x)$*, then* $\mathrm{PC}(x)^c$ *is the $\mathcal{T}$-parabolic closure of x^c.*

Proof. Let v be the target of v, and let $H \subset \mathcal{G}(v,v)$ be a $\mathcal{T}$-parabolic subgroup containing x^c. By definition, $H^{c^{-1}} \subset \mathcal{G}(u,u)$ is a $\mathcal{T}$-parabolic subgroup containing x. Thus, we have $\mathrm{PC}(x) \subset H^{c^{-1}}$ and $\mathrm{PC}(x)^c \subset H$. Since $\mathrm{PC}(x)^c$ is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(v,v)$ which contains x^c, we have that it is the $\mathcal{T}$-parabolic closure of x. □

Theorem 5.2.17 (Existence of $\mathcal{T}$-parabolic closure). *Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, and let $\mathcal{T}$ be a support-preserving shoal for $\mathcal{G}$. Every endomorphism x in $\mathcal{G}$ admits a $\mathcal{T}$-parabolic closure* $\mathrm{PC}(x)$.

Proof. Let $x \in \mathcal{G}$ be an endomorphism. Applying iterated swaps, we can conjugate x to a recurrent endomorphism x^c. By Theorem 5.2.14, $\mathrm{SPC}(x^c)$ is the $\mathcal{T}$-parabolic closure of x^c. By Lemma 5.2.16, $\mathrm{SPC}(x^c)^{c^{-1}}$ is the $\mathcal{T}$-parabolic closure of x. □

Proposition 5.2.18 ($\mathcal{T}$-parabolic closure of a power). *Assume that $\mathcal{T}$ is support-preserving. If $x \in \mathcal{G}$ is an endomorphism, and if m is a nonzero integer, then we have* $\mathrm{PC}(x) = \mathrm{PC}(x^m)$.

Proof. Since $\mathrm{PC}(x)$ is a group, we always have $x^m \in \mathrm{PC}(x)$ and $\mathrm{PC}(x^m) \subset \mathrm{PC}(x)$. Thus we only have to show that $x \in \mathrm{PC}(x^m)$. In the case $m = -1$, this is obvious and we have $\mathrm{PC}(x) = \mathrm{PC}(x^{-1})$. In particular, it is sufficient to prove the result when m is positive.

Assume at first that x is positive, and let $\mathcal{G}_\delta := \mathrm{SCPC}(x^m)$. We have $x \preccurlyeq x^m$ in $\mathcal{C}$ and $x \in \mathcal{C}_\delta$ by Proposition 5.1.10, thus $x \in \mathrm{PC}(x^m)$. Then, if x is negative, x^{-1} is positive, and we have

$$\mathrm{PC}(x^m) = \mathrm{PC}((x^m)^{-1}) = \mathrm{PC}((x^{-1})^m) = \mathrm{PC}(x^{-1}) = \mathrm{PC}(x).$$

Now, assume that x is a recurrent endomorphism which is neither positive nor negative. For $k \geqslant 0$, we write $\mathrm{sw}^k(x) = a_k^{-1} b_k$ the reduced left-fraction decomposition of $\mathrm{sw}^k(x)$. By Theorem 3.3.10 (reduced left-fraction decomposition of powers), the reduced left-fraction decomposition of x^m is given by

$$x^m = (a_0 \cdots a_{m-1})^{-1} b_0 \cdots b_{m-1}.$$

By Proposition 5.1.14 (compatibility of standard parabolic subgroupoids), we have $a_0 \cdots a_{m-1} \in \mathrm{SCPC}(x^m)$ and $b_0 \cdots b_{m-1} \in \mathrm{SCPC}(x^m)$. By Proposition 5.1.10, we have $a_0, b_0 \in \mathrm{SCPC}(x^m)$ and $a_0^{-1} b_0 = x \in \mathrm{SCPC}(x^m)$.

Lastly, assume that x is an arbitrary endomorphism. By applying iterated swaps, we obtain an element $c \in \mathcal{G}$ such that x^c is recurrent. We have that $(x^m)^c = (x^c)^m$ is a recurrent endomorphism by Corollary 3.3.11 (swap and powers). By the previous points, we have

$$\mathrm{PC}(x) = \mathrm{PC}(x^c)^{c^{-1}} = \mathrm{PC}((x^c)^m)^{c^{-1}} = \mathrm{PC}(x^m),$$

which terminates the proof. □

Corollary 5.2.19. *Assume that $\mathcal{T}$ is support-preserving. Let $x \in \mathcal{G}$ be an endomorphism, and let H be a $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$ which contains x. If $y \in \mathcal{G}$ is such that $y^m = x$ for some nonzero integer m, then we have $y \in H$.*

Proof. Since $x \in H$, we have $\mathrm{PC}(x) \subset H$. By Proposition 5.2.18, we have $\mathrm{PC}(y) = \mathrm{PC}(x)$, and thus $y \in \mathrm{PC}(y) \subset H$. □

5.2.2 Minimal positive conjugators

So far, we have investigated the consequences of support-preservingness, but we did not detail how to actually check whether or not a given shoal is support-preserving. As the definition of support-preservingness hinges on conjugate positive endomorphisms, we are interested in describing the conjugacy of positive endomorphism.

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix a shoal $\mathcal{T}$ for $\mathcal{G}$.

Let $x \in \mathcal{C}$ be a positive endomorphism. By Proposition 3.3.8, the set $\Gamma := \mathsf{R}(x)$ of recurrent conjugates of x in $\mathcal{G}$ is simply the set of positive conjugates of x. Recall from Section 4.3 that the *conjugacy graph* $\mathcal{S}_\Gamma$ associated to x is defined as follows

- The object set of $\mathcal{S}_\Gamma$ is the set Γ of positive conjugates of x.
- For $y, z \in \Gamma$, $\mathcal{S}_\Gamma(y, z) = \{s \in \mathcal{S} \mid y^s = z\}$.

This oriented graph is endowed with a germ structure by Lemma 4.3.5. However, since Γ is not finite a priori, we do not know whether or not the associated groupoid $\mathcal{G}_\Gamma$ is a Garside groupoid. Nonetheless, Corollary 4.3.7 (conjugacy graph and conjugacy category) allows us to identify $\mathcal{G}_\Gamma$ with a full subgroupoid of $\mathrm{Conj}(\mathcal{G})$. As such, we obtain that $\mathcal{G}_\Gamma$ is generated by the atoms of $\mathcal{C}_\Gamma$. The atoms of $\mathcal{S}_\Gamma$ starting from an endomorphism y are called the *minimal positive conjugators* for y. In other words, a minimal positive conjugator for some endomorphism $x \in \mathcal{C}(u, u)$ is a nontrivial morphism $\rho \in \mathcal{C}(u, -)$ such that $x^\rho \in \mathcal{C}$ and such that $x^h \notin \mathcal{C}$ when $1_u \prec h \prec \rho$.

An important property of minimal positive conjugators is that we only need to show that they preserve the $\mathcal{T}$-standard parabolic closure to show that a shoal $\mathcal{T}$ is support-preserving.

Proposition 5.2.20. *The shoal $\mathcal{T}$ is support-preserving if and only if for every positive endomorphism $x \in \mathcal{C}$, and every minimal positive conjugator ρ for x, one has*

$$\mathrm{SPC}(x)^\rho = \mathrm{SPC}(x^\rho).$$

Proof. The only if statement is immediate, so let us assume that conjugation by minimal positive conjugators preserves the $\mathcal{T}$-standard parabolic closure. Let $x \in \mathcal{C}(u, u), y \in \mathcal{C}(v, v)$ be conjugate positive endomorphisms, and let $\alpha \in \mathcal{G}(u, v)$ be such that $x^\alpha = y$. By Lemma 5.2.10, it is sufficient to consider the case where α is positive.

Let Γ be the set of positive conjugates of x. By definition, α can be seen as an element of the category $\mathcal{C}_\Gamma$. By the discussion above, α can then be decomposed as a product $\alpha_1 \cdots \alpha_r$ of minimal positive conjugators (i.e. of atoms of $\mathcal{S}_\Gamma$). We then have

$$\mathrm{SPC}(x)^\alpha = \mathrm{SPC}(x)^{\alpha_1 \cdots \alpha_r} = \mathrm{SPC}(x^{\alpha_1})^{\alpha_2 \cdots \alpha_r} = \ldots = \mathrm{SPC}(x^{\alpha_1 \cdots \alpha_r}) = \mathrm{SPC}(x^\alpha),$$

which terminates the proof. □

In practice, we will want to apply this proposition to prove that a given shoal is support-preserving. Thus we will be interested in computing minimal positive conjugators of positive endomorphisms. This is a rather classical procedure, explained in [GM22] and [FG03b] in the case of a Garside monoid.

We fix $x \in \mathcal{C}(u,u)$ a positive endomorphism. Recall from Corollary 4.3.2 that, if $s,t \in \mathcal{C}(u,-)$ are such that $x^s, x^t \in \mathcal{C}$, then $x^{s\wedge t} \in \mathcal{C}$. In other words the set of positive conjugators of x is stable under left-gcd. For any $f \in \mathcal{C}(u,-)$, the set

$$\{\alpha \in \mathcal{C}(u,-) \mid f \preccurlyeq \alpha \text{ and } x^\alpha \in \mathcal{C}\}$$

is nonempty (it contains $\Delta^k(u)$ where $k = \sup(f)$), and it is closed under left-gcd. By Noetherianity, this set contains a unique $\preccurlyeq$-minimal element, that we denote $\rho_f(x)$.

We clearly have $f \preccurlyeq g \Rightarrow \rho_f(x) \preccurlyeq \rho_g(x)$. Thus a minimal positive conjugator for x can always be written $\rho_a(x)$, where $a \in \mathcal{C}(u,-)$ is an atom. However, a positive conjugator of the form $\rho_a(x)$ may not be a minimal positive conjugator, as it may be left-divided by some other $\rho_b(x)$ where $b \in \mathcal{C}(u,-)$ is another atom.

Let us try and compute $\rho_f(x)$ starting from x and f. The element $c := \rho_f(x)$ is such that $f \preccurlyeq c \preccurlyeq xc$. As we always have $f \preccurlyeq c$ and thus $xf \preccurlyeq xc$, we deduce that $f \vee xf \preccurlyeq xc$. Since $f \vee xf = f \vee x \vee xf$, multiplying by x^{-1} yields

$$x^{-1}(f \vee x \vee xf) = x^{-1}(f \vee x) \vee f = (x\backslash f) \vee f \preccurlyeq c.$$

We set $c_1 := x\backslash f \vee f$. We have $xf \preccurlyeq xc_1 \preccurlyeq xc$ and thus $f \preccurlyeq c_1 \preccurlyeq c$. We can apply the same reasoning to c_1, computing $c_2 := x\backslash c_1 \vee c_1$ and obtaining $f \preccurlyeq c_1 \preccurlyeq c_2 \preccurlyeq c$ and so on. We obtain a sequence

$$f = c_0 \preccurlyeq c_1 \preccurlyeq \cdots \preccurlyeq c$$

of positive left-divisors of $c = \rho_f(x)$, where $c_{i+1} = x\backslash c_i \vee c_i$ for every $i \geqslant 0$. They are called the *converging prefixes* of $\rho_f(x)$. The following lemma shows that one can compute $\rho_f(x)$ by computing converging prefixes until the sequence stabilizes.

Lemma 5.2.21. *Let $u \in \mathrm{Ob}(\mathcal{G})$, $x \in \mathcal{C}(u,u)$, and let $f \in \mathcal{C}(u,-)$. Let also $(c_i)_{i\geqslant 0}$ be the sequence of converging prefixes of $\rho_f(x)$. For $m \geqslant 0$, we have*

$$c_m = c_{m+1} \Leftrightarrow c_m = \rho_f(x) \Leftrightarrow \forall n \geqslant m, c_n = c_m.$$

Furthermore, the sequence $(c_i)_{i\geqslant 0}$ stabilizes, i.e. the above conditions are realized for some integer $m \geqslant 0$.

Proof. By construction, the sequence $(c_i)_{i\geqslant 0}$ is an increasing sequence in the finite poset of left-divisors of $\rho_f(x)$, and thus it stabilizes. Let now $m \geqslant 0$, we have

$$\begin{aligned} c_m = c_{m+1} &\Leftrightarrow c_m = x\backslash c_m \vee c_m \\ &\Leftrightarrow x\backslash c_m \preccurlyeq c_m \\ &\Leftrightarrow x \vee c_m \preccurlyeq xc_m \\ &\Leftrightarrow c_m \preccurlyeq xc_m. \end{aligned}$$

Thus $c_m = c_{m+1}$ if and only if c_m is a positive conjugator of x.

Now, if $c_m = \rho_f(x)$, then c_m is by definition a positive conjugator of x. Conversely, if c_m is a positive conjugator of x, then $f \preccurlyeq c_m$ implies $\rho_f(x) \preccurlyeq c_m$ and $c_m = \rho_f(x)$ since $c_m \preccurlyeq \rho_f(x)$ always holds.

Lastly, if $c_m = \rho_f(x)$, then for all $n \geqslant m$, we have $\rho_f(x) \preccurlyeq c_m \preccurlyeq c_n \preccurlyeq \rho_f(x)$, thus $c_n = c_m$. Conversely, if $c_n = c_m$ for all $n \geqslant m$, then $c_m = c_{m+1}$ and $c_m = \rho_f(x)$. $\square$

Now, assume that $f = a$ is a simple morphism (for instance an atom). Since $\rho_a(x) \preccurlyeq \Delta(u)$, all the converging prefixes are simple morphisms, we can in particular compute them by only computing lcms of simple morphisms. Let us write $x = x_1 \cdots x_r$ as a product of simple morphisms of $\mathcal{C}$ (not necessarily the greedy normal form). We consider the following diagram, where each of the squares is a right-lcm.

$$\begin{tikzcd}
\cdot \arrow[r, "x_1"] \arrow[d, "c_j := s_{j,0}"'] & \cdot \arrow[r, "x_2"] \arrow[d, "s_{j,1}"'] & \cdot \arrow[r, dotted] \arrow[d, "s_{j,2}"] & \cdot \arrow[r, "x_r"] \arrow[d, "s_{j,r-1}"] & \cdot \arrow[d, "s_{j,r}"'] \\
\cdot \arrow[r, "y_1"'] & \cdot \arrow[r, "y_2"'] & \cdot \arrow[r, dotted] & \cdot \arrow[r, "y_r"'] & \cdot
\end{tikzcd}$$

Since concatenating right-lcm squares still gives a right-lcm (Lemma 2.1.31), we deduce that $x \backslash c_j = s_{j,r}$. The elements $s_{j,i} = (x_1 \cdots x_i) \backslash c_j$ are called the *pre-minimal conjugators*. Notice that they depend on the decomposition chosen for x. Notice also that the pre-minimal conjugators do not necessarily divide $\rho_a(x)$, while the converging prefixes do.

5.2.3 Interlude: proof of support-preservingness on an example

In the last section, we detailed a general strategy for proving that a given shoal for a Garside groupoid is support-preserving. In this short section, we detail an example of how to use these techniques to actually prove that a shoal is support-preserving.

Consider the monoid $M := \langle s, t, u \mid ustus = stust, sustu = tusts, tust = ustu \rangle^+$. By [Pic00, Example 13], it is a Garside monoid with Garside element $\Delta = (stu)^3$. There are 90 simple elements in M, among which the only balanced elements are $\{1, s, t, u, t^2, u^2, \Delta\}$. One easily deduce that the parabolic Garside elements are $\{1, s, t^2, u^2, \Delta\}$. The shoal of all standard parabolic subgroups of $G = G(M)$ is given by

$$\mathcal{T} := \{1, \langle s \rangle, \langle t \rangle, \langle u \rangle, G\}\,.$$

Let $x \in M$ be a positive element, we have either

- $x = 1$, in which case $\mathrm{SPC}(x) = \{1\}$.
- $x = s^n$ for some $n \geqslant 1$, in which case $\mathrm{SPC}(x) = \langle s \rangle$.
- $x = t^n$ for some $n \geqslant 1$, in which case $\mathrm{SPC}(x) = \langle t \rangle$.
- $x = u^n$ for some $n \geqslant 1$, in which case $\mathrm{SPC}(x) = \langle u \rangle$.
- None of the above, in which case $\mathrm{SPC}(x) = G$.

In order to check the support-preservingness of $\mathcal{T}$, we compute the minimal positive conjugators of powers of the atoms, as in Section 5.2.2.

<u>Minimal positive conjugators for powers of s :</u> Let $n \geqslant 1$ be an integer, we obviously have $\rho_s(s^n) = s$ is a minimal positive conjugator of s^n. In order to compute $\rho_t(s^n)$ and $\rho_u(s^n)$, we will make use of the following diagrams:

$$\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "t"'] & \cdot \arrow[r, "s"] \arrow[d, "tust"] & \cdot \arrow[r, "s"] \arrow[d, "tust"] & \cdot \arrow[r, dotted] \arrow[d, "tust"] & {} \\
\cdot \arrow[r, "usts"'] & \cdot \arrow[r, "s"'] & \cdot \arrow[r, "s"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "s"] \arrow[d, "u"'] & \cdot \arrow[r, "s"] \arrow[d, "tust"] & \cdot \arrow[r, "s"] \arrow[d, "tust"] & \cdot \arrow[r, dotted] \arrow[d, "tust"] & {} \\
\cdot \arrow[r, "stus"'] & \cdot \arrow[r, "s"'] & \cdot \arrow[r, "s"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}$$

where all the squares are right-lcms. In order to compute $\rho_t(s^n)$, we compute its sequence of converging prefixes. We set $c_0 := t$, and

$$c_1 := s^n\backslash t \vee t = tust \vee t = tust.$$

Since c_1 commutes with s, it is a positive conjugator of s^n, and we have $\rho_t(s^n) = c_1 = tust$. Similarly, we compute the converging prefixes of $\rho_u(s^n)$. We set $c_0 := u$, and

$$c_1 := s^n\backslash u \vee u = tust \vee u = tust = ustu = \rho_u(s^n).$$

We obtain that the minimal positive conjugators of s^n are $s, tust$, which both commute with s.

<u>Minimal positive conjugators for powers of t :</u> Let $n \geqslant 1$ be an integer, we obviously have $\rho_t(t^n) = t$ is a minimal positive conjugator of t^n. In order to compute $\rho_s(t^n)$ and $\rho_u(t^n)$, we will make use of the following diagrams:

$$\begin{tikzcd}
\cdot \arrow[r, "t"] \arrow[d, "s"'] & \cdot \arrow[r, "t"] \arrow[d, "usts"] & \cdot \arrow[r, "t"] \arrow[d, "uststus"] & \cdot \arrow[r, "t"] \arrow[d, "uststus"] & \cdot \arrow[r, dotted] \arrow[d, "uststus"] & {} \\
\cdot \arrow[r, "tust"'] & \cdot \arrow[r, "tust"'] & \cdot \arrow[r, "t"'] & \cdot \arrow[r, "t"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "t"] \arrow[d, "u"'] & \cdot \arrow[r, "t"] \arrow[d, "ust"] & \cdot \arrow[r, "t"] \arrow[d, "ust"] & \cdot \arrow[r, dotted] \arrow[d, "ust"] & {} \\
\cdot \arrow[r, "stu"'] & \cdot \arrow[r, "u"'] & \cdot \arrow[r, "u"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}$$

where all the squares are right-lcms. In order to compute $\rho_u(t^n)$, we compute its sequence of converging prefixes. We set $c_0 := u$ and

$$c_1 := t^n\backslash u \vee u = ust \vee u = ust.$$

Since $t^{ust} = u$, we have that c_1 is a positive conjugator for t^n. Thus we have $c_1 = ust = \rho_u(t^n)$ with $(t^n)^{ust} = u^n$. Similarly, we compute the converging prefixes of $\rho_s(t^n)$. We set $c_0 := s$, and we either have $c_1 = usts \vee s$ if $n = 1$, and $c_1 = uststus \vee s$ if $n > 1$. In either case, we have $ust = \rho_u(t^n) \preccurlyeq c_1 \preccurlyeq \rho_s(t^n)$, thus $\rho_s(t^n)$ is either equal to $\rho_u(t^n)$ or not a minimal positive conjugator.

We obtain that the minimal positive conjugators of t^n are t, ust, with $(t^n)^t = t$ and $(t^n)^{ust} = u^n$.

<u>Minimal positive conjugators for powers of u :</u> Let $n \geqslant 1$ be an integer, we obviously have $\rho_u(u^n) = u$ is a minimal positive conjugator of u^n. In order to compute $\rho_s(u^n)$ and $\rho_t(u^n)$, we will make use of the following diagrams:

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "s"'] & \cdot \arrow[r, "u"] \arrow[d, "stus"] & \cdot \arrow[r, "u"] \arrow[d, "stus"] & \cdot \arrow[r, dotted] \arrow[d, "stus"] & {} \\
\cdot \arrow[r, "tust"'] & \cdot \arrow[r, "t"'] & \cdot \arrow[r, "t"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "t"'] & \cdot \arrow[r, "u"] \arrow[d, "stu"] & \cdot \arrow[r, "u"] \arrow[d, "stus"] & \cdot \arrow[r, dotted] \arrow[d, "stus"] & {} \\
\cdot \arrow[r, "ust"'] & \cdot \arrow[r, "st"'] & \cdot \arrow[r, "t"'] & \cdot \arrow[r, dotted] & {}
\end{tikzcd}$$

where all the squares are right-lcms. In order to compute $\rho_s(u^n)$, we compute its sequence of converging prefixes. We set $c_0 := s$ and

$$c_1 := u^n \backslash s \vee s = stus \vee s = stus.$$

Since $u^{stus} = t$, we have that c_1 is a positive conjugator for u^n. Thus we have $c_1 = stus = \rho_s(u^n)$ with $(u^n)^{stus} = t^n$. Similarly, we compute the converging prefixes of $\rho_t(u^n)$. We set $c_0 := t$, and we either have $c_1 = stu \vee t = stust$ if $n = 1$, and $c_1 = stus \vee t = stust$ if $n > 1$. In either case, we have $stus = \rho_s(u^n) \preccurlyeq c_1 \preccurlyeq \rho_t(u^n)$, thus $\rho_t(u^n)$ is either equal to $\rho_s(u^n)$ or not a minimal positive conjugator.

We obtain that the minimal positive conjugators of u^n are $u, stus$, with $(u^n)^u = u$ and $(u^n)^{stus} = t^n$.

The various computations of minimal positive conjugators of powers of atoms allow us to compute the conjugacy graphs of powers of atoms, as in the following proposition:

Proposition 5.2.22. *Let $n \geqslant 1$ be an integer.*

(a) We have $\mathsf{R}(s^n) = \{s^n\}$, and the atoms of the associated conjugacy graph are given by

$$\begin{tikzcd} s^n \arrow[loop left, "s"] \arrow[loop right, "tust"] \end{tikzcd}$$

(b) We have $\mathsf{R}(t^n) = \mathsf{R}(u^n) = \{t^n, u^n\}$, and the atoms of the associated conjugacy graph are given by

$$\begin{tikzcd} t^n \arrow[loop left, "t"] \arrow[r, bend left, "ust"] & u^n \arrow[l, bend left, "stus"] \arrow[loop right, "u"] \end{tikzcd}$$

Corollary 5.2.23. *An element $x \in M$ is conjugate in G to a power of an atom of M if and only if it is itself a power of an atom.*

Let now $x \in M$, and let ρ be a minimal positive conjugator for x, with $y := x^\rho$. If x is not a power of an atom of M, then neither is y by Corollary 5.2.23. In this case we have

$$\mathrm{SPC}(y) = G = G^\rho = \mathrm{SPC}(x)^\rho.$$

Conversely, if x is a power of an atom, say a^n, then $y = b^n$ is also a power of an atom by Proposition 5.2.22, and we have $a^\rho = b$. In this case, we have

$$\mathrm{SPC}(y) = \langle b \rangle = \langle a \rangle^\rho = \mathrm{SPC}(x)^\rho.$$

Since conjugation by ρ preserves the standard parabolic closure in both case, we obtain that (G, M, Δ) is support-preserving.

Since G is homogeneous (the defining relations of M are homogeneous), applying Theorem 5.2.33 (intersection of parabolic subgroups) below gives us the following result:

Proposition 5.2.24. *The parabolic subgroups of the Garside group (G, M, Δ) are stable under intersection.*

5.2.4 Systems of conjugacy representatives

In this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism ϕ. We also fix a shoal $\mathcal{T}$ for $\mathcal{G}$.

In [GM22, Section 6.1], the authors show that, in a homogeneous Garside group, the conjugacy of parabolic subgroups is somehow characterized by their center. More precisely, they define, for a parabolic subgroup H, a distinguished element $z_H \in Z(H)$ such that H is conjugate to H' if and only if z_H and $z_{H'}$ are conjugate.

Their strategy is to first define z_H when H is a standard parabolic subgroup, and then to extend the definition to arbitrary parabolic subgroups by conjugacy. It is hard to mimic their arguments in the categorical case, even under homogeneity assumptions. For instance the integer e of [GM22, Definition 6.1] may now depend on the choice of an object, as in the following example:

Example 5.2.25. Consider the following oriented graph $\mathcal{S}$:

$$\begin{tikzcd} u \arrow[r, bend right, "b"'] & v \arrow[l, bend right, "a"'] \arrow[r, bend right, "c"'] & w \arrow[l, bend right, "d"'] \end{tikzcd}$$

endowed with the relations $ab = cd$. The category $\mathcal{C}$ presented by this data is a Garside category with $\Delta(u) = bc$, $\Delta(v) = ab$ and $\Delta(w) = da$. Let $\mathcal{G} := \mathcal{G}(\mathcal{C})$ be the enveloping groupoid of $\mathcal{C}$. By taking $\{a, c\}$ as a Schreier transversal rooted in v, we see that $\mathcal{G}(v, v)$ is generated by $ab = cd = \Delta(v)$. Thus $\Delta(v) = \Delta^1(v)$ is the smallest power of $\Delta(v)$ which lies in $Z(\mathcal{G}(v, v))$. However, the Garside automorphism of $\mathcal{C}$ has order 2.

To palliate this problem, we define abstractly what properties the elements z_H should have in Definition 5.2.26, and we deduce similar properties as in [GM22, Section 6]. Later, in Section 5.3, we will show that, if $(\mathcal{G}, \mathcal{C}, \Delta)$ is endowed with a support-preserving shoal $\mathcal{T}$ and a corresponding system of elements z_H, then applying one of the constructions of Chapter 4 to $\mathcal{G}$ yields a Garside groupoid naturally endowed with a shoal related to $\mathcal{T}$, and we will construct a corresponding system of elements z_H.

Definition 5.2.26 (System of conjugacy representatives). A *system of conjugacy representatives* for $\mathcal{T}$ is the data, for every $\mathcal{G}_\delta \in \mathcal{T}$, of a natural transformation $z_\delta : 1_{\mathcal{C}_\delta} \Rightarrow 1_{\mathcal{C}_\delta}$ such that

- For $\mathcal{G}_\delta(u, u)$ and $\mathcal{G}_{\delta'}(v, v)$ two $\mathcal{T}$-standard parabolic subgroups in $\mathcal{G}$, and $f \in \mathcal{G}(u, v)$, we have
$$(\mathcal{G}_\delta(u, u))^f = \mathcal{G}_{\delta'}(v, v) \Leftrightarrow (z_\delta(u))^f = z_{\delta'}(v).$$
- For $\mathcal{G}_\delta(u, u)$ a $\mathcal{T}$-standard parabolic subgroup in $\mathcal{G}$, we have $\mathrm{SPC}(z_\delta(u)) = \mathcal{G}_\delta(u, u)$.

Let $\mathcal{G}_\delta(u, u)$ be a $\mathcal{T}$-standard parabolic subgroup. By definition, $z_\delta(u)$ is a positive endomorphism of u, which belongs to $Z(\mathcal{G}_\delta(u, u))$. Furthermore, for $f \in \mathcal{G}_\delta(u, v)$, we have $(z_\delta(u))^f = z_\delta(v)$ by definition of a natural transformation $1_{\mathcal{C}_\delta} \Rightarrow 1_{\mathcal{C}_\delta}$.

As explained above, the definition of [GM22, Section 6] can be reformulated by stating that the element z_H introduced there form a system of conjugacy representatives for the shoal $\mathcal{T}$ of all parabolic subgroup of a homogeneous Garside group.

Proposition 5.2.27. *[GM22, Proposition 6.2] Let (G, M, Δ) be a homogeneous Garside group, endowed with the shoal $\mathcal{T}$ of all its standard parabolic subgroups. For $G_\delta \in \mathcal{T}$, we define $z_\delta := \delta^k$,* *where k is the smallest positive integer such that δ^k is central in G_δ. The set $\{z_\delta \mid G_\delta \in \mathcal{T}\}$ is a system of conjugacy representatives for $\mathcal{T}$.*

From now on, we fix a system of conjugacy representatives $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ for $\mathcal{T}$. We will show similar results as the ones in [GM22, Section 6], with similar proofs. First, we define elements z_H for arbitrary $\mathcal{T}$-parabolic subgroups.

Definition 5.2.28. Let H be a $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$. Let $f \in \mathcal{G}$ with source $v \in \mathrm{Ob}(\mathcal{G})$ be such that $H^f = \mathcal{G}_\delta(v,v)$ is a $\mathcal{T}$-standard parabolic subgroup. We define $z_H := (z_\delta(v))^{f^{-1}}$.

In particular, we have $z_{\mathcal{G}_\delta(u,u)} = z_\delta(u)$ by definition. We first show that the definition of z_H does not depend on the choice of $\mathcal{G}_\delta(v,v)$ in the above definition.

Lemma 5.2.29. *Let $u \in \mathrm{Ob}(\mathcal{G})$, and let $H \subset \mathcal{G}(u,u)$ be a $\mathcal{T}$-parabolic subgroup. The definition of z_H does not depend on the choice of a $\mathcal{T}$-standard parabolic conjugate of H.*

Proof. Let $v, v' \in \mathrm{Ob}(\mathcal{G})$, and let $f \in \mathcal{G}(u,v)$, $f' \in \mathcal{G}(u,v')$ be such that $H^f = \mathcal{G}_\delta(v,v)$ and $H^{f'} = \mathcal{G}_{\delta'}(v',v')$. We have $(\mathcal{G}_\delta(v,v))^{f^{-1}f'} = \mathcal{G}_{\delta'}(v',v')$, and thus $(z_\delta(v))^{f^{-1}f'} = z_{\delta'}(v')$ by definition of a system of conjugacy representatives. We deduce that $(z_\delta(u))^{f^{-1}} = (z_{\delta'}(v'))^{f'^{-1}}$ which shows the result. □

In order to show that the elements z_H characterize the conjugacy of $\mathcal{T}$-parabolic subgroups, we need to assume that the shoal $\mathcal{T}$ is support-preserving. Under this assumption, we easily show that any $\mathcal{T}$-parabolic subgroup H is the parabolic closure of the associated element z_H.

Proposition 5.2.30. *Assume that $\mathcal{T}$ is support-preserving, and let H be a $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$. We have $\mathrm{PC}(z_H) = H$.*

Proof. First, if $H = \mathcal{G}_\delta(u,u)$ is $\mathcal{T}$-standard, then $H = \mathrm{SPC}(z_H)$ by definition. Furthermore, as $z_\delta(u)$ is positive by definition, we have $\mathrm{SPC}(z_H) = \mathrm{PC}(z_H)$ by Theorem 5.2.14 ($\mathcal{T}$-parabolic closure of recurrent endomorphisms). Now, let $f \in \mathcal{G}(u,v)$ be such that $H^f = \mathcal{G}_\delta(v,v)$ is $\mathcal{T}$-standard. We have

$$\mathrm{PC}(z_H) = \mathrm{PC}\left((z_\delta(v))^{f^{-1}}\right) = \mathrm{PC}(z_\delta(v))^{f^{-1}} = (\mathcal{G}_\delta(v,v))^{f^{-1}} = H.$$

□

Proposition 5.2.31 (Characterization of conjugacy of parabolic subgroups). *Assume that $\mathcal{T}$ is support-preserving. Let H_1, H_2 be two $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$. For $f \in \mathcal{G}$, we have $(H_1)^f = H_2$ if and only if $(z_{H_1})^f = z_{H_2}$.*

Proof. Let $u_1, u_2 \in \mathrm{Ob}(\mathcal{G})$ be such that $H_i \subset \mathcal{G}(u_i,u_i)$ for $i = 1,2$, and let $f_i \in \mathcal{G}(u_i,v_i)$ such that $(H_i)^{f_i} = \mathcal{G}_{\delta_i}(v_i,v_i)$ for $i = 1,2$. First, assume that $(H_1)^f = H_2$. We have

$$(\mathcal{G}_{\delta_1}(v,v))^{f_1^{-1}ff_2} = (H_1)^{ff_2} = (H_2)^{f_2} = \mathcal{G}_{\delta_2}(v_2,v_2),$$

and thus $(z_{\delta_1}(v_1))^{f_1^{-1}ff_2} = z_{\delta_2}(v_2)$ by definition. We then have

$$(z_{H_1})^f = (z_{\delta_1}(v_1))^{f_1^{-1}f} = (z_{\delta_2}(v_2))^{f_2^{-1}} = z_{H_2}.$$

Conversely, assume that $(z_{H_1})^f = z_{H_2}$. By Proposition 5.2.30, we then have

$$(H_1)^f = \mathrm{PC}(z_{H_1})^f = \mathrm{PC}(z_{H_2}) = H_2.$$

□

Proposition 5.2.32. *Assume that $\mathcal{T}$ is support-preserving, and let H be a $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$. The parabolic subgroup H is standard if and only if z_H is positive.*

Proof. First, if $H = \mathcal{G}_\delta(u,u)$ is standard, we have that $z_H = z_\delta(u)$ is positive. Conversely, assume that z_H is positive. We have $H = \mathrm{PC}(z_H) = \mathrm{SPC}(z_H)$ by Theorem 5.2.14 ($\mathcal{T}$-parabolic closure of recurrent endomorphisms). In particular, H is a standard parabolic subgroup in $\mathcal{G}$. □

In [GM22], the elements z_H are introduced not only to study the conjugacy of parabolic subgroups, but also to show that the intersection of two parabolic subgroups is again a parabolic subgroup, under suitable assumptions. We will not directly show that, under suitable assumptions on $\mathcal{G}$ and $\mathcal{T}$, the intersection of $\mathcal{T}$-parabolic subgroups is again $\mathcal{T}$-parabolic. Rather, we will deduce in Section 5.3 that the property "$\mathcal{T}$-parabolic subgroups are stable under intersection" remains true under most of the constructions of Chapter 4. Our starting point is then that the intersection of parabolic subgroups in a support-preserving homogeneous Garside group is again parabolic.

Theorem 5.2.33 (Intersection of parabolic subgroups). *Let (G, M, Δ) be a homogeneous Garside group, such that the shoal of all its standard parabolic subgroups is support-preserving. Any intersection of parabolic subgroups of G is again a parabolic subgroup of G.*

Proof. Let I be a set, and let $(H_i)_{i\in I}$ be a family of parabolic subgroups of G. We want to show that $\bigcap_{i\in I} H_i$ is a parabolic subgroup of G. The case where $|I| = 2$ (i.e. the intersection of two parabolic subgroups) is [GM22, Theorem 6.11]. From this we easily deduce the case where I is finite. (Note that in [GM22, Section 6], the authors make the assumptions that no square of a nontrivial simple element is again a simple element. However, this assumption is not used in the proof of [GM22, Theorem 6.11]).

The case where I is infinite is an adaptation of the argument at the beginning of [GM22, Section 6.3]. If H is a parabolic subgroup of G, conjugate to a standard parabolic subgroup G_δ, we can define the rank of H as $\mathrm{rk}(H) := \ell(\delta)$ (where ℓ is a length function on M). By [GM22, Proposition 6.2], this does not depend on the choice of the standard parabolic subgroup G_δ.

If $H \subset H'$ are two parabolic subgroups, then we claim that $\mathrm{rk}(H) \leqslant \mathrm{rk}(H')$, with $H = H'$ if and only if $\mathrm{rk}(H) = \mathrm{rk}(H')$. Up to replacing H, H' by conjugates, we can assume that $H' = G_{\delta'}$ is standard. We can then consider the central element $z_H \in H$. Since z_H is conjugate (by definition) to a positive element, we can apply iterated swaps to z_H to obtain $c \in G_{\delta'}$ such that $(z_H)^c \in M_\delta$. Furthermore, by Theorem 5.2.14 ($\mathcal{T}$-parabolic closure of recurrent endomorphisms) and Lemma 5.2.16, we have

$$H^c = \mathrm{PC}(z_H)^c = \mathrm{SPC}((z_H)^c) =: G_\delta.$$

We then have $G_\delta \subset G_{\delta'}$, thus $\delta \preccurlyeq \delta'$ and $\mathrm{rk}(H) = \ell(\delta) \leqslant \ell(\delta') = \mathrm{rk}(H')$. We then have $\mathrm{rk}(H) = \mathrm{rk}(H')$ if and only if $\delta = \delta'$, which is equivalent to

$$H = (G_\delta)^{c^{-1}} = (G_{\delta'})^{c^{-1}} = G_{\delta'}.$$

Now, the set

$$\left\{ \mathrm{rk}(H) \,\middle|\, H = \bigcap_{i\in I'} H_i,\ I' \subset I \text{ is finite} \right\}$$

is a set of (possibly zero) integers, and we can then consider $I' \subset I$ finite such that $\mathrm{rk}(\bigcap_{i\in I'} H_i)$ is minimal. We write $H := \bigcap_{i\in I'} H_i$. For $i \in I$, the set $I' \cup \{i\}$ is also finite, thus $\mathrm{rk}(H \cap H_i) \geqslant \mathrm{rk}(H)$ by definition, and $\mathrm{rk}(H \cap H_i) \leqslant \mathrm{rk}(H)$ since $H \cap H_i \subset H$. We then have $\mathrm{rk}(H \cap H_i) = \mathrm{rk}(H)$, $H \cap H_i = H$ and thus $H \subset H_i$. Since this is true for all $i \in I$, we obtain that $H = \bigcap_{i\in I} H_i$ is a parabolic subgroup of G. □

5.3 Shoals attached to different constructions

Let $(\mathcal{G}, \mathcal{C}, \Delta)$ be a Garside groupoid, endowed with a shoal $\mathcal{T}$. If $(\mathcal{G}', \mathcal{C}', \Delta')$ is a Garside groupoid obtained from $(\mathcal{G}, \mathcal{C}, \Delta)$ using one of the constructions of Chapter 4 (groupoid of fixed points, groupoid of cosets,...) we want to endow $(\mathcal{G}', \mathcal{C}', \Delta')$ with a shoal $\mathcal{T}'$ somehow related to $\mathcal{T}$, and to deduce properties of $\mathcal{T}'$ from properties of $\mathcal{T}$ (support-preservingness, existence of a system of conjugacy representatives). In this section, we describe for the first three constructions of Chapter 4 an associated construction of a shoal.

In the particular case of dual braid monoids attached to well-generated complex reflection groups, we will describe shoals for divided groupoids and for groupoids of periodic elements in Section 9.2.4. Note that, as groupoids of periodic elements are isomorphic to a Garside groupoids attached to a conjugacy graph by Proposition 4.5.12 (and the subsequent discussion), we can use the shoal defined in Section 5.3.4 in order to study parabolic subgroups of centralizers of regular elements in general.

Throughout this section, we fix a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$, with set of simples $\mathcal{S}$ and Garside automorphism $\mathcal{T}$. We also fix a shoal $\mathcal{T}$ for $\mathcal{G}$.

5.3.1 Shoals for standard parabolic subgroupoids

Before we actually get to the constructions of Chapter 4, we quickly describe how to define a shoal for a standard parabolic subgroupoid starting from a shoal of the ambient category.

In this subsection, we fix an element $(\mathcal{G}_{\delta_0}, \mathcal{C}_{\delta_0}, \delta_0)$ of $\mathcal{T}$. We denote by $\mathcal{S}_{\delta_0}$ its set of simples, and by φ_0 its Garside automorphism.

Proposition 5.3.1 (Shoal for standard parabolic subgroupoid). *Consider the set*

$$\mathcal{T}_{\delta_0} = \{\mathcal{G}_\delta \mid \mathcal{G}_\delta \in \mathcal{T},\ \mathcal{G}_\delta \subset \mathcal{G}_{\delta_0}\} = \{\mathcal{G}_\delta \cap \mathcal{G}_{\delta_0} \mid \mathcal{G}_\delta \in \mathcal{T}\}.$$

If $\mathcal{T}_{\delta_0}$ is stable under φ_0, then it is a shoal for $(\mathcal{G}_{\delta_0}, \mathcal{C}_{\delta_0}, \delta_0)$. Furthermore, if $\mathcal{T}$ is support-preserving, then so is $\mathcal{T}_{\delta_0}$.

Proof. First, since $\mathcal{G}_{\delta_0} \in \mathcal{T}$, the second definition of $\mathcal{T}_{\delta_0}$ makes sense and gives the same set as the first one. We have $\mathcal{G}_{\delta_0} = \mathcal{G}_\delta \cap \mathcal{G}_{\delta_0} \in \mathcal{T}_{\delta_0}$. Then, we have $\{1_u\}_{u\in \mathrm{Ob}(\mathcal{G}_{\delta_0})} = \{1_u\}_{u\in\mathrm{Ob}(\mathcal{G})} \cap \mathcal{G}_{\delta_0}$. Lastly, if $\mathcal{G}_\delta, \mathcal{G}_{\delta'} \in \mathcal{T}$, then

$$(\mathcal{G}_\delta \cap \mathcal{G}_{\delta_0}) \cap (\mathcal{G}_{\delta'} \cap \mathcal{G}_{\delta_0}) = \mathcal{G}_\delta \cap \mathcal{G}_{\delta'} \cap \mathcal{G}_{\delta_0} \in \mathcal{T}_{\delta_0},$$

and $\mathcal{T}_{\delta_0}$ is stable under intersection. If $\mathcal{T}_{\delta_0}$ is stable under φ_0, then it is a shoal for $(\mathcal{G}_{\delta_0}, \mathcal{C}_{\delta_0}, \delta_0)$.

Now, assume that $\mathcal{T}$ is support-preserving. By definition of $\mathcal{T}_{\delta_0}$, the $\mathcal{T}_{\delta_0}$-standard parabolic subgroups in $\mathcal{G}_{\delta_0}$ are exactly the $\mathcal{T}$-standard parabolic subgroups in $\mathcal{G}$ which are included in $\mathcal{G}_{\delta_0}$. Since the standard parabolic closure of an endomorphism is the intersection of the standard parabolic subgroups which contain it, we deduce that the $\mathcal{T}_{\delta_0}$-standard parabolic closure of an endomorphism is equal to its $\mathcal{T}$-standard parabolic closure. The support-preservingness of $\mathcal{T}_{\delta_0}$ is then an immediate consequence of the support-preservingness of $\mathcal{T}$. $\square$

Note that, if $(\mathcal{G}_{\delta_0}, \mathcal{C}_{\delta_0}, \delta_0)$ does not belong to $\mathcal{T}$, then we do not have $\mathcal{G}_{\delta_0} \in \mathcal{T}_{\delta_0}$ a priori with the first definition, and we cannot apply the second definition of $\mathcal{T}_{\delta_0}$ since we do not know if the intersection of elements of $\mathcal{T}$ with $\mathcal{G}_{\delta_0}$ are standard parabolic subgroupoids.

Example 5.3.2. In Proposition 5.3.1, the assumption that $\mathcal{T}_{\delta_0}$ is stable under φ_0 is important. Consider for instance the monoid $M := \langle s,t,u \mid sts = tst, tut = utu, su = us\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with $\Delta = stsuts$, whose Garside automorphism ϕ swaps s and u. We denote $G := G(M)$. The parabolic Garside elements of M are $\{1, s, t, u, su, sts, tut, \Delta\}$. One readily checks that the set $\mathcal{T} := \{\{1\}, G_t, G_{sts}, G_{utu}, G\}$ is a shoal for G (with $G_{sts} \cap G_{utu} = G_{sts\wedge utu} = G_t$).

The standard parabolic submonoid M_{sts} is an Artin-Tits monoid of type A_2, whose Garside automorphism φ swaps s and t. In particular, the set $\mathcal{T}_{sts}$ of Proposition 5.3.1, which is given by $\mathcal{T}_{sts} = \{\{1\}, G_t, G_{sts}\}$ is not a shoal for G_{sts} as $\varphi(G_t) = G_s \notin \mathcal{T}_{sts}$.

In the case of a Garside group, the situation is easier to describe, as in the following corollary:

Corollary 5.3.3. *Let (G, M, Δ) be a Garside group, and let $\mathcal{T}$ be the shoal of all standard parabolic subgroups of G. The set $\mathcal{T}_\delta$ associated to a standard parabolic subgroup $(G_\delta, M_\delta, \delta)$ is the shoal of all standard parabolic subgroups of $(G_\delta, M_\delta, \delta)$.*

Proof. By Corollary 5.1.16, $\mathcal{T}$ is in bijection with the set of parabolic Garside elements of G. Under this bijection, $\mathcal{T}_\delta$ becomes the set of parabolic Garside elements of G which divide δ. This set is also the set of parabolic Garside elements of G_δ by construction. As it contains $1, \delta$, and as it is stable under left-gcd and under the action of the Garside automorphism of G_δ, we obtain the result. $\square$

Assume now that $\mathcal{T}_{\delta_0}$ is stable under φ_0. By definition, the $\mathcal{T}_{\delta_0}$-standard parabolic subgroups in $\mathcal{G}_{\delta_0}$ are exactly the $\mathcal{T}$-standard parabolic subgroups in $\mathcal{G}$ which are included in $\mathcal{G}_{\delta_0}$. We want to generalize this property to all $\mathcal{T}_{\delta_0}$-parabolic subgroups (and not only standard ones). It is always true that $\mathcal{T}_{\delta_0}$-parabolic subgroups in $\mathcal{G}_{\delta_0}$ are in particular $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$, but support-preservingness will be needed to prove the converse statement.

Proposition 5.3.4 (Parabolic subgroups of a parabolic subgroup).
Assume that $\mathcal{T}_{\delta_0}$ is stable under φ_0, and let $u \in \mathrm{Ob}(\mathcal{G}_{\delta_0})$. The inclusion map $\mathcal{G}_{\delta_0}(u,u) \to \mathcal{G}(u,u)$ induces an inclusion map

$$\mathcal{P}_{\mathcal{T}_{\delta_0}}(\mathcal{G}_{\delta_0}(u,u)) \to \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)).$$

Furthermore, if $\mathcal{T}$ is support-preserving, then the image of this map is the set

$$\{H \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)) \mid H \subset \mathcal{G}_{\delta_0}(u,u)\}.$$

Proof. First, let $H \subset \mathcal{G}_{\delta_0}(u,u)$ be a $\mathcal{T}_{\delta_0}$-parabolic subgroup. By definition, there is a morphism $f \in \mathcal{G}_{\delta_0}(u,v)$ such that $H^f = \mathcal{G}_\delta(v,v)$ is a $\mathcal{T}_{\delta_0}$-standard parabolic subgroup of $\mathcal{G}_{\delta_0}(v,v)$. By definition of $\mathcal{T}_{\delta_0}$, $\mathcal{G}_\delta$ is in particular a standard parabolic subgroupoid of $\mathcal{G}$, and $\mathcal{G}_\delta(v,v)$ is a standard parabolic subgroup of $\mathcal{G}(v,v)$. Thus $H = (\mathcal{G}_\delta(v,v))^{f^{-1}}$ is a parabolic subgroup of $\mathcal{G}(u,u)$, which is included in $\mathcal{G}_{\delta_0}(u,u)$ by definition.

Assume now that $\mathcal{T}$ is support-preserving, and let $H \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u))$ be included in $\mathcal{G}_{\delta_0}(u,u)$. We show that H is also a $\mathcal{T}_{\delta_0}$-parabolic subgroup of $\mathcal{G}_{\delta_0}(u,u)$. Let $f \in \mathcal{G}(u,v)$ be such that $H^f = \mathcal{G}_\delta(v,v)$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(v,v)$, and let $k > 0$ be the smallest integer such that $\delta^k(v) \in \mathcal{G}_\delta(v,v)$. We have $\mathcal{G}_\delta(v,v) = \mathrm{SPC}(\delta^k(v))$ by Corollary 5.2.7. We write $x := (\delta^k(v))^{f^{-1}} \in H$. Since x is conjugate in $\mathcal{G}$ to the positive endomorphism $\delta^k(v)$, we can apply iterated swaps to x to obtain a positive conjugate $y = x^c$ of x, with $x, c \in \mathcal{G}_{\delta_0}$ by Corollary 5.1.12. By support-preservingness of $\mathcal{T}$, we then have

$$\mathrm{SPC}(y) = \mathrm{SPC}(\delta^k(v))^{f^{-1}c} = (\mathcal{G}_\delta(v,v))^{f^{-1}c} = H^c.$$

In particular, H is a $\mathcal{T}_{\delta_0}$-parabolic subgroup of $\mathcal{G}_{\delta_0}$ as it is conjugate in $\mathcal{G}_{\delta_0}$ to the $\mathcal{T}_{\delta_0}$-standard parabolic subgroup $\mathrm{SPC}(y)$. □

Corollary 5.3.5. *Assume that $\mathcal{T}_{\delta_0}$ is stable under φ_0, and let $x \in \mathcal{G}_{\delta_0}$ be an endomorphism. Assume further that $\mathcal{T}$ is support-preserving, and let us denote by $\mathrm{PC}_{\delta_0}(x)$ (resp. $\mathrm{PC}(x)$) the $\mathcal{T}_{\delta_0}$-parabolic closure of x in $\mathcal{G}_{\delta_0}$ (resp. the $\mathcal{T}$-parabolic closure of x in $\mathcal{G}$). We have $\mathrm{PC}_{\delta_0}(x) = \mathrm{PC}(x)$.*

Proof. Let $u \in \mathrm{Ob}(\mathcal{G}_{\delta_0})$ be such that $x \in \mathcal{G}_{\delta_0}(u,u)$. We have $\mathrm{PC}(x) \subset \mathcal{G}_{\delta_0}(u,u)$ by definition. In particular $\mathrm{PC}(x)$ is a $\mathcal{T}_{\delta_0}$-parabolic subgroup which contains x by Proposition 5.3.4, and $\mathrm{PC}_{\delta_0}(x) \subset \mathrm{PC}(x)$. Conversely, $\mathrm{PC}_{\delta_0}(x)$ is a $\mathcal{T}$-parabolic subgroup which contains x also by Proposition 5.3.4, and $\mathrm{PC}(x) \subset \mathrm{PC}_{\delta_0}(x)$. □

Proposition 5.3.6 (System of conjugacy representatives). *Assume that $\mathcal{T}_{\delta_0}$ is stable under φ_0, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$. The set $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}_{\delta_0}\}$ is a system of conjugacy representatives for $\mathcal{T}_{\delta_0}$.*

Proof. By construction, we have $\mathcal{G}_\delta \in \mathcal{T}_{\delta_0}$ if and only if $\mathcal{G}_\delta \subset \mathcal{G}_{\delta_0}$ and $\mathcal{G}_\delta \in \mathcal{T}$. By definition, for $\mathcal{G}_\delta \in \mathcal{T}_{\delta_0}$, z_δ is still a natural transformation from $1_{\mathcal{C}_\delta}$ to itself. Then, let $u, v \in \mathrm{Ob}(\mathcal{G}_{\delta_0})$, and let $\mathcal{G}_\delta(u,u)$, $\mathcal{G}_{\delta'}(v,v)$ be two $\mathcal{T}_{\delta_0}$-standard parabolic subgroups in $\mathcal{G}_{\delta_0}$. For $f \in \mathcal{G}_{\delta_0}(u,v)$, we have

$$(\mathcal{G}_\delta(u,u))^f = \mathcal{G}_\delta(v,v) \Leftrightarrow (z_\delta(u))^f = z_{\delta'}(v),$$

by definition of a system of conjugacy representatives. Lastly, for $\mathcal{G}_\delta(u,u)$ a $\mathcal{T}_{\delta_0}$-standard parabolic subgroup in $\mathcal{G}_{\delta_0}$, we have that $\mathcal{G}_\delta(u,u)$ is the $\mathcal{T}$-standard parabolic closure of $z_\delta(u)$ in $\mathcal{G}$. By construction of $\mathcal{T}_{\delta_0}$, $\mathcal{G}_\delta(u,u)$ is also the $\mathcal{T}_{\delta_0}$-standard parabolic closure of $z_\delta(u)$. □

Corollary 5.3.7. *Assume that $\mathcal{T}_{\delta_0}$ is stable under φ_0, and let $u \in \mathrm{Ob}(\mathcal{G}_{\delta_0})$. Assume further that $\mathcal{T}$ is support-preserving. If $\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u))$ is stable under intersection, then so is $\mathcal{P}_{\mathcal{T}_{\delta_0}}(\mathcal{G}_{\delta_0}(u,u))$.*

Proof. Let $\{H_i\}_{i\in I}$ be a family in $\mathcal{P}_{\mathcal{T}_{\delta_0}}(\mathcal{G}_{\delta_0}(u,u))$. By Proposition 5.3.4, the H_i are $\mathcal{T}$-parabolic subgroups of $\mathcal{G}(u,u)$ included in $\mathcal{G}_{\delta_0}(u,u)$. By assumption, $H := \bigcap_{i\in I} H_i$ is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(u,u)$ included in $\mathcal{G}_{\delta_0}(u,u)$, and thus a $\mathcal{T}_{\delta_0}$-parabolic subgroup of $\mathcal{G}_{\delta_0}(u,u)$, again by Proposition 5.3.4. □

5.3.2 Shoals for groupoids of fixed points

In this subsection, we fix an automorphism $\psi : \mathcal{G} \to \mathcal{G}$ which preserves $\mathcal{C}$ and Δ (see Section 4.1), and we assume that $\mathcal{G}^\psi$ is nonempty. We also assume that $\mathcal{T}$ is globally stable by ψ, meaning that $\psi(\mathcal{G}_\delta) = \mathcal{G}_{\psi(\delta)} \in \mathcal{T}$ for all $\mathcal{G}_\delta \in \mathcal{T}$.

Recall from Theorem 4.1.11 (Garside groupoid of fixed points) that the groupoid of fixed points $(\mathcal{G}^\psi, \mathcal{C}^\psi, \Delta)$ is a Garside groupoid. We will try and construct a shoal for $\mathcal{G}^\psi$. Before doing so, we give a few general compatibility results between ψ and standard parabolic subgroupoids in $\mathcal{G}$ (not necessarily belonging to $\mathcal{T}$). We begin by showing an analogue of Lemma 5.1.25, replacing the Garside automorphism ϕ with ψ.

Lemma 5.3.8. *Let δ be a parabolic Garside map for $(\mathcal{G}, \mathcal{C}, \Delta)$. The map $\psi(\delta)$ defined by $\psi(\delta)(u) = \psi(\delta(\psi^{-1}(u)))$ whenever $\delta(\psi^{-1}(u))$ is defined, is a parabolic Garside map for $(\mathcal{G}, \mathcal{C}, \Delta)$. Furthermore, we have $\mathcal{G}_{\psi(\delta)} = \psi(\mathcal{G}_\delta)$ and $\mathcal{C}_{\psi(\delta)} = \psi(\mathcal{C}_\delta)$.*

Proof. First, we have $\psi(\delta)(u) \in \mathcal{S}(\psi(u), -)$ whenever $\psi(\delta)(u)$ is defined since ψ preserves $\mathcal{S}$ globally. Then, let $s \in \mathcal{S}(u, v)$, we have

$$\begin{aligned} s \preccurlyeq \psi(\delta)(u) &\Leftrightarrow \psi^{-1}(s) \preccurlyeq \delta(\psi^{-1}(u)) \\ &\Leftrightarrow \delta(\varphi^{-1}\psi^{-1}(v)) \succcurlyeq \psi^{-1}(s) \\ &\Leftrightarrow \psi(\delta)(\psi\varphi^{-1}\psi^{-1}(v)) \succcurlyeq s, \end{aligned}$$

and thus $\psi(\delta)$ is a balanced map. If $s, t \in \mathrm{Div}(\psi(\delta))$ are such that $st \in \mathcal{S}$, then we have $\psi^{-1}(s), \psi^{-1}(t) \in \mathrm{Div}(\delta)$, and $\psi^{-1}(s)\psi^{-1}(t) = \psi^{-1}(st) \in \mathcal{S}$. Thus $\psi^{-1}(st) \in \mathrm{Div}(\delta)$ since δ is a parabolic Garside map and $st \in \mathrm{Div}(\psi(\delta))$. We obtain that $\psi(\delta)$ is a parabolic Garside map by Lemma 5.1.2. By definition, $\mathcal{G}_{\psi(\delta)}$ is generated (as a subgroupoid of $\mathcal{G}$) by $\mathrm{Div}(\psi(\delta))$. We obtain that $\mathcal{G}_{\psi(\delta)} = \mathcal{G}_\delta$ since $\mathrm{Div}(\psi(\delta)) = \psi(\mathrm{Div}(\delta))$. We obtain similarly that $\mathcal{C}_{\psi(\delta)} = \psi(\mathcal{C}_\delta)$. □

Lemma 5.3.9. *Let $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ be a standard parabolic subgroupoid of $\mathcal{G}$. The following assertions are equivalent:*

(i) $\psi(\mathcal{C}_\delta) = \mathcal{C}_\delta$.

(ii) $\psi(\mathcal{G}_\delta) = \mathcal{G}_\delta$.

(iii) $\forall u \in \mathrm{Ob}(\mathcal{C}_\delta),\ \psi(\delta(u)) = \delta(\psi(u))$.

In this case, we say that ψ preserves *$(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$.*

Proof. $(i) \Rightarrow (ii)$ is because $\mathcal{G}_\delta$ (resp. $\psi(\mathcal{G}_\delta)$) is the enveloping groupoid of $\mathcal{C}_\delta$ (resp. of $\psi(\mathcal{C}_\delta)$).

$(ii) \Rightarrow (i)$ By Lemma 5.3.8, we have $\psi(\mathcal{G}_\delta) = \mathcal{G}_{\psi(\delta)}$. By Corollary 5.1.15, we have $\psi(\mathcal{G}_\delta) \cap \mathcal{C} = \psi(\mathcal{C}_\delta)$ and $\mathcal{G}_\delta \cap \mathcal{C} = \mathcal{C}_\delta$. The result follows immediately from the assumption that $\psi(\mathcal{C}) = \mathcal{C}$.

$(ii) \Leftrightarrow (iii)$ Again by Lemma 5.3.8, we have $\psi(\mathcal{G}_\delta) = \mathcal{G}_{\psi(\delta)}$. By Corollary 5.1.16, we have $\psi(\mathcal{G}_\delta) = \mathcal{G}_\delta$ if and only if $\psi(\delta) = \delta$, which is precisely (iii). □

Let $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ be a standard parabolic subgroupoid of $\mathcal{G}$. Using Lemma 5.3.9, we can on the one hand see ψ as an automorphism of $\mathcal{G}_\delta$ which preserves $\mathcal{C}_\delta$ and δ, and consider the groupoid of fixed points $(\mathcal{G}_\delta)^\psi$ (provided that it is nonempty). On the other hand, we can consider δ as a parabolic Garside map in $\mathcal{G}^\psi$ (provided that $\delta(u)$ is defined for at least one $u \in \mathrm{Ob}(\mathcal{G}^\psi)$), yielding a standard parabolic subgroupoid $(\mathcal{G}^\psi)_\delta$. The following corollary shows that the two constructions give the same Garside groupoid.

Corollary 5.3.10. *Let $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ be standard parabolic subgroupoid of $\mathcal{G}$ preserved by ψ. The automorphism ψ restricts to an automorphism of $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ which preserves δ, and we have*

(a) The groupoid $(\mathcal{G}_\delta)^\psi$ is nonempty if and only if $\delta(u)$ is defined for some $u \in \mathrm{Ob}(\mathcal{G}^\psi)$.

(b) If $(\mathcal{G}_\delta)^\psi$ is nonempty, then the restriction of δ to $\mathrm{Ob}(\mathcal{C}^\psi)$ *is a parabolic Garside map of $\mathcal{C}^\psi$. Furthermore, we have $(\mathcal{C}_\delta)^\psi = (\mathcal{C}^\psi)_\delta = \mathcal{C}_\delta \cap \mathcal{C}^\psi$ and $(\mathcal{G}_\delta)^\psi = (\mathcal{G}^\psi)_\delta = \mathcal{G}_\delta \cap \mathcal{G}^\psi$.*

Proof. The first point is a direct consequence of Lemma 5.3.9. Indeed, since $\psi(\mathcal{C}_\delta) = \mathcal{C}_\delta$, ψ restricts to an automorphism of $\mathcal{C}_\delta$. The fact that ψ, restricted to $\mathcal{C}_\delta$, preserves δ is point (iii) of Lemma 5.3.9.

(b) is immediate. Now, for point (c), $\delta_{|\mathcal{C}^\psi}$ is a partial map $\mathrm{Ob}(\mathcal{C}^\psi) \to \mathcal{C}^\psi$. If $u \in \mathrm{Ob}(\mathcal{C}^\psi)$ is such that $\delta(u)$ is defined, then $\psi(\delta(u)) = \delta(\psi(u)) = \delta(u)$ by Lemma 5.3.9 (iii) and $\delta(u) \in \mathcal{S}^\psi(u,-)$. In particular, the target $\varphi(u)$ of $\delta(u)$ also lies in $\mathrm{Ob}(\mathcal{C}^\psi)$. Since φ is injective, and since $\mathrm{Ob}(\mathcal{C}^\psi)$ is finite, we obtain that φ is bijective on $\mathrm{Ob}(\mathcal{C}^\psi)$.

Let now $u, v \in \mathrm{Ob}(\mathcal{C}^\psi)$, and let $s \in \mathcal{S}^\psi(u,v)$. We have $s \preccurlyeq \delta(u)$ in $\mathcal{C}$, if and only if $\delta(\varphi^{-1}(v)) \succcurlyeq s$ in $\mathcal{C}$. Since $\delta(\varphi^{-1}(v))$ also lies in $\mathcal{C}^\psi$, we obtain that $s \preccurlyeq \delta(u)$ in $\mathcal{C}^\psi$ if and only if $\delta(\varphi^{-1}(v)) \succcurlyeq s$ in $\mathcal{C}^\psi$ by Lemma 4.1.7 and $\delta_{|\mathcal{C}^\psi}$ is balanced. Furthermore, we see that the set of divisors of $\delta_{|\mathcal{C}^\psi}$ is $\mathcal{S}_\delta \cap \mathcal{S}^\psi$. Let $s, t \in \mathcal{S}_\delta \cap \mathcal{S}^\psi$ be such that $st \in \mathcal{S}$, we have $st \in \mathcal{S}_\delta$ by definition of δ, and $st \in \mathcal{S}^\psi$ since ψ is a functor. Thus $st \in \mathcal{S}_\delta \cap \mathcal{S}^\psi$, and $\delta_{|\mathcal{C}^\psi}$ is a parabolic Garside map of $\mathcal{C}^\psi$ by Lemma 5.1.2.

Now, by definition, the subcategory $(\mathcal{C}^\psi)_\delta$ of $\mathcal{C}^\psi$ is generated by the divisors of $\delta_{|\mathcal{C}^\psi}$, that is by $\mathcal{S}_\delta \cap \mathcal{S}^\psi$. Likewise, the subcategory $(\mathcal{C}_\delta)^\psi$ of $\mathcal{C}_\delta$ is generated by $\mathcal{S}_\delta^\psi = \mathcal{S}_\delta \cap \mathcal{S}^\psi$. Thus $(\mathcal{C}_\delta)^\psi = (\mathcal{C}^\psi)_\delta$. We also obtain that $(\mathcal{G}_\delta)^\psi = (\mathcal{G}^\psi)_\delta$ (they are both generated by $\mathcal{S}_\delta \cap \mathcal{S}^\psi$ as subgroupoids of $\mathcal{G}$). We then have $(\mathcal{G}_\delta)^\psi \subset \mathcal{G}_\delta \cap \mathcal{G}^\psi$ by construction. Lastly, the fact that $(\mathcal{C}_\delta)^\psi = \mathcal{C}_\delta \cap \mathcal{C}^\psi$ comes from Theorem 4.1.11 (Garside groupoid of fixed points) and Proposition 5.1.8 (compatibility of standard parabolic subcategories). The fact that $(\mathcal{G}_\delta)^\psi = \mathcal{G}_\delta \cap \mathcal{G}^\psi$ comes from Theorem 4.1.11 and Proposition 5.1.14 (compatibility of standard parabolic subgroupoids). □

Using this result, we can now define a shoal for $(\mathcal{G}^\psi, \mathcal{C}^\psi, \Delta)$. Recall that we assumed $\mathcal{T}$ to be globally stable under ψ.

Proposition 5.3.11 (Shoal for groupoid of fixed points). *The set*

$$\mathcal{T}^\psi := \{(\mathcal{G}^\psi)_\delta \mid \mathcal{G}_\delta \in \mathcal{T} \text{ is preserved by } \psi \text{ and } (\mathcal{G}^\psi)_\delta \neq \varnothing\}$$

is a shoal for $\mathcal{G}^\psi$. Furthermore the $\mathcal{T}^\psi$-standard parabolic subgroup in $\mathcal{G}^\psi$ are exactly the intersection with $\mathcal{G}^\psi$ of the $\mathcal{T}$-standard parabolic subgroups in $\mathcal{G}$ which are preserved by ψ. In particular, if $\mathcal{T}$ is globally preserved by ψ and support-preserving, then $\mathcal{T}^\psi$ is also support-preserving.

Proof. First, we have $\mathcal{G}^\psi = \mathcal{G} \cap \mathcal{G}^\psi \in \mathcal{T}^\psi$ as $\mathcal{G} \in \mathcal{T}$. Then, if $\mathcal{G}_\delta = \{1_u\}_{u \in \mathrm{Ob}(\mathcal{G})} \in \mathcal{T}$, we have $(\mathcal{G}_\delta)^\psi = (\mathcal{G}^\psi)_\delta = \{1_u\}_{u \in \mathrm{Ob}(\mathcal{G}^\psi)} \in \mathcal{T}^\psi$. Now, let $(\mathcal{G}^\psi)_\delta \in \mathcal{T}$. Since ψ and ϕ commute by Corollary 4.1.2, we have that $\phi(\mathcal{G}^\psi) = \mathcal{G}^\psi$. By Corollary 5.3.10, we obtain that

$$\phi((\mathcal{G}^\psi)_\delta) = \phi(\mathcal{G}^\psi \cap \mathcal{G}_\delta) = \mathcal{G}^\psi \cap \mathcal{G}_{\phi(\delta)} = (\mathcal{G}^\psi)_{\phi(\delta)}.$$

This last subgroupoid is nonempty as the image of a nonempty subgroupoid of $\mathcal{G}$. It belongs to $\mathcal{T}^\psi$ since $\mathcal{G}_{\phi(\delta)} \in \mathcal{T}$ by definition of a shoal.

Lastly, let $(\mathcal{G}^\psi)_\delta$ and $(\mathcal{G}^\psi)_{\delta'}$ be two elements of $\mathcal{T}^\psi$. By Corollary 5.3.10, and since $\mathcal{T}$ is a shoal, we have

$$(\mathcal{G}^\psi)_\delta \cap (\mathcal{G}^\psi)_{\delta'} = \mathcal{G}^\psi \cap \mathcal{G}_\delta \cap \mathcal{G}_{\delta'} = \mathcal{G}^\psi \cap \mathcal{G}_{\delta \wedge \delta'}.$$

Since both δ and δ' are preserved by ψ, we deduce that $\delta \wedge \delta'$ is also preserved by ψ. Thus $\mathcal{G}^\psi \cap \mathcal{G}_{\delta\wedge\delta'} \in \mathcal{T}^\psi$, which is then stable under intersection.

Let now $u \in \mathrm{Ob}(\mathcal{G}^\psi)$. Let $H \subset \mathcal{G}^\psi(u,u)$ be a $\mathcal{T}^\psi$-standard parabolic subgroup. There is some $\mathcal{G}_\delta \in \mathcal{T}$ which is preserved by ψ, and such that $H = (\mathcal{G}^\psi)_\delta(u,u) = \mathcal{G}^\psi \cap \mathcal{G}_\delta(u,u)$. In particular, H is the intersection with $\mathcal{G}^\psi$ of a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$. Conversely, let $H \subset \mathcal{G}(u,u)$ be a $\mathcal{T}$-standard parabolic subgroup such that $\psi(H) = H$, and let $\mathcal{G}_\delta \in \mathcal{T}$ be such that $\mathcal{G}_\delta(u,u) = H$. Since $\mathcal{T}$ is a shoal and globally preserved by ψ, we can define

$$\mathcal{G}_{\delta'} := \bigcap_{k\geqslant 0} \mathcal{G}_{\psi^k(\delta)} \in \mathcal{T}.$$

By construction, we have $\psi(\mathcal{G}_{\delta'}) = \mathcal{G}_{\delta'}$ and $H = \mathcal{G}_{\delta'}(u,u)$. In particular, $H \cap \mathcal{G}^\psi$ is a $\mathcal{T}^\psi$-standard parabolic subgroup of $\mathcal{G}^\psi(u,u)$ by definition of $\mathcal{T}^\psi$.

Lastly, assume that $\mathcal{T}$ is support-preserving. For and endomorphism $x \in \mathcal{G}^\psi$, we denote by $\mathrm{SPC}_\psi(x)$ the $\mathcal{T}^\psi$-standard parabolic closure of x in $\mathcal{G}^\psi$, and by $\mathrm{SPC}(x)$ the $\mathcal{T}$-standard parabolic closure of x in $\mathcal{G}$. By the second point, we have $\psi(\mathrm{SPC}(x)) = \mathrm{SPC}(x)$ and $\mathrm{SPC}(x)\cap\mathcal{G}^\psi = \mathrm{SPC}_\psi(x)$. Let $x \in \mathcal{C}^\psi(u,u)$, $y \in \mathcal{C}^\psi(v,v)$, $\alpha \in \mathcal{G}^\psi(u,v)$ be such that $x^\alpha = y$. Since $\mathcal{T}$ is support-preserving, we have

$$\begin{aligned}\mathrm{SPC}_\psi(x)^\alpha &= (\mathrm{SPC}(x) \cap \mathcal{G}^\psi(u,u))^\alpha \\ &= \mathrm{SPC}(x)^\alpha \cap (\mathcal{G}^\psi(u,u))^\alpha \\ &= \mathrm{SPC}(y) \cap \mathcal{G}^\psi(v,v) = \mathrm{SPC}_\psi(y),\end{aligned}$$

and $\mathcal{T}^\psi$ is support-preserving. □

In order to describe $\mathcal{T}^\psi$-parabolic subgroups in $\mathcal{G}^\psi$ in terms of $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$, we will need to assume that the shoal $\mathcal{T}$ admits a system of conjugacy representatives. Before getting to this part, we first show how the parabolic closure for a shoal of the form $\mathcal{T}^\psi$ relates to the parabolic closure defined for the shoal $\mathcal{T}$.

Corollary 5.3.12. *Assume that $\mathcal{T}$ is support-preserving, and let $x \in \mathcal{G}^\psi$ be an endomorphism. Let us denote by $\mathrm{PC}_\psi(x)$ (resp. $\mathrm{PC}(x)$) the $\mathcal{T}^\psi$-parabolic closure of x in $\mathcal{G}^\psi$ (resp. the $\mathcal{T}$-parabolic closure of x in $\mathcal{G}$). We have $\mathrm{PC}_\psi(x) = \mathrm{PC}(x) \cap \mathcal{G}^\psi$.*

Proof. Let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$ be the source of x, and let $c \in \mathcal{G}^\psi(u,v)$ be a conjugating element for iterated swaps such that x^c is recurrent. By Theorem 4.1.11 (Garside groupoid of fixed points), x^c is also recurrent as an endomorphism in $\mathcal{G}$. By construction of the parabolic closure (Theorem 5.2.14), we have $\mathrm{PC}(x^c) = \mathrm{SPC}(x^c)$ and $\mathrm{PC}_\psi(x^c) = \mathrm{SPC}_\psi(x^c)$. We saw in the proof of Proposition 5.3.11 that $\mathrm{SPC}_\psi(x^c) = \mathrm{SPC}(x^c) \cap \mathcal{G}^\psi(v,v)$. We then obtain that

$$\begin{aligned}\mathrm{PC}_\psi(x) &= (\mathrm{PC}_\psi(x^c))^{c^{-1}} = (\mathrm{SPC}_\psi(x^c))^{c^{-1}} = (\mathrm{SPC}(x^c) \cap \mathcal{G}^\psi(v,v))^{c^{-1}} \\ &= (\mathrm{SPC}(x^c))^{c^{-1}} \cap (\mathcal{G}^\psi(v,v))^{c^{-1}} = (\mathrm{PC}(x^c))^{c^{-1}} \cap \mathcal{G}^\psi(u,u) \\ &= \mathrm{PC}(x) \cap \mathcal{G}^\psi(u,u),\end{aligned}$$

which is precisely the desired result. □

Using this Lemma, we can construct a system of conjugacy representatives for $\mathcal{T}^\psi$ starting from a system of conjugacy representatives for $\mathcal{T}$, under some compatibility condition: We say that a system of conjugacy representatives $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ is *preserved by ψ* if we have

$$\forall \mathcal{G}_\delta \in \mathcal{T}, u \in \mathrm{Ob}(\mathcal{G}_\delta),\ \psi(z_\delta(u)) = z_{\psi(\delta)}(\psi(u)).$$

Proposition 5.3.13 (System of conjugacy representatives). *Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$, which is preserved by ψ. The set $\{z_\delta \mid (\mathcal{G}^\psi)_\delta \in \mathcal{T}^\psi\}$ is a system of conjugacy representatives for $\mathcal{T}^\psi$.*

Proof. Let $(\mathcal{G}^\psi)_\delta \in \mathcal{T}^\psi$. For $u \in \mathrm{Ob}((\mathcal{G}^\psi)_\delta)$, we have $\psi(z_\delta(u)) = z_{\psi(\delta)}(\psi(u)) = z_\delta(u)$ since $\psi(\mathcal{G}_\delta) = \mathcal{G}_\delta$. Since z_δ induces a natural transformation $1_{\mathcal{C}_\delta} \Rightarrow 1_{\mathcal{C}_\delta}$, each term of which lies in $(\mathcal{C})^\psi$, it restricts to a natural transformation $1_{(\mathcal{C}_\delta)^\psi} \Rightarrow 1_{(\mathcal{C}_\delta)^\psi}$.

Let now $H = \mathcal{G}^\psi(u,u) \cap \mathcal{G}_\delta(u,u)$ be a $\mathcal{T}^\psi$-standard parabolic subgroup of $\mathcal{G}^\psi(u,u)$. We have to show that $H = \mathrm{SPC}_\psi(z_\delta(u))$. Since $z_\delta(u)$ is positive (both in $\mathcal{G}^\psi$ and in $\mathcal{G}$), we can apply Theorem 5.2.14 ($\mathcal{T}$-parabolic closure of recurrent endomorphisms) and Corollary 5.3.12 to get

$$\begin{aligned}\mathrm{SPC}_\psi(z_\delta(u)) &= \mathrm{PC}_\psi(z_\delta(u))\\ &= \mathrm{PC}(z_\delta(u)) \cap \mathcal{G}^\psi(u,u)\\ &= \mathrm{SPC}(z_\delta(u)) \cap \mathcal{G}^\psi(u,u)\\ &= \mathcal{G}_\delta(u,u) \cap \mathcal{G}^\psi(u,u) = H.\end{aligned}$$

Lastly, let $u, v \in \mathrm{Ob}(\mathcal{G}^\psi)$, and let H (resp. H') be a $\mathcal{T}^\psi$-standard parabolic subgroup of $\mathcal{G}^\psi(u,u)$ (resp. $\mathcal{G}^\psi(v,v)$). We have to show that $H^f = H'$ if and only if $(z_H)^f = z_{H'}$. By Proposition 5.3.11, we have $\mathcal{G}_\delta, \mathcal{G}_{\delta'} \in \mathcal{T}$ such that $\psi(\mathcal{G}_\delta) = \mathcal{G}_\delta$ and $\psi(\mathcal{G}_{\delta'}) = \mathcal{G}_{\delta'}$, and such that

$$H = \mathcal{G}^\psi(u,u) \cap \mathcal{G}_\delta(u,u) \text{ and } H' = \mathcal{G}^\psi(v,v) \cap \mathcal{G}_{\delta'}(v,v).$$

If $H^f = H'$, then in particular, we have $(z_\delta(u))^f \in H' \subset \mathcal{G}_{\delta'}(v,v)$. Since $\mathcal{T}$ is support-preserving, we deduce that

$$(\mathcal{G}_\delta(u,u))^f = \mathrm{SPC}(z_\delta(u))^f = \mathrm{PC}(z_\delta(u)^f) \subset \mathcal{G}_{\delta'}(v,v).$$

Similarly, we have $(z_{\delta'}(v))^{f^{-1}} \in \mathcal{G}_\delta(u,u)$ and $(\mathcal{G}_{\delta'}(v,v))^{f^{-1}} \subset \mathcal{G}_\delta(u,u)$. Thus $(\mathcal{G}_\delta(u,u))^f = \mathcal{G}_{\delta'}(v,v)$, and $(z_\delta(u))^f = z_{\delta'}(v)$ by definition of a system of conjugacy representatives. Conversely, if $(z_\delta(u))^f = z_{\delta'}(v)$, then $H^f = \mathrm{PC}_\psi(z_\delta(u))^f = \mathrm{PC}_\psi(z_{\delta'}(v)) = H'$ by the second part of the proof. □

Now, we want to determine the $\mathcal{T}^\psi$-parabolic subgroups in $\mathcal{G}^\psi$ in terms of the $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$. We give a first partial result.

Lemma 5.3.14. *Let H be a $\mathcal{T}^\psi$ parabolic subgroup in $\mathcal{G}^\psi$. There is a $\mathcal{T}$-parabolic subgroup $\widetilde{H}$ in $\mathcal{G}$ such that $\psi(\widetilde{H}) = \widetilde{H}$ and $H = \widetilde{H} \cap \mathcal{G}^\psi$.*

Proof. Let $u \in \mathrm{Ob}(\mathcal{G}^\psi)$ be such that $H \subset \mathcal{G}^\psi(u,u)$, and let $f \in \mathcal{G}^\psi(u,v)$ be such that $H^f = (\mathcal{G}^\psi)_\delta(v,v)$ is a $\mathcal{T}^\psi$-standard parabolic subgroup. We define $\widetilde{H} := (\mathcal{G}_\delta(v,v))^{f^{-1}}$, it is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(u,u)$ by definition. Furthermore, since $\psi(\mathcal{G}_\delta(v,v)) = \mathcal{G}_\delta(v,v)$ and since $\psi(f) = f$, we have $\psi(\widetilde{H}) = \widetilde{H}$. Lastly, we have

$$H = ((\mathcal{G}^\psi)_\delta(v,v))^{f^{-1}} = (\mathcal{G}_\delta(v,v) \cap \mathcal{G}^\psi(v,v))^{f^{-1}} = \widetilde{H} \cap \mathcal{G}^\psi(u,u).$$

□

Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$, which is preserved by ψ. If H is a $\mathcal{T}^\psi$-parabolic subgroup in $\mathcal{G}^\psi$, written

$H = \widetilde{H} \cap \mathcal{G}^\psi(u,u)$ as in the above lemma, then we have $z_H = z_{\widetilde{H}}$. Now, for $u \in \mathrm{Ob}(\mathcal{G}^\psi)$, we define

$$(\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)))^\psi := \{H \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)) \mid \psi(H) = H\}.$$

Since the system of conjugacy representatives we consider is preserved by ψ, we can give an easy characterization of this set in terms of the elements z_H, as in the following lemma:

Lemma 5.3.15. *Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$ which is preserved by ψ. For H a $\mathcal{T}$-parabolic subgroup in $\mathcal{G}$, we have $z_{\psi(H)} = \psi(z_H)$. In particular, we have $\psi(H) = H$ if and only if $\psi(z_H) = z_H$.*

Proof. Let $u \in \mathrm{Ob}(\mathcal{G})$ be such that $H \subset \mathcal{G}(u,u)$. By definition, there is some $f \in \mathcal{G}(u,v)$ such that $H^f = \mathcal{G}_\delta(v,v)$ is a $\mathcal{T}$-standard parabolic subgroup, and we have $z_H = (z_\delta(v))^{f^{-1}}$. We also have

$$\psi(H)^{\psi(f)} = \psi(H^f) = \mathcal{G}_{\psi(\delta)}(\psi(v),\psi(v)).$$

By definition of $z_{\psi(H)}$, we then have

$$z_{\psi(H)} = (z_{\psi(\delta)}(\psi(v)))^{\psi(f)^{-1}} = (\psi(z_\delta(v)))^{\psi(f^{-1})} = \psi\left((z_\delta(v))^{f^{-1}}\right) = \psi(z_H).$$

Now, we have $\psi(H) = H$ if and only if 1_u conjugates $\psi(H)$ to H. By Proposition 5.2.31 (characterization of conjugacy of parabolic subgroups), this is equivalent to $z_{\psi(H)} = \psi(z_H) = z_H$. ☐

Under the above assumptions, we are able to completely describe the parabolic subgroups in $\mathcal{G}^\psi$ using sets of the form $(\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)))^\psi$. We do not know whether or not the following results also hold without requiring the existence of a system of conjugacy representatives preserved by ψ.

Proposition 5.3.16 (Parabolic subgroups in groupoid of fixed points). *Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$ which is preserved by ψ. Let also $u \in \mathrm{Ob}(\mathcal{G}^\psi)$.*
For $H \in (\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)))^\psi$, the intersection $H \cap \mathcal{G}^\psi(u,u)$ is a $\mathcal{T}^\psi$-parabolic subgroup of $\mathcal{G}^\psi(u,u)$. Furthermore, the map

$$\begin{array}{rcl}(\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)))^\psi & \longrightarrow & \mathcal{P}_{\mathcal{T}^\psi}(\mathcal{G}^\psi(u,u)) \\ H & \longmapsto & H \cap \mathcal{G}^\psi(u,u)\end{array}$$

is a bijection. The inverse bijection sends $H \in \mathcal{P}_{\mathcal{T}^\psi}(\mathcal{G}^\psi(u,u))$ to the $\mathcal{T}$-parabolic closure of z_H in $\mathcal{G}$.

Proof. Let $H \in \mathcal{G}(u,u)$ be a $\mathcal{T}$-parabolic subgroup such that $\psi(H) = H$. By Lemma 5.3.15, we have $\psi(z_H) = z_H$, and by Corollary 5.3.12, we then have

$$H \cap \mathcal{G}^\psi(u,u) = \mathrm{PC}(z_H) \cap \mathcal{G}^\psi(u,u) = \mathrm{PC}_\psi(z_H),$$

which is a $\mathcal{T}^\psi$-parabolic subgroup of $\mathcal{G}^\psi(u,u)$. Furthermore, H is indeed equal to $\mathrm{PC}(z_{H\cap\mathcal{G}^\psi(u,u)}) = \mathrm{PC}(z_H)$. Conversely, let H be a $\mathcal{T}^\psi$-parabolic subgroup of $\mathcal{G}^\psi(u,u)$, we can consider $z_H \in \mathcal{G}^\psi(u,u)$ given by the system of conjugacy representatives $\{z_\delta \mid (\mathcal{G}^\psi)_\delta \in \mathcal{T}^\psi\}$. The parabolic subgroup $\mathrm{PC}(z_H)$ of $\mathcal{G}(u,u)$ is stable under ψ since $\psi(z_H) = z_H$, and we have $H = \mathrm{PC}_\psi(z_H) = \mathrm{PC}(z_H) \cap \mathcal{G}^\psi(u,u)$, again by Corollary 5.3.12. ☐

Corollary 5.3.17. *Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$ which is preserved by ψ. Let also $u \in \mathrm{Ob}(\mathcal{G}^\psi)$. If $\mathcal{P}_\mathcal{T}(\mathcal{G}(u,u))$ is stable under intersection, then so is $\mathcal{P}_{\mathcal{T}^\psi}(\mathcal{G}^\psi(u,u))$.*

Proof. Let $\{H_i\}_{i\in I}$ be a family in $\mathcal{P}_{\mathcal{T}^\psi}(\mathcal{G}^\psi(u,u))$. By Proposition 5.3.16, we can, for each $i \in I$, consider some $\widetilde{H}_i \in \mathcal{P}_\mathcal{T}(\mathcal{G}(u,u))$ such that $\psi(\widetilde{H}_i) = \widetilde{H}_i$ and $\widetilde{H}_i \cap \mathcal{G}^\psi(u,u) = H_i$. By assumption, the intersection $\widetilde{H} = \bigcap_{i\in I} \widetilde{H}_i$ is a parabolic subgroup of $\mathcal{G}(u,u)$, which is stable under ψ (since all the $\widetilde{H}_i$ are). We then have that

$$\bigcap_{i\in I} H_i = \bigcap_{i\in I} \widetilde{H}_i \cap \mathcal{G}^\psi(u,u) = \widetilde{H} \cap \mathcal{G}^\psi(u,u)$$

is a parabolic subgroup, again by Proposition 5.3.16. □

Example 5.3.18. Consider the monoid $M = \langle s,t,u, \mid sts = tst, tut = utu, st = ts\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. The Garside automorphism ϕ acts on the atoms by swapping s and u. We saw in Example 4.1.12 that the fixed point monoid M^ϕ is generated by $x := t$ and $y := su$, with relation $xyxy = yxyx$ (Artin-Tits monoid of type B_2).

The parabolic Garside elements in M are $\{1, s, t, u, sts, su, tut, \Delta\}$. Among these elements, the ones that are invariant under ϕ are $1, t, su, \Delta$, thus the parabolic submonoids of M^ϕ are $\{1\}$, $\langle t\rangle^+$, $\langle su\rangle^+$ and $G(M)^\phi$. On the other hand, in the Garside monoid $\langle x, y \mid xyxy = yxyx\rangle^+$, the parabolic Garside elements are $1, x, y, \Delta = xyxy$.

5.3.3 Shoals for groupoids of cosets

In this subsection, we fix an object u of $\mathcal{G}$, along with a finite index subgroup $H \subset \mathcal{G}(u,u)$.

Recall from section 4.2 that we have a Garside groupoid $(\mathcal{G}_H, \mathcal{C}_H, \Delta_H)$, where $\mathcal{G}_H$ is the groupoid of cosets associated to H, and where $\mathcal{C}_H$ is made of the morphisms $f_{[g]} \in \mathcal{G}_H$ where $f \in \mathcal{C}$. Recall also that $\mathcal{G}_H$ comes equipped with a functor $\pi : \mathcal{G}_H \to \mathcal{G}$ which induces an isomorphism $\mathcal{G}_H([g],[g]] \simeq H^g \subset \mathcal{G}(v,v)$ for $g \in \mathcal{G}(u,v)$. We will try and construct a shoal for $\mathcal{G}_H$ using the functor π.

Lemma 5.3.19. *Let δ be a parabolic Garside map in $(\mathcal{G}, \mathcal{C}, \Delta)$. For $g \in \mathcal{G}(u,v)$ such that $\delta(v)$ is defined, we set*

$$\delta_H([g]) := \delta(v)_{[g]} : [g] \to [g\delta(v)] = [\delta(u)\varphi(g)].$$

The partial map δ_H is a parabolic Garside map in $(\mathcal{G}_H, \mathcal{C}_H, \Delta_H)$. Furthermore, we have $(\mathcal{G}_H)_{\delta_H} = \pi^{-1}(\mathcal{G}_\delta)$ and $(\mathcal{C}_H)_{\delta_H} = \pi^{-1}(\mathcal{C}_\delta)$.

Proof. By Theorem 4.2.2 (Garside groupoid of cosets), the set of simples of $\mathcal{G}_H$ is $\mathcal{S}_H = \pi^{-1}(\mathcal{S})$. In particular, $\delta_H([g])$ is always a simple morphism with source $[g]$. Then, for $s_{[g]} \in \mathcal{S}_H([g], -)$, we have $s_{[g]} \preccurlyeq \delta_H([g])$ if and only if $s \preccurlyeq \delta(v)$ by Lemma 4.2.1 (preservation of lattice structures) (where v is the target of g). Thus $\mathrm{Div}(\delta_H) = \pi^{-1}(\mathrm{Div}(\delta))$. Likewise, we have $\mathrm{Div}_R(\delta_H) = \pi^{-1}(\mathrm{Div}_R(\delta))$, and δ_H is balanced since δ is balanced.

Now, let $s_{[g]}, t_{[gs]}$ be two simple morphisms in $\mathcal{G}_H$ such that $s_{[g]}, t_{[gs]} \in \mathrm{Div}(\delta_H)$ and $s_{[g]}t_{[gs]} = (st)_{[g]} \in \mathcal{S}_H$. We have $s, t \in \mathrm{Div}(\delta)$ and $st \in \mathcal{S}$. Thus $st \in \mathrm{Div}(\delta)$ since δ is a parabolic Garside map, and $(st)_{[g]} \in \pi^{-1}(\mathrm{Div}(\delta)) = \mathrm{Div}(\delta_H)$. Thus δ_H is a parabolic Garside map in $\mathcal{G}_H$.

Lastly, since $(\mathcal{C}_H)_{\delta_H}$ is generated by $(\mathcal{S}_H)_{\delta_H} = \pi^{-1}(\mathcal{S}_\delta) \subset \pi^{-1}(\mathcal{C}_\delta)$, we have $(\mathcal{C}_H)_{\delta_H} \subset \pi^{-1}(\mathcal{C}_\delta)$. Conversely, let $f_{[g]} \in \pi^{-1}(\mathcal{C}_\delta)$. We write $s_1 \cdots s_r$ for the greedy normal form of $f = \pi(f_{[g]})$ in $\mathcal{C}$. By assumption, we have $f \in \mathcal{C}_\delta$ and thus $s_i \in \mathcal{S}_\delta$ for all $i \in [\![1, r]\!]$ by Proposition 5.1.8 (compatibility of standard parabolic subcategories). For $i \in [\![1, r]\!]$, we then have $s_{i[gs_1 \cdots s_{i-1}]} \in \pi^{-1}(\mathcal{S}_\delta) = (\mathcal{S}_H)_{\delta_H}$ and thus

$$f_{[g]} = s_{1[g]} s_{2[gs_1]} \cdots s_{r[gs_1 \cdots s_{r-1}]} \in (\mathcal{C}_H)_{\delta_H}.$$

A similar argument using Proposition 5.1.14 (compatibility of standard parabolic subgroupoids) proves that $(\mathcal{G}_H)_{\delta_H} = \pi^{-1}(\mathcal{G}_\delta)$. □

Proposition 5.3.20 (Shoal for groupoid of cosets). *The set*

$$\mathcal{T}_H = \{\pi^{-1}(\mathcal{G}_\delta) = \mathcal{G}_{\delta_H} \mid \mathcal{G}_\delta \in \mathcal{T}\}$$

is a shoal for $\mathcal{G}_H$. Furthermore, for $g \in \mathcal{G}(u, v)$, the isomorphism $\mathcal{G}_H([g], [g]) \simeq H^g$ induced by π identifies the $\mathcal{T}_H$-standard parabolic subgroups of $\mathcal{G}_H([g], [g])$ with the groups of the form $H^g \cap P$, where P is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(v, v)$. In particular, if $\mathcal{T}$ is support-preserving, then so is $\mathcal{T}_H$.

Proof. We have $\pi^{-1}(\mathcal{G}_\delta) = (\mathcal{G}_H)_{\Delta_H} = \mathcal{G}_H \in \mathcal{T}_H$, and $\pi^{-1}(\{1_u\}_{u \in \mathrm{Ob}(\mathcal{G})}) = \{1_{[g]}\}_{[g] \in \mathrm{Ob}(\mathcal{G}_H)}$. Let now $\pi^{-1}(\mathcal{G}_\delta) \in \mathcal{T}_H$. By Theorem 4.2.2, we have $\phi_H(\pi^{-1}(\mathcal{G}_\delta)) = \pi^{-1}(\phi(\mathcal{G}_\delta)) \in \mathcal{T}_H$. Lastly, let $\mathcal{G}_1$, $\mathcal{G}_2$ be two elements of $\mathcal{T}_H$ such that $\mathcal{G}_1 \cap \mathcal{G}_2$ is nonempty. If we denote $\mathcal{G}_1 = \pi^{-1}(\mathcal{G}_{\delta_1})$ and $\mathcal{G}_2 = \pi^{-1}(\mathcal{G}_{\delta_2})$, then $\mathcal{G}_1 \cap \mathcal{G}_2 = \pi^{-1}(\mathcal{G}_{\delta_1} \cap \mathcal{G}_{\delta_2})$ lies in $\mathcal{T}_H$ by definition, and $\mathcal{T}_H$ is a shoal.

Now, let $g \in \mathcal{G}(u, v)$, and let $(\mathcal{G}_H)_{\delta_H}([g], [g]) \subset \mathcal{G}_H([g], [g])$ be a $\mathcal{T}_H$-parabolic subgroup. By construction, we have

$$(\mathcal{G}_H)_{\delta_H}([g], [g]) = \{f_{[g]} \in \pi^{-1}(\mathcal{G}_\delta) \mid [gf] = [g]\} = \{f \in \mathcal{G}_\delta \mid f \in H^g\} = \mathcal{G}_\delta(v, v) \cap H^g.$$

Conversely, if $\mathcal{G}_\delta(v, v)$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(v, v)$, we have

$$\pi((\mathcal{G}_H)_{\delta_H}([g], [g])) = \mathcal{G}_\delta(v, v) \cap H^g.$$

Lastly, let $g \in \mathcal{G}(u, v), g' \in \mathcal{G}(u, v')$, and let $x_{[g]}, y_{[g']}$ be two positive endomorphisms in $\mathcal{G}_H$ which are conjugate by some $\alpha_{[g]} \in \mathcal{G}_H([g], [g'])$. If we denote by SPC_H the $\mathcal{T}_H$-standard parabolic closure, the above point proves that $\pi(\mathrm{SPC}_H(x_{[g]})) = \mathrm{SPC}(x) \cap H^g$. If $\mathcal{T}$ is support-preserving, then we have

$$\begin{aligned}
\pi(\mathrm{SPC}_H(x_{[g]})^{\alpha_{[g]}}) &= \pi(\mathrm{SPC}_H(x_{[g]}))^\alpha \\
&= \mathrm{SPC}(x)^\alpha \cap H^{ga} \\
&= \mathrm{SPC}(y) \cap H^{g'} \\
&= \pi(\mathrm{SPC}_H(y_{[g']})).
\end{aligned}$$

Since π restricted to $\mathcal{G}_H([g'], [g'])$ is an isomorphism, we deduce that $\mathrm{SPC}_H(x_{[g]})^{\alpha_{[g]}} = \mathrm{SPC}_H(y_{[g']})$, and $\mathcal{T}_H$ is support-preserving. □

Now, we would like to determine the $\mathcal{T}_H$-parabolic subgroups in $\mathcal{G}_H$ in terms of the $\mathcal{T}$-parabolic subgroups in $\mathcal{G}$.

Proposition 5.3.21 (Parabolic subgroups of a finite index subgroup). *Let $g \in \mathcal{G}(u,v)$. The natural isomorphism $\mathcal{G}_H([g],[g]) \simeq H^g$ induces a bijective map*

$$\mathcal{P}_{\mathcal{T}_H}(\mathcal{G}_H([g],[g])) \to \{P \cap H^g \mid P \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))\}.$$

Furthermore, if $\mathcal{T}$ is support-preserving, then the natural map

$$\begin{array}{rcl} \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v)) & \longrightarrow & \{P \cap H^g \mid P \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))\} \\ P & \longmapsto & P \cap H^g \end{array}$$

is bijective.

Proof. Let $E := \{P \cap H^g \mid P \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))\}$. Let $P \subset \mathcal{G}_H([g],[g])$ be a $\mathcal{T}_H$-parabolic subgroup. By definition, there is some $f_{[g]} \in \mathcal{G}_H([g],[g'])$ such that $P^{f_{[g]}} = (\mathcal{G}_H)_{\delta_H}([g'],[g'])$. Taking the image under π yields

$$\begin{aligned} \pi(P) &= \pi\left(((\mathcal{G}_H)_{\delta_H}([g'],[g']))^{f_{[g]}^{-1}}\right) = \pi\left((\mathcal{G}_H)_{\delta_H}([g'],[g'])\right)^{f^{-1}} \\ &= \left(\mathcal{G}_\delta(v',v') \cap H^{g'}\right)^{f^{-1}} = (\mathcal{G}_\delta(v',v'))^{f^{-1}} \cap H^g, \end{aligned}$$

where v' is the target of g'. In particular, the isomorphism $\mathcal{G}_H([g],[g]) \simeq H^g$ induces a map $\mathcal{P}_{\mathcal{T}_H}(\mathcal{G}_H([g],[g])) \to E$. Since this map stems from a group isomorphism, it is injective and we just have to show that its image is precisely E.

Let us show that this map is surjective. Let $g \in \mathcal{G}(u,v)$, and let $P \subset \mathcal{G}(v,v)$ be a $\mathcal{T}$-parabolic subgroup. By definition, there is some $f \in \mathcal{G}(v,v')$ such that $P^f = \mathcal{G}_\delta(v',v')$. By Proposition 5.3.20, there is a $\mathcal{T}_H$-standard parabolic subgroup of $\mathcal{G}_H([gf],[gf])$ whose image under π is $\mathcal{G}_\delta(v',v') \cap H^{gf}$. Conjugating this $\mathcal{T}_H$-standard parabolic subgroup by $f_{[gf]}^{-1}$ yields a $\mathcal{T}_H$-parabolic subgroup of $\mathcal{G}_H([g],[g])$ whose image under π is $(\mathcal{G}_\delta(v',v'))^{f^{-1}} \cap H^g = P \cap H^g$.

Lastly, assume that $\mathcal{T}$ is support-preserving. The natural map $\mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v)) \to E$ is obviously surjective, and we just have to show that it is injective. Let then $P, P' \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))$ be such that $P \cap H^g = P' \cap H^g$. Let $f \in \mathcal{G}(v,v')$ be such that $P^f = \mathcal{G}_\delta(v',v')$ is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(v',v')$. By Corollary 5.2.7, there is a positive endomorphism $x \in \mathcal{G}(v',v')$ such that $\mathcal{G}_\delta(v',v') = \mathrm{SPC}(x) = \mathrm{PC}(x)$. If we write $y := x^f$, then $P = \mathrm{PC}(y)$ by to Lemma 5.2.16. Since H^g has finite index in $\mathcal{G}(v,v)$, there is some positive power y^m of y which lies in H^g. We then have $y^m \in P' \cap H^g \subset P'$ and $P = \mathrm{PC}(y^m) \subset P'$ by Proposition 5.2.18 ($\mathcal{T}$-parabolic closure of a power). Likewise, we deduce that $P' \subset P$ and $P = P'$. □

A first consequence of this proposition is that the natural isomorphism $\mathcal{G}_H([g],[g]) \simeq \mathcal{G}(v,v)$ somehow preserves parabolic closure.

Corollary 5.3.22. *Assume that $\mathcal{T}$ is support-preserving, and let $x \in \mathcal{G}_H$ be an endomorphism with source $[g] \in \mathrm{Ob}(\mathcal{G}_H)$. Let us denote by $\mathrm{PC}_H(x)$ (resp. $\mathrm{PC}(\pi(x))$) the $\mathcal{T}_H$-parabolic closure of x in $\mathcal{G}_H$ (resp. the $\mathcal{T}$-parabolic closure of $\pi(x)$ in $\mathcal{G}$). The natural isomorphism $\mathcal{G}_H([g],[g]) \simeq H^g$ identifies $\mathrm{PC}_H(x)$ with $\mathrm{PC}(\pi(x)) \cap H^g$.*

Proof. Let v be the target of g in $\mathcal{G}$. Since $\mathcal{T}$ is support-preserving, there is a unique $\mathcal{T}$-parabolic subgroup P of $\mathcal{G}(v,v)$ such that $P \cap H^g = \pi(\mathrm{PC}_H(x))$, and we have to show that $P = \mathrm{PC}(\pi(x))$. Since P is a parabolic subgroup which contains $\pi(x)$, we naturally have $\mathrm{PC}(\pi(x)) \subset P$ and $\mathrm{PC}(\pi(x)) \cap H^g \subset P \cap H^g$. Conversely, $\mathrm{PC}(\pi(x)) \cap H^g$ is identified with a $\mathcal{T}_H$-parabolic subgroup of $\mathcal{G}_H([g],[g])$ containing x, thus this parabolic subgroup contains $\mathrm{PC}_H(x)$, and we have $P \cap H^g \subset \mathrm{PC}(\pi(x)) \cap H^g$. Thus we have $P \cap H^g = \mathrm{PC}(\pi(x)) \cap H^g$ and $P = \mathrm{PC}(\pi(x))$ by Proposition 5.3.21. $\square$

We also obtain as a corollary that parabolic subgroups in $\mathcal{G}_H$ are stable under intersection if the parabolic subgroups in $\mathcal{G}$ are stable under intersection.

Corollary 5.3.23. *Assume that $\mathcal{T}$ is support-preserving, and let $g \in \mathcal{G}(u,v)$. If $\mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))$ is stable under intersection, then so is $\mathcal{P}_{\mathcal{T}_H}(\mathcal{G}_H([g],[g]))$.*

Proof. Let $\{P_i\}_{i\in I}$ be a family in $\mathcal{P}_{\mathcal{T}_H}(\mathcal{G}_H([g],[g]))$. For $i \in I$, the isomorphism $\mathcal{G}_H([g],[g]) \simeq \mathcal{G}(v,v) \cap H^g$ identifies P_i with $P_i' \cap H^g$, where $P_i' \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(v,v))$. It also identifies $P := \bigcap_{i\in I} P_i$ with

$$\bigcap_{i\in I} P_i' \cap H^g = H^g \cap \left(\bigcap_{i\in I} P_i'\right).$$

By assumption, $P' := \bigcap_{i\in I} P_I'$ is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(v,v)$, and thus P is a $\mathcal{T}_H$-parabolic subgroup of $\mathcal{G}_H([g],[g])$. $\square$

Let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$. Let $(\mathcal{G}_H)_{\delta_H} \in \mathcal{T}_H$, and let $g \in \mathcal{G}(u,v)$ be such that $\delta_H([g])$ is defined. Since H^g has finite index in $\mathcal{G}(v,v)$, there is a smallest positive integer n such that $(z_\delta(v)^n)_{[g]} \in \mathcal{G}_H([g],[g])$, and we can set $z_{\delta_H}([g]) := (z_\delta(v)^n)_{[g]}$.

Proposition 5.3.24 (System of conjugacy representatives). *Assume that $\mathcal{T}$ is support-preserving, and let $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$ be a system of conjugacy representatives for $\mathcal{T}$. The set $\{z_{\delta_H} \mid (\mathcal{G}_H)_{\delta_H} \in \mathcal{T}_H\}$ constructed above is a system of conjugacy representatives for $\mathcal{T}_H$.*

Proof. Let $(\mathcal{G}_H)_{\delta_H} \in \mathcal{T}_H$. First, we have to show that z_{δ_H} is a natural transformation from $1_{(\mathcal{C}_H)_{\delta_H}}$ to itself. Let $g \in \mathcal{G}(u,v)$ be such that $\delta_H([g])$ is defined, and let $f \in \mathcal{G}(v,v')$ be such that $f_{[g]} \in (\mathcal{G}_H)_{\delta_H}$. By Lemma 5.3.19, we have $f \in \mathcal{G}_\delta$, and thus $z_\delta(v)^k f = f z_\delta(v')^k$ for all integer k since z_δ is a natural transformation $1_{\mathcal{C}_\delta} \Rightarrow 1_{\mathcal{C}_\delta}$. Let $n > 0$ be an integer such that $(z_\delta(v)^n)_{[g]} \in \mathcal{G}_H([g],[g])$. We have

$$(z_\delta(v)^n f)_{[g]} = (z_\delta(v)^n)_{[g]} f_{[g]} = (f z_\delta(v')^n)_{[g]} = f_{[g]}(z_\delta(v')^n)_{[gf]} \in \mathcal{G}_H([gf],[gf]).$$

If $n' > 0$ is the smallest integer such that $(z_\delta(v')^{n'})_{[gf]} \in \mathcal{G}_H([gf],[gf])$, then we have $n' \leqslant n$. Likewise, using f^{-1}, we obtain $n \leqslant n'$ and $n = n'$. We then have

$$\begin{aligned}
z_{\delta_H}([g]) f_{[g]} &= (z_\delta(v)^n)_{[g]} f_{[g]} \\
&= (z_\delta(v)^n f)_{[g]} \\
&= (f z_\delta(v')^n)_{[g]} \\
&= f_{[g]}(z_\delta(v')^n)_{[gf]} \\
&= f_{[g]} z_{\delta_H}([gf]),
\end{aligned}$$

and z_{δ_H} is a natural transformation $1_{(\mathcal{G}_H)_{\delta_H}} \Rightarrow 1_{(\mathcal{G}_H)_{\delta_H}}$, which restricts to a natural transformation $1_{(\mathcal{C}_H)_{\delta_H}} \Rightarrow 1_{(\mathcal{C}_H)_{\delta_H}}$.

Let $g \in \mathcal{G}(u,v)$ and $g' \in \mathcal{G}(u,v')$, and let $(\mathcal{G}_H)_{\delta_H}([g],[g])$ and $(\mathcal{G}_H)_{\delta_H'}([g'],[g'])$ be two $\mathcal{T}_H$-standard parabolic subgroups in $\mathcal{G}_H$. For $f_{[g]} \in \mathcal{G}_H([g],[g'])$, and by Proposition 5.3.21, we have

$$
\begin{aligned}
((\mathcal{G}_H)_{\delta_H}([g],[g]))^{f_{[g]}} = (\mathcal{G}_H)_{\delta'_H}([g'],[g']) &\Leftrightarrow \pi\left(((\mathcal{G}_H)_{\delta_H}([g],[g]))^{f_{[g]}}\right) = \pi\left((\mathcal{G}_H)_{\delta'_H}([g'],[g'])\right) \\
&\Leftrightarrow (\mathcal{G}_\delta(v,v) \cap H^g)^f = \mathcal{G}_{\delta'}(v',v') \cap H^{g'} \\
&\Leftrightarrow (\mathcal{G}_\delta(v,v))^f \cap H^{gf} = \mathcal{G}_{\delta'}(v',v') \cap H^{g'} \\
&\Leftrightarrow (\mathcal{G}_\delta(v,v))^f = \mathcal{G}_{\delta'}(v',v').
\end{aligned}
$$

By definition of a system of conjugacy representatives, this is also equivalent to $(z_\delta(v))^f = z_{\delta'}(v')$. Let $n > 0$ (resp. $n' > 0$) be the smallest integer such that $(z_\delta(v)^n)_{[g]} \in \mathcal{G}_H([g],[g])$ (resp. $(z_{\delta'}(v')^{n'})_{[g']} \in \mathcal{G}_H([g'],[g'])$). If $(z_\delta(v))^f = z_{\delta'}(v')$, then $(z_\delta(v)^n)^f = z_{\delta'}(v')^n \in H^{gf}$. In particular, $((z_\delta(v)^n)^f)_{[g']} = (z_{\delta'}(v')^n) \in \mathcal{G}_H([g'],[g'])$, and $n' \leqslant n$. Likewise, using f^{-1}, we obtain $n \leqslant n'$ and $n = n'$. In particular, we have $(z_{\delta_H}([g]))^{f_{[g]}} = z_{\delta'_H}([g'])$. Conversely, assume that $(z_{\delta_H}([g]))^{f_{[g]}} = z_{\delta'_H}([g'])$, we have $(z_\delta(v)^n)^f = z_{\delta'}(v)^{n'}$. By support-preservingness of $\mathcal{T}$, and by Proposition 5.2.18 ($\mathcal{T}$-parabolic closure of a power), we have

$$
(\mathcal{G}_\delta(v,v))^f = \mathrm{PC}(z_\delta(v))^f = \mathrm{PC}(z_\delta(v)^n)^f = \mathrm{PC}(z'_\delta(v)^{n'}) = \mathrm{PC}(z_{\delta'}(v)) = \mathcal{G}_{\delta'}(v',v'),
$$

and thus $f_{[g]}$ conjugates $(\mathcal{G}_H)_{\delta_H}([g],[g])$ to $(\mathcal{G}_H)_{\delta'_H}([g'],[g'])$.

Lastly, we have to show that $(\mathcal{G}_H)_{\delta_H}([g],[g])$ is the $\mathcal{T}_H$-standard parabolic closure of $z_{\delta_H}([g])$. By Proposition 5.3.20, we have that $\mathcal{T}_H$ is support-preserving. Since $z_{\delta_H}([g])$ is positive, we have to show that $(\mathcal{G}_H)_{\delta_H}([g],[g])$ is the $\mathcal{T}$-parabolic closure of $z_{\delta_H}([g])$, which is a consequence of Corollary 5.3.22 and Proposition 5.2.18. □

Corollary 5.3.25. *Under the assumptions of Proposition 5.3.24, let $g \in \mathcal{G}(u,v)$, and let $P_1, P_2 \in \mathcal{P}_{\mathcal{T}_H}(\mathcal{G}_H([g],[g]))$. Let also $P'_1, P'_2 \in \mathcal{P}_T(\mathcal{G}(v,v))$ be such that P_i is identified with $P'_i \cap H^g$ by the isomorphism $\mathcal{G}_H([g],[g]) \simeq H^g$ for $i = 1,2$. We have $z_{P_1} z_{P_2} = z_{P_2} z_{P_1}$ if and only if $z_{P'_1} z_{P'_2} = z_{P'_2} z_{P'_1}$.*

Proof. First, let $f_{[g]} \in \mathcal{G}_H([g],[g'])$ such that $P_1^{f_{[g]}} = (\mathcal{G}_H)_{\delta_H}([g'],[g'])$ is a standard parabolic subgroup. By definition, we have $(z_{P_1})^{f_{[g]}} = z_{\delta_H}([g'])$. We also saw in the proof of Theorem 5.3.21 that $P_1'^f = \mathcal{G}_\delta(v',v')$, where v' is the target of g in $\mathcal{G}$. Since, by construction, $z_{\delta_H}([g]) = (z_\delta(v)^n)_{[g]}$ for some integer n, we obtain that the isomorphism $P_1 \simeq P'_1 \cap H^g$ identifies z_{P_1} with $z^n_{P'_1}$. Likewise, we obtain that z_{P_2} is identified with $z^m_{P'_2}$ for some integer m. It is then sufficient to show that $z^n_{P'_1}$ and $z^m_{P'_2}$ commute in $\mathcal{G}_H([g],[g])$ if and only if $z_{P'_1}$ and $z_{P'_2}$ do.

The if part is immediate. Conversely, assume that $z^n_{P'_1}$ and $z^m_{P'_2}$ commute. By Proposition 5.2.30, we have $P'_i = \mathrm{PC}(z_{P'_i})$ for $i = 1,2$. By Proposition 5.2.18 ($\mathcal{T}$-parabolic closure of a power), we obtain

$$
P_2'^{(z^n_{P'_1})} = \mathrm{PC}(z_{P'_2})^{(z^n_{P'_1})} = \mathrm{PC}(z^m_{P'_2})^{(z^n_{P'_1})} = \mathrm{PC}(z^m_{P'_2}) = P'_2,
$$

and thus $z_{P'_2} z^n_{P'_1} = z^n_{P'_1} z_{P'_2}$ by Proposition 5.2.31 (characterization of conjugacy of parabolic subgroups). Likewise, we then obtain

$$
P_1'^{z_{P'_2}} = \mathrm{PC}(z^n_{P'_1})^{z_{P'_2}} = \mathrm{PC}(z^n_{P'_1}) = P'_1,
$$

and thus $z_{P'_1} z_{P'_2} = z_{P'_2} z_{P'_1}$. □

In Example 5.1.30, we saw that the intersection of two standard parabolic subgroupoids in a Garside groupoid may not be a standard parabolic subgroupoid. We constructed the notion of a shoal to get around this problem. In the following example, we define a shoal for the Garside category of Example 5.1.30 using the construction of a shoal for a finite index subgroup.

Example 5.3.26. Consider the monoid $M = \langle a, b, c \mid a^3 = b^3 = c^3 \rangle^+$. It is a Garside monoid with Garside element $\Delta = a^3$. The normal closure N in the group $G(M)$ of $\{a^2, ab^{-1}, ac^{-1}\}$ is a finite index subgroup, and the associated groupoid of coset $(\mathcal{G}, \mathcal{C}, \Delta_N)$ is simply the groupoid considered in Example 5.1.30, generated by the oriented graph

$$\begin{tikzcd} u \arrow[r, bend left=60, "a"] \arrow[r, bend left=10, "b"] \arrow[r, bend right=60, "c"'] & v \arrow[l, bend right=40, "\alpha"'] \arrow[l, bend left=10, "\beta"] \arrow[l, bend left=40, "\gamma"] \end{tikzcd}$$

and endowed with the relations $a\alpha a = b\beta b = c\gamma c$ and $\alpha a\alpha = \beta b\beta = \gamma c\gamma$.

One readily checks that the parabolic Garside elements of M are 1 and Δ, thus the shoal of all standard parabolic subgroups of $G(M)$ is $\{\{1\}, G(M)\}$. The shoal $\mathcal{T}_N$ for $(\mathcal{G}, \mathcal{C}, \Delta_N)$ is then simply $\{\{1_u, 1_v\}, \mathcal{G}\}$. The $\mathcal{T}_N$-parabolic subgroups of $\mathcal{G}(u, u)$ are then $\{\{1_u\}, \mathcal{G}(u, u)\}$, and they are stable under intersection.

5.3.4 Shoals for conjugacy graphs

In this subsection, we fix a finite conjugacy set Γ for some endomorphism in $\mathcal{G}$.

In Section 4.3, we defined a Garside groupoid $(\mathcal{G}_\Gamma, \mathcal{C}_\Gamma, \Delta_\Gamma)$, which is equivalent to the centralizer of some $x \in \Gamma$ in $\mathcal{G}$. This groupoid naturally comes equipped with two functors, on the one hand a functor $I : \mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$, which is an isomorphism on its image by Corollary 4.3.7 (conjugacy graph and conjugacy category), and on the other hand a functor $\pi : \mathcal{G}_\Gamma \to \mathcal{G}$, which preserves normal forms, gcds and lcms by Proposition 4.3.9 (preservation of lattice structures). Furthermore, if we denote by p the natural projection $\mathrm{Conj}(\mathcal{G}) \to \mathcal{G}$, then we have $p \circ I = \pi$.

In order to define a shoal of standard parabolic subgroups of $\mathcal{G}_\Gamma$, we will need several assumptions on $\mathcal{T}$ (on top of support-preservingness). Before we work under such assumptions, we give a few general result on some parabolic Garside maps existing in $\mathcal{G}_\Gamma$.

Let δ be a parabolic Garside map in $\mathcal{G}$. In order to define an associated parabolic Garside map in $\mathcal{G}_\Gamma$, we first need to define the associated object set. We define

$$\mathcal{O}_\delta := \{x \in \Gamma \mid u \in \mathrm{Ob}(\mathcal{G}_\delta) \text{ and } \exists k > 0, \delta^k(u) \in \mathcal{C}_\mathcal{G}(x)\},$$

where u denotes the source of x. This set can be difficult to compute in practice, but under further assumptions on the shoal we consider on $\mathcal{G}$, we will be able to give a more convenient characterization.

Lemma 5.3.27. *Let δ be a parabolic Garside map in $\mathcal{G}$ such that the set $\mathcal{O}_\delta$ is nonempty. The map δ_Γ defined for $x \in \mathcal{O}_\delta$ by $\delta_\Gamma(x) := (\delta(u))_x$ (where u is the source of x in $\mathcal{G}$) is a parabolic Garside map for $\mathcal{G}$.*

Proof. First, let $x \in \mathcal{O}_\delta$, and let u be the source of x in $\mathcal{G}$. By definition, there is a positive integer k such that $\delta^k(u)x = x\delta^k(u)$. By definition of a conjugacy set, $\alpha(\delta^k(u)) = \delta(u)$ is such that $y := x^{\delta(u)} \in \Gamma$. In particular, $\delta_\Gamma(x) = (\delta(u))_x$ exists in $\mathcal{S}_\Gamma$.

Let $x, y \in \mathcal{O}_\delta$ with u, v their respective sources in $\mathcal{G}$, and let $s_x \in \mathcal{S}_\Gamma(x,y)$. Since $y \in \mathcal{O}_\delta$, there is an integer k such that $y\delta^k(v) = \delta^k(v)y$. Thus $y^{\delta^{k-1}(v)} = y^{\delta^{-1}(v)} \in \mathcal{O}_\delta$. By Proposition 4.3.9 (preservation of lattice structures), we have

$$\begin{aligned} s_x \preccurlyeq \delta_\Gamma(x) &\Leftrightarrow s \preccurlyeq \delta(u) \Leftrightarrow \delta(\varphi^{-1}(v)) \succcurlyeq s \\ &\Leftrightarrow \delta(\varphi^{-1}(v))_{y^{(\delta(\varphi^{-1}(v))^{-1}}} \succcurlyeq s_x \\ &\Leftrightarrow \delta(\varphi^{-1}(v))_{y^{\delta^{-1}(v)}} \succcurlyeq s_x \\ &\Leftrightarrow \delta_\Gamma(y^{\delta^{-1}(v)}) \succcurlyeq s_x, \end{aligned}$$

and thus δ_Γ is a balanced map. Furthermore, we also obtain that $\mathrm{Div}(\delta_\Gamma)$ is made of the $s_x \in \mathcal{S}_\Gamma$ such that $x \in \mathcal{O}_\delta$ and $s \in \mathcal{S}_\delta$.

Lastly, let $s_x, t_y \in \mathrm{Div}(\delta_\Gamma)$ be such that $s_x t_y \in \mathcal{S}_\Gamma$. We have $st \in \mathcal{S}_\delta$ since δ is a parabolic Garside map, and $(st)_x \in \mathrm{Div}(\delta_\Gamma)$ since $x \in \mathcal{O}_\delta$ and since $st \in \mathcal{S}_\delta$. Thus δ_Γ is a parabolic Garside map. □

As stated before, the groupoid $\mathcal{G}_\Gamma$ comes equipped with two functors I and π. The two following lemmas describe how these functors acts on standard parabolic subgroupoids of the form $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$.

Lemma 5.3.28. *Let δ be a parabolic Garside map in $\mathcal{G}$ such that $\mathcal{O}_\delta$ is nonempty. The functor $I : \mathcal{G}_\Gamma \to \mathrm{Conj}(\mathcal{G})$ identifies $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$ (resp. $(\mathcal{C}_\Gamma)_{\delta_\Gamma}$) with the subcategory of $p^{-1}(\mathcal{G}_\delta)$ (resp. of $p^{-1}(\mathcal{C}_\delta)$) whose objects are the elements of $\mathcal{O}_\delta$.*

Proof. By definition, for $x \in \Gamma$, we have $I(x) = x$ as an endomorphism in $\mathcal{G}$. Thus the object set of $I((\mathcal{G}_\Gamma)_{\delta_\Gamma})$ is $\mathcal{O}_\delta$.

For $f \in \mathcal{C}_\Gamma(x, -)$ with $x \in \mathcal{O}_\delta$, we have $I(f) \in p^{-1}(\mathcal{C}_\delta)$ if and only if $\pi(f) = p(I(f)) \in \mathcal{G}_\delta$ since $\pi = p \circ I$. We have $f \in (\mathcal{C}_\Gamma)_{\delta_\Gamma}$ if and only if f is a product of elements of $\mathrm{Div}(\delta_\Gamma)$. Furthermore, we saw in the proof of Lemma 5.3.27 that the set of divisors of δ_Γ is given by

$$\mathrm{Div}(\delta_\Gamma) = \{s_x \mid x \in \mathcal{O}_\delta \text{ and } s = \pi(s_x) \in \mathrm{Div}(\delta)\}.$$

Thus, if f is a product of elements of $\mathrm{Div}(\delta_\Gamma)$, then $\pi(f)$ is a product of elements of $\mathrm{Div}(\delta)$, and $\pi(f) \in \mathcal{C}_\delta$. Conversely, assume that $\pi(f)$ lies in $\mathcal{C}_\delta$. Since π preserves greedy normal forms by Proposition 4.3.10 (preservation of normal forms), we have that f is a product of elements of $\pi^{-1}(\mathcal{S}_\delta)$, say $f = s_1 \cdots s_r$. Since the source x of s_1 lies in $\mathcal{O}_\delta$, $s_1 \in \mathrm{Div}(\delta_\Gamma)$, and its target belongs to $\mathcal{O}_\delta$. By an immediate induction, we obtain that $s_i \in \mathrm{Div}(\delta_\Gamma)$ for all $i \in [\![1, r]\!]$, and thus f is a product of elements of $\mathrm{Div}(\delta_\Gamma)$. A similar reasoning proves the result for $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$. □

Lemma 5.3.29. *Let δ be a parabolic Garside map in $\mathcal{G}$ such that $\mathcal{O}_\delta$ is nonempty. Let also $x \in \Gamma$, and let u be the source of x in $\mathcal{G}$. The group isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_{\mathcal{G}}(x)$ induced by π identifies $(\mathcal{G}_\Gamma)_{\delta_\Gamma}(x,x)$ with $\mathcal{G}_\delta(u,u) \cap C_{\mathcal{G}}(x)$.*

Proof. Recall that π induces a group isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_{\mathcal{G}}(x)$ after Remark 4.3.8. Let $f \in (\mathcal{G}_\Gamma)_{\delta_\Gamma}(x,x)$, we can write $a^{-1}b$ for the reduced left-fraction decomposition of f. We have $a, b \in (\mathcal{C}_\Gamma)_{\delta_\Gamma}$, and thus both a and b can be written as a product of elements of $\mathrm{Div}(\delta_\Gamma)$. We saw in the proof of Lemma 5.3.27 that the set of divisors of δ_Γ is given by

$$\mathrm{Div}(\delta_\Gamma) = \{s_x \mid x \in \mathcal{O}_\delta \text{ and } s = \pi(s_x) \in \mathrm{Div}(\delta)\}.$$

In particular, both $\pi(a)$ and $\pi(b)$ can be written as products of elements of $\mathrm{Div}(\delta)$, and $\pi(f) = \pi(a)^{-1}\pi(b) \in \mathcal{G}_\delta(u,u)$.

Conversely, let $g \in \mathcal{G}_\delta(u,u) \cap C_{\mathcal{G}}(x)$. There is a unique $f \in \mathcal{G}_\Gamma(x,x)$ such that $\pi(f) = g$. Since $g \in \mathcal{G}_\delta(u,u)$, all the terms of the symmetric normal form of g lie in $\mathrm{Div}(\delta)$. Since π preserves symmetric normal forms, we obtain that all the terms of the symmetric normal form of f in $\mathcal{G}_\Gamma$ lie in $\pi^{-1}(\mathrm{Div}(\delta))$. In the proof of Lemma 5.3.27, we saw that an element of $\pi^{-1}(\mathrm{Div}(\delta))$ belongs to $\mathrm{Div}(\delta_\Gamma)$ if and only if either its source or its target belongs to $\mathcal{O}_\delta$. Since $x \in \mathcal{O}_\delta$, an immediate induction proves that all the terms of the symmetric normal form of f in $\mathcal{G}_\Gamma$ belongs to $\mathrm{Div}(\delta_\Gamma)$, and thus $f \in (\mathcal{G}_\Gamma)_{\delta_\Gamma}(x,x)$ as claimed. □

Another easy result on the parabolic Garside maps defined using Lemma 5.3.27 is the action of the Garside automorphism ϕ_Γ of $\mathcal{G}_\Gamma$.

Lemma 5.3.30. *Let δ be a parabolic Garside map in $\mathcal{G}$ such that $\mathcal{O}_\delta$ is nonempty. The parabolic Garside map $\phi_\Gamma(\delta_\Gamma)$ is equal to $(\phi(\delta))_\Gamma$.*

Proof. First, the object set of $(\mathcal{G}_\Gamma)_{\phi_\Gamma(\delta_\Gamma)}$ is equal to $\phi_\Gamma(\mathcal{O}_\delta)$ by definition. Let $x \in \Gamma$, we have

$$\begin{aligned} x \in \mathcal{O}_{\phi(\delta)} &\Leftrightarrow \exists k > 0 \mid \phi(\delta)^k(u)x = x\phi(\delta)^k(u) \\ &\Leftrightarrow \exists k > 0 \mid \phi(\delta^k(\phi^{-1}(u)))x = x\phi(\delta^k(\phi^{-1}(u))) \\ &\Leftrightarrow \exists k > 0 \mid \delta^k(\phi^{-1}(u))\phi^{-1}(x) = \phi^{-1}(x)\delta^k(\phi^{-1}(u)) \\ &\Leftrightarrow \phi^{-1}(x) \in \mathcal{O}_\delta, \end{aligned}$$

and thus $\mathcal{O}_{\phi(\delta)} = \phi_\Gamma(\mathcal{O}_\delta)$ is also the object set of $(\mathcal{G}_\Gamma)_{(\phi(\delta))_\Gamma}$. For $x \in \mathcal{O}_{\phi(\delta)}$, we have

$$\begin{aligned} \phi_\Gamma(\delta_\Gamma)(x) &= \phi_\Gamma(\delta_\Gamma(\phi_\Gamma^{-1}(x))) = \phi_\Gamma\left(\delta(\phi^{-1}(u))_{\phi^{-1}(x)}\right) \\ &= \left(\phi(\delta(\phi^{-1}(u)))\right)_x = (\phi(\delta))(u)_x \\ &= (\phi(\delta))_\Gamma(x). \end{aligned}$$

And thus the two considered parabolic Garside maps are equal. □

A first intuition to define a shoal for $\mathcal{G}_\Gamma$ would be to consider all Garside maps of the form δ_Γ, where $\mathcal{G}_\delta \in \mathcal{T}$. This gives a set of standard parabolic subgroupoids which is stable under ϕ, and which contains both $\mathcal{G}_\Gamma = (\mathcal{G}_\Gamma)_{\Delta_\Gamma}$ and $\{1_x\}_{x \in \mathrm{Ob}(\mathcal{G}_\Gamma)}$. However, it is hard to see whether or not it is stable under intersection. For instance, one should be able to relate $\mathcal{O}_\delta \cap \mathcal{O}_{\delta'}$ and $\mathcal{O}_{\delta \wedge \delta'}$ for $\mathcal{G}_\delta, \mathcal{G}_{\delta'} \in \mathcal{T}$. This is where we add assumptions on the shoal $\mathcal{T}$.

From now on, we assume that $\mathcal{T}$ is support-preserving, and that defining $z_\delta := \delta^n$ for $\mathcal{G}_\delta \in \mathcal{T}$, where n is the order of φ, gives a system of conjugacy representatives $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$. Under this assumption, we can give a characterization of the set $\mathcal{O}_\delta$ associated to $\mathcal{G}_\delta \in \mathcal{T}$.

Lemma 5.3.31. *Let $x \in \Gamma$, and let u be the source of x in $\mathcal{G}$. For $\mathcal{G}_\delta \in \mathcal{T}$ such that $u \in \mathrm{Ob}(\mathcal{G}_\delta)$, the following assertions are equivalent:*

(i) $x \in \mathcal{O}_\delta$.

(ii) *x normalizes $\mathcal{G}_\delta(u,u)$.*

(iii) *x commutes with $z_\delta(u)$.*

Proof. $(iii) \Rightarrow (i)$ is immediate since $z_\delta(u)$ has the form $\delta^n(u)$.
$(ii) \Leftrightarrow (iii)$ comes from Proposition 5.2.31 (characterization of conjugacy of parabolic subgroups) since $z_\delta(u) = z_{\mathcal{G}_\delta(u,u)}$.
$(i) \Rightarrow (ii)$ Let $k > 0$ be such that $x\delta^k(u) = \delta^k(u)x$, and let a, b be integers such that $na = kb$, where $z_\delta(u) = \delta^n(u)$. We have that x commutes with $\delta^{kb}(u) = (z_\delta(u))^a$. Applying Lemma 5.2.16, Proposition 5.2.18 ($\mathcal{T}$-parabolic closure of a power) and Proposition 5.2.30 then yields

$$\begin{aligned}(\mathcal{G}_\delta(u,u))^x &= \mathrm{PC}(z_\delta(u))^x = \mathrm{PC}((z_\delta(u))^a)^x \\ &= \mathrm{PC}((z_\delta(u))^a) = \mathrm{PC}(z_\delta(u)) \\ &= \mathcal{G}_\delta(u,u),\end{aligned}$$

and x normalizes $\mathcal{G}_\delta(u,u)$. □

In order to define a shoal for $\mathcal{G}_\Gamma$, we need to consider decompositions of standard parabolic subgroupoids as disjoint unions, as in Lemma 5.1.26 (standard parabolic subgroupoids and disjoint unions). Indeed, we will consider the set of disjoint factors of parabolic subgroupoids of $\mathcal{G}_\Gamma$ of the form $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$ with $\mathcal{G}_\delta \in \mathcal{T}$.

Definition 5.3.32 (Shoal for conjugacy graph). We define a set $\mathcal{T}_\Gamma$ of subgroupoids of $\mathcal{G}_\Gamma$ in the following way: A subgroupoid $\mathcal{G}_1$ of $\mathcal{G}_\Gamma$ belongs to $\mathcal{T}_\Gamma$ if and only if there exists a subgroupoid $\mathcal{G}_2$ of $\mathcal{G}_\Gamma$ such that $\mathcal{G}_1 \cap \mathcal{G}_2 = \varnothing$ and $\mathcal{G}_1 \sqcup \mathcal{G}_2 = (\mathcal{G}_\Gamma)_{\delta_\Gamma}$, with $\mathcal{G}_\delta \in \mathcal{T}$.

By Lemma 5.1.26 (standard parabolic subgroupoids and disjoint unions), the elements of $\mathcal{T}_\Gamma$ are indeed standard parabolic subgroupoids of $\mathcal{G}_\Gamma$, and $\mathcal{T}_\Gamma$ is stable under disjoint factors. We have $\mathcal{G}_\Gamma = (\mathcal{G}_\Gamma)_{\Delta_\Gamma} \in \mathcal{T}_\Gamma$, and $\{1_x\}_{x\in\Gamma} \in \mathcal{T}_\Gamma$ (by taking $\delta(u) = 1_u$ for all $u \in \mathrm{Ob}(\mathcal{G})$). Moreover, if $\mathcal{G}_1 \in \mathcal{T}_\Gamma$ is a disjoint factor of some $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$, then $\phi_\Gamma(\mathcal{G}_1)$ is a disjoint factor of $\phi_\Gamma((\mathcal{G}_\Gamma)_{\delta_\Gamma}) = (\mathcal{G}_\Gamma)_{\phi(\delta)_\Gamma}$ by Lemma 5.3.30. Thus $\phi(\mathcal{G}_1) \in \mathcal{T}_\Gamma$ since $\mathcal{G}_{\phi(\delta)} \in \mathcal{T}$. The complicated part in showing that $\mathcal{T}_\Gamma$ is actually a shoal for $\mathcal{G}_\Gamma$ is showing that it is stable under intersection. We start by showing that the intersection of two elements of $\mathcal{T}_\Gamma$ of the form $(\mathcal{G}_\Gamma)_{\delta_{1\Gamma}}$ and $(\mathcal{G}_\Gamma)_{\delta_{2\Gamma}}$ lies in $\mathcal{T}_\Gamma$.

Lemma 5.3.33. *Let $\mathcal{G}_{\delta_1}, \mathcal{G}_{\delta_2} \in \mathcal{T}$. Then the intersection $(\mathcal{G}_\Gamma)_{\delta_{1\Gamma}} \cap (\mathcal{G}_\Gamma)_{\delta_{2\Gamma}}$ belongs to $\mathcal{T}_\Gamma$.*

Proof. The object set of $(\mathcal{G}_\Gamma)_{\delta_{1\Gamma}} \cap (\mathcal{G}_\Gamma)_{\delta_{2\Gamma}}$ is $\mathcal{O}_{\delta_1} \cap \mathcal{O}_{\delta_2}$, which is nonempty if and only if the intersection we consider is nonempty. In this case, the intersection $\mathcal{G}_{\delta_1} \cap \mathcal{G}_{\delta_2}$ is nonempty, and we can define

$$\mathcal{G}_\delta := \mathcal{G}_{\delta_1} \cap \mathcal{G}_{\delta_2} \in \mathcal{T}$$

since $\mathcal{T}$ is a shoal. We claim that $\mathcal{O}_{\delta_1} \cap \mathcal{O}_{\delta_2} \subset \mathcal{O}_\delta$. Indeed, by Lemma 5.3.31, an element $x \in \Gamma$ belongs to $\mathcal{O}_{\delta_1} \cap \mathcal{O}_{\delta_2}$ if and only if it normalizes both $\mathcal{G}_{\delta_1}(u,u)$ and $\mathcal{G}_{\delta_2}(u,u)$, where u is the source of x in $\mathcal{G}$. In this case, x normalizes $\mathcal{G}_\delta(u,u) = \mathcal{G}_{\delta_1}(u,u) \cap \mathcal{G}_{\delta_2}(u,u)$, and thus $x \in \mathcal{O}_\delta$, again by Lemma 5.3.31.

Let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) denote the full subcategory of $(\mathcal{G}_\Gamma)_{\delta_\Gamma}$ whose objects are the elements of $\mathcal{O} := \mathcal{O}_{\delta_1} \cap \mathcal{O}_{\delta_2}$ (resp. of $\mathcal{O}' := \mathcal{O}_\delta \setminus \mathcal{O}$). We have $\mathcal{G}_1 \cap \mathcal{G}_2 = \varnothing$ by construction. In order to show that $\mathcal{G}_1 \sqcup \mathcal{G}_2 = (\mathcal{G}_\Gamma)_{\delta_\Gamma}$, it is sufficient to show that, for $f \in (\mathcal{G}_\Gamma)_{\delta_\Gamma}$, the source of f belongs to $\mathcal{O}$ if and only if the target of f belongs to $\mathcal{O}$.

Let $x, y \in \Gamma$, and let $s \in \mathcal{S}_\Gamma(x,y) \cap \mathrm{Div}(\delta_\Gamma)$. If $x \in \mathcal{O}$, then $s \preccurlyeq \delta_{1\Gamma}(x)$ and $s \preccurlyeq \delta_{2\Gamma}(x)$. In particular, $y \in \mathcal{O}_{\delta_1}$ and $y \in \mathcal{O}_{\delta_2}$ and $y \in \mathcal{O}$. Likewise, if $y \in \mathcal{O}$, then $x \in \mathcal{O}$. By an immediate induction, we obtain that, for $f \in (\mathcal{G}_\Gamma)_{\delta_\Gamma}$, the source of f belongs to $\mathcal{O}$ if and only if its target belongs to $\mathcal{O}$.

By Lemma 5.3.28, $\mathcal{G}_1$ is equal to $(\mathcal{G}_\Gamma)_{\delta_{1\Gamma}} \cap (\mathcal{G}_\Gamma)_{\delta_{2\Gamma}}$ since the functor I identifies both categories with the full subcategory of $p^{-1}(\mathcal{G}_\delta)$ whose objects set is $\mathcal{O}$. By definition, we then obtain that $\mathcal{G}_1 = (\mathcal{G}_\Gamma)_{\delta_{1\Gamma}} \cap (\mathcal{G}_\Gamma)_{\delta_{2\Gamma}} \in \mathcal{T}_\Gamma$. □

We can now show that $\mathcal{T}_\Gamma$ is a support-preserving shoal for $\mathcal{G}_\Gamma$. Note that the following proposition relies on the assumptions that $\mathcal{T}$ is a support-preserving shoal endowed with a particular system of conjugacy representatives.

Proposition 5.3.34 (Shoal for conjugacy graph). *The set $\mathcal{T}_\Gamma$ is a support-preserving shoal for $\mathcal{G}_\Gamma$. Furthermore, for $x \in \Gamma$, The group isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_\mathcal{G}(x)$ induced by π identifies the $\mathcal{T}_\Gamma$-standard parabolic subgroups of $\mathcal{G}_\Gamma(x,x)$ with the groups of the form $C_\mathcal{G}(x) \cap H$, where H is a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$ which is normalized by x.*

Proof. In order to show that $\mathcal{T}_\Gamma$ is a shoal, it remains to show that the intersection of two elements of $\mathcal{T}_\Gamma$, if nonempty, also lies in $\mathcal{T}_\Gamma$. Let $\mathcal{G}_1, \mathcal{G}_1'$ be two elements of $\mathcal{T}_\Gamma$. Let $\mathcal{G}_2, \mathcal{G}_2'$ be such that $\mathcal{G}_1 \sqcup \mathcal{G}_2 = (\mathcal{G}_\Gamma)_{\delta_\Gamma}$ and $\mathcal{G}_1' \sqcup \mathcal{G}_2' = (\mathcal{G}_\Gamma)_{\delta_\Gamma'}$, where $\mathcal{G}_\delta, \mathcal{G}_{\delta'} \in \mathcal{T}$. We have

$$(\mathcal{G}_\Gamma)_{\delta_\Gamma} \cap (\mathcal{G}_\Gamma)_{\delta_\Gamma'} = \mathcal{G}_1 \cap \mathcal{G}_1' \sqcup \mathcal{G}_1 \cap \mathcal{G}_2' \sqcup \mathcal{G}_2 \cap \mathcal{G}_1' \sqcup \mathcal{G}_2 \cap \mathcal{G}_2'.$$

By Lemma 5.3.33, there is a subcategory $\mathcal{G}_3$ of $\mathcal{G}_\Gamma$ such that

$$(\mathcal{G}_\Gamma)_{\delta_\Gamma} \cap (\mathcal{G}_\Gamma)_{\delta_\Gamma'} \sqcup \mathcal{G}_3 = (\mathcal{G}_\Gamma)_{\delta_{1\Gamma}},$$

with $\mathcal{G}_{\delta_1} \in \mathcal{T}$. The subcategory $\mathcal{G}_4 := \mathcal{G}_1 \cap \mathcal{G}_2' \sqcup \mathcal{G}_2 \cap \mathcal{G}_1' \sqcup \mathcal{G}_2 \cap \mathcal{G}_2' \sqcup \mathcal{G}_3$ is then a subcategory of $\mathcal{G}_\Gamma$ such that

$$(\mathcal{G}_1 \cap \mathcal{G}_1') \sqcup \mathcal{G}_4 = (\mathcal{G}_\Gamma)_{\delta_\Gamma} \cap (\mathcal{G}_\Gamma)_{\delta_\Gamma'} \sqcup \mathcal{G}_3 = (\mathcal{G}_\Gamma)_{\delta_{1\Gamma}},$$

and thus $\mathcal{G}_1 \cap \mathcal{G}_1'$ belongs to $\mathcal{T}_\Gamma$, which is then a shoal.

We then show the statement on $\mathcal{T}_\Gamma$-standard parabolic subgroups. Let $x \in \Gamma$ have source u in $\mathcal{G}$, and let $H \subset \mathcal{G}_\Gamma(x,x)$ be a $\mathcal{T}_\Gamma$-standard parabolic subgroup. We have $H = \mathcal{G}_1(x,x)$, where $\mathcal{G}_1 \in \mathcal{T}_\Gamma$. There is a subgroupoid $\mathcal{G}_2$ of $\mathcal{G}_\Gamma$, and some $\mathcal{G}_\delta \in \mathcal{T}$ such that $\mathcal{G}_1 \sqcup \mathcal{G}_2 = (\mathcal{G}_\Gamma)_{\delta_\Gamma}$. We then have $H = \mathcal{G}_1(x,x) = (\mathcal{G}_\Gamma)_{\delta_\Gamma}(x,x)$. By Lemma 5.3.29, the isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_\mathcal{G}(x)$ identifies H with $\mathcal{G}_\delta(u,u) \cap C_\Gamma(x)$. Furthermore, $\mathcal{G}_\delta(u,u)$ is normalized by x by Lemma 5.3.31.

Conversely, let $\mathcal{G}_\delta(u,u)$ be a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$ which is normalized by x. By Lemma 5.3.31, we have $x \in \mathcal{O}_\delta$, and $(\mathcal{G}_\Gamma)_{\delta_\Gamma}(x,x)$ is a $\mathcal{T}_\Gamma$-standard parabolic subgroup of $\mathcal{G}_\Gamma(x,x)$, which is identified with $\mathcal{G}_\delta(u,u) \cap C_\mathcal{G}(x)$ by the isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_\mathcal{G}(x)$, again after Lemma 5.3.29.

Lastly, we show that $\mathcal{T}_\Gamma$ is support-preserving. Let $x \in \Gamma$ with source u in $\mathcal{G}$, and let $g \in \mathcal{C}_\Gamma(x,x)$. We denote by $\mathrm{SPC}_\Gamma(g)$ (resp. $\mathrm{SPC}(\pi(g))$) the $\mathcal{T}_\Gamma$-standard parabolic closure of g in $\mathcal{G}_\Gamma(x,x)$ (resp. the $\mathcal{T}$-standard parabolic closure of $\pi(g)$ in $\mathcal{G}(u,u)$, where u is the source of x in $\mathcal{G}$). We show that $\pi(\mathrm{SPC}_\Gamma(g)) = \mathrm{SPC}(\pi(g)) \cap C_\mathcal{G}(x)$. First, let H be a $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(u,u)$ such that $H \cap C_\mathcal{G}(x) = \pi(\mathrm{SPC}_\Gamma(g))$. Since $\pi(g) \in \pi(\mathrm{SPC}_\Gamma(g))$, we have $\pi(g) \in H$ and $\mathrm{SPC}(\pi(g)) \subset H$. Thus $\mathrm{SPC}(\pi(g)) \cap C_\mathcal{G}(x) \subset H \cap C_\mathcal{G}(x) = \pi(\mathrm{SPC}_\Gamma(g))$. Conversely, since $\mathcal{T}$ is support-preserving, we have

$$\mathrm{SPC}(\pi(g))^x = \mathrm{PC}(\pi(g))^x = \mathrm{PC}(\pi(g)^x) = \mathrm{PC}(\pi(g)) = \mathrm{SPC}(\pi(g)),$$

and $\mathrm{SPC}(g)$ is normalized by x. There is then a $\mathcal{T}_\Gamma$-standard parabolic subgroup H of $\mathcal{G}_\Gamma(x,x)$ such that $\pi(H) = \mathrm{SPC}(\pi(g)) \cap C_\mathcal{G}(x)$. We have $g \in H$, and thus $\mathrm{SPC}_\Gamma(g) \subset H$, and $\pi(\mathrm{SPC}_\Gamma(g)) \subset \pi(H) = \mathrm{SPC}(\pi(g)) \cap C_\mathcal{G}(x)$.

Let now $y \in \Gamma$ and $h \in \mathcal{C}_\Gamma(y,y)$. Let also $\alpha \in \mathcal{C}_\Gamma(x,y)$ such that $g^\alpha = h$. By support-preservingness of $\mathcal{T}$, we have

$$\begin{aligned}
\pi(\mathrm{SPC}_\Gamma(g)^\alpha) &= \pi(\mathrm{SPC}_\Gamma(g))^{\pi(\alpha)} \\
&= (\mathrm{SPC}(\pi(g)) \cap C_\Gamma(x))^{\pi(\alpha)} \\
&= \mathrm{SPC}(\pi(g))^{\pi(\alpha)} \cap C_\Gamma(x)^{\pi(\alpha)} \\
&= \mathrm{SPC}(\pi(h)) \cap C_\Gamma(y) \\
&= \pi(\mathrm{SPC}_\Gamma(h)).
\end{aligned}$$

Since π is injective on the set of subgroups of $\mathcal{G}_\Gamma(y,y)$, we deduce that $\mathrm{SPC}_\Gamma(g)^\alpha = \mathrm{SPC}_\Gamma(h)$, and $\mathcal{T}_\Gamma$ is support-preserving. □

Corollary 5.3.35. *Let $x \in \Gamma$, and let $g \in \mathcal{G}_\Gamma(x,x)$. Let us denote by $\mathrm{PC}_\Gamma(g)$ (resp. $\mathrm{PC}(\pi(g))$) the $\mathcal{T}_\Gamma$-parabolic closure of g in $\mathcal{G}_\Gamma$ (resp. the $\mathcal{T}$-parabolic closure of g in $\mathcal{G}$). The natural isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_\mathcal{G}(x)$ identifies $\mathrm{PC}_\Gamma(g)$ with $\mathrm{PC}(\pi(g)) \cap C_\mathcal{G}(x)$.*

Proof. Let $c \in \mathcal{G}_\Gamma(x,y)$ be a conjugating element for iterated swaps such that g^c is recurrent. Since π preserves symmetric normal forms by Proposition 4.3.10 (preservation of normal forms), it commutes with iterated swaps, and $\pi(g^c)$ is recurrent in $\mathcal{G}$. By construction of the parabolic closure (Theorem 5.2.14), we have $\mathrm{PC}_\Gamma(g^c) = \mathrm{SPC}_\Gamma(g^c)$ and $\mathrm{PC}(\pi(g^c)) = \mathrm{SPC}(\pi(g^c))$. We saw in the proof of Proposition 5.3.34 that $\pi(\mathrm{SPC}_\Gamma(g^c)) = \mathrm{SPC}(\pi(g^c)) \cap C_\mathcal{G}(y)$, where y is the source of g^c in $\mathcal{G}_\Gamma$. We then obtain that

$$\begin{aligned}
\pi(\mathrm{PC}_\Gamma(g)) &= \pi\left((\mathrm{PC}_\Gamma(g^c))^{c^{-1}}\right) = \pi\left(\mathrm{PC}_\Gamma(g^c)\right)^{\pi(c^{-1})} = \pi\left(\mathrm{SPC}_\Gamma(g^c)\right)^{\pi(c^{-1})} \\
&= (\mathrm{SPC}(\pi(g^c)) \cap C_\mathcal{G}(y))^{\pi(c^{-1})} \\
&= (\mathrm{SPC}(\pi(g^c)))^{\pi(c^{-1})} \cap (C_\mathcal{G}(y))^{\pi(c^{-1})} = \mathrm{PC}(\pi(g)) \cap C_\mathcal{G}(x),
\end{aligned}$$

which is precisely the desired result. □

Proposition 5.3.36 (Parabolic subgroups in conjugacy graph). *Let $x \in \Gamma$ with source u in $\mathcal{G}$. The natural isomorphism $\mathcal{G}_\Gamma(x,x) \simeq C_\mathcal{G}(x)$ induces a bijective map*

$$\mathcal{P}_{\mathcal{T}_\Gamma}(\mathcal{G}_\Gamma(x,x)) \to \{H \cap C_\mathcal{G}(x) \mid H \in \mathcal{P}_\mathcal{T}(\mathcal{G}(u,u)), H^x = H\}.$$

Furthermore, the natural map $H \mapsto H \cap C_\mathcal{G}(x)$, defined on $\{H \in \mathcal{P}_\mathcal{T}(\mathcal{G}(u,u)) \mid H^x = H\}$ is bijective.

Proof. Since π induces an isomorphism $\mathcal{G}_\Gamma(x,x) \simeq \mathcal{G}(u,u)$, the map on $\mathcal{P}_{\mathcal{T}_\Gamma}(\mathcal{G}_\Gamma(x,x))$ induced by π is injective, and we only have to show that its image is the set

$$\{H \cap C_\mathcal{G}(x) \mid H \in \mathcal{P}_\mathcal{T}(\mathcal{G}(u,u)), H^x = H\}.$$

Let $H \in \mathcal{P}_{\mathcal{T}_\Gamma}(\mathcal{G}_\Gamma(x,x))$. By definition, there is some $f \in \mathcal{G}_\Gamma(x,y)$ such that H^f is a $\mathcal{T}_\Gamma$-standard parabolic subgroup of $\mathcal{G}_\Gamma(y,y)$. By Proposition 5.3.34, we have

$$\pi(H^f) = C_\mathcal{G}(y) \cap \mathcal{G}_\delta(v,v)$$

for some $\mathcal{T}$-standard parabolic subgroup of $\mathcal{G}(v,v)$ which is normalized by y (where v denotes the source of y in $\mathcal{G}$). We then have

$$\begin{aligned}\pi(H) &= \pi(H^f)^{\pi(f^{-1})}\\ &= C_{\mathcal{G}}(y)^{\pi(f^{-1})} \cap (\mathcal{G}_\delta(v,v))^{\pi(f^{-1})}\\ &= C_{\mathcal{G}}(x) \cap (\mathcal{G}_\delta(v,v))^{\pi(f^{-1})}.\end{aligned}$$

Furthermore, we have

$$\left((\mathcal{G}_\delta(v,v))^{\pi(f^{-1})}\right)^x = (\mathcal{G}_\delta(v,v))^{\pi(f^{-1})x} = (\mathcal{G}_\delta(v,v))^{y\pi(f^{-1})} = (\mathcal{G}_\delta(v,v))^{\pi(f^{-1})},$$

and $\pi(H)$ is the intersection with $C_{\mathcal{G}}(x)$ of some $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(u,u)$ which is normalized by x. Conversely, let $H \subset \mathcal{G}(u,u)$ be a $\mathcal{T}$-parabolic subgroup such that $H^x = H$. By Proposition 5.2.30, we have $z_H \in C_{\mathcal{G}}(x)$, and there is a unique $z \in \mathcal{G}_\Gamma(x,x)$ such that $\pi(z) = z_H$. The $\mathcal{T}_\Gamma$-parabolic closure H' of z in $\mathcal{G}_\Gamma(x,x)$ is a $\mathcal{T}_\Gamma$-parabolic subgroup, and we have $\pi(H') = H \cap C_{\mathcal{G}}(x)$ by Corollary 5.3.35. In particular, $H \cap C_{\mathcal{G}}(x)$ is the image under π of a $\mathcal{T}_\Gamma$-parabolic subgroup in $\mathcal{G}_\Gamma$.

Lastly, the natural map $H \mapsto H \cap C_{\mathcal{G}}(x)$ defined on the set $\{H \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u)) \mid H^x = H\}$ is obviously surjective, and we just have to show that it is injective. Let then $H, H' \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u))$ be normalized by x and such that $H \cap C_{\mathcal{G}}(x) = H' \cap C_{\mathcal{G}}(x)$. We have $z_H, z_{H'} \in C_{\mathcal{G}}(x)$ by Lemma 5.3.31, and thus $z_H \in H \cap C_{\mathcal{G}}(x) \subset H'$, which proves that $H = \mathrm{PC}(z_H) \subset H'$. Likewise, we have $z'_H \in H$ and $H' \subset H$, thus $H = H'$. □

Corollary 5.3.37. *Let $x \in \Gamma$ with source u in $\mathcal{G}$. If the set $\mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u))$ is stable under intersection, then so is $\mathcal{P}_{\mathcal{T}_\Gamma}(\mathcal{G}_\Gamma(x,x))$.*

Proof. Let $\{H_i\}_{i\in T}$ be a family in $\mathcal{P}_{\mathcal{T}_\Gamma}(\mathcal{G}_\Gamma(x,x))$. By Proposition 5.3.36, we can, for each $i \in I$, consider some $\widetilde{H_i} \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}(u,u))$, normalized by x, such that $\pi(H_i) = \widetilde{H_i} \cap C_{\mathcal{G}}(x)$. By assumption, the intersection $\widetilde{H} = \bigcap_{i\in I} \widetilde{H_i}$ is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(u,u)$, which is normalized by x by construction. We then have that

$$\pi\left(\bigcap_{i\in I} H_i\right) = \bigcap_{i\in I} \pi(H_i) = \bigcap_{i\in I} \widetilde{H_i} \cap C_{\mathcal{G}}(x) = \widetilde{H} \cap C_{\mathcal{G}}(x).$$

Since $\widetilde{H}$ is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}(u,u)$ which is normalized by x, the intersection of the H_i is a $\mathcal{T}_\Gamma$-parabolic subgroup of $\mathcal{G}_\Gamma(x,x)$ by Proposition 5.3.36. □

Lastly, we define a set of conjugacy representatives for $\mathcal{T}_\Gamma$ using the set $\{z_\delta \mid \mathcal{G}_\delta \in \mathcal{T}\}$.

Let $\mathcal{G}_1 \in \mathcal{T}_\Gamma$, and let $\mathcal{G}_2 \in \mathcal{T}_\Gamma$ such that $\mathcal{G}_1 \cap \mathcal{G}_2 = \varnothing$ and $\mathcal{G}_1 \sqcup \mathcal{G}_2 = (\mathcal{G}_\Gamma)_{\delta_\Gamma}$ for some $\mathcal{G}_\delta \in \mathcal{T}$. For $x \in \mathcal{O}_\delta$ with source u in $\mathcal{G}$. By Lemma 5.3.31, we have $z_\delta(u) \in C_{\mathcal{G}}(x)$, and there is a unique $z_{\delta_\Gamma}(x) \in \mathcal{G}_\Gamma(x,x)$ such that $\pi(z_{\delta_\Gamma}(x)) = z_\delta(u)$. This allows us to define a natural transformation $z_{\delta_\Gamma} : 1_{(\mathcal{C}_\Gamma)_{\delta_\Gamma}} \Rightarrow 1_{(\mathcal{C}_\Gamma)_{\delta_\Gamma}}$, which we can restrict to a natural transformation $z_{\mathcal{C}_1} : 1_{\mathcal{C}_1} \Rightarrow 1_{\mathcal{C}_1}$, where $\mathcal{C}_1 := \mathcal{G}_1 \cap \mathcal{C}_\Gamma$.

Proposition 5.3.38 (System of conjugacy representatives). *The set $\{z_{\mathcal{C}_1} \mid \mathcal{G}_1 \in \mathcal{T}_\Gamma\}$ constructed above is a system of conjugacy representatives for $\mathcal{T}_\Gamma$.*

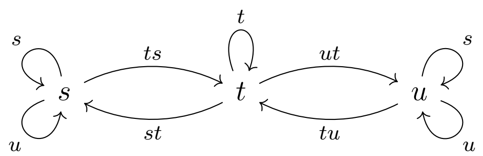
Proof. Let $x \in \Gamma$ (resp. $y \in \Gamma$) with source u (resp. v) in $\mathcal{G}$. Let also $H \subset \mathcal{G}_\Gamma(x,x)$ (resp. $H' \subset \mathcal{G}_\Gamma(y,y)$) be two $\mathcal{T}_\Gamma$-standard parabolic subgroups. For $f \in \mathcal{G}_\Gamma(x,y)$, we have $H^f = H'$ if and only if $\pi(H)^{\pi(f)} = \pi(H')$. By Proposition 5.3.36, there are unique $\mathcal{T}$-standard parabolic subgroups P, P', respectively of $\mathcal{G}(u,u)$ and of $\mathcal{G}(v,v)$, such that $\pi(H) = C_{\mathcal{G}}(x) \cap P$ and $\pi(H') = C_{\mathcal{G}}(y) \cap P'$. Since $\pi(f)$ conjugates $C_{\mathcal{G}}(x)$ to $C_{\mathcal{G}}(y)$, we have that $\pi(f)$ conjugates $\pi(H)$ to $\pi(H')$ if and only if it conjugates P to P'. By definition of a system of conjugacy representatives, this is equivalent to $z_P^{\pi(f)} = z_{P'}$. Since z_H (resp. $z_{H'}$) is the unique lift in $\mathcal{G}_\Gamma(x,x)$ (resp. in $\mathcal{G}_\Gamma(y,y)$) of z_P (resp. of $z_{P'}$), we then obtain that f conjugates H to H' if and only if $z_H^f = z_{H'}$.

Furthermore, by Corollary 5.3.35, we have

$$\pi(\mathrm{SPC}_\Gamma(z_H)) = \pi(\mathrm{PC}_\Gamma(z_H)) = \mathrm{PC}(z_P) \cap C_{\mathcal{G}}(x) = P \cap C_{\mathcal{G}}(x),$$

and thus $\mathrm{SPC}_\Gamma(z_H) = H$ by Proposition 5.3.36, and $\{z_{\mathcal{C}_1} \mid \mathcal{G}_1 \in \mathcal{T}_\Gamma\}$ is a system of conjugacy representatives. □

Example 5.3.39. Consider the monoid $M = \langle s,t,u \mid sts = tst, tut = utu, su = us\rangle^+$ (Artin-Tits monoid of type A_3). It is a Garside monoid with Garside element $\Delta = stsuts$. In Example 4.3.12, we computed the conjugacy graph Γ attached to the super-summit set of s. The atoms of the category $\mathcal{C}_\Gamma$ are given by



We deduced that $C_{G(M)}(s)$ is generated by $w := s, x := u, y := tsst$, with the following presentation:

$$C_{G(M)}(s) = \langle w\rangle \times \langle x,y \mid xyxy = yxyx\rangle.$$

Now, the parabolic Garside elements of M are $\{1, s, t, u, sts, su, tut, \Delta\}$. Since M is a homogeneous monoid, Proposition 5.2.27 proves that a system of conjugacy representatives for the shoal $\mathcal{T}$ of all standard parabolic subgroups of $G(M)$ is given by

$$\{1, s, t, u, (sts)^2, su, (tut)^2, \Delta^2\}.$$

Among those, the ones which commute with s are $\{1, s, u, (sts)^2, su, \Delta^2\}$. By Lemma 5.3.31 and Proposition 5.3.34, the $\mathcal{T}_\Gamma$ standard parabolic subgroups of $C_{G(M)}(s)$ are given by

$$\begin{array}{lll} \langle 1\rangle \cap C_{G(M)}(s) = \langle 1\rangle, & \langle s\rangle \cap C_{G(M)}(s) = \langle w\rangle, & \langle u\rangle \cap C_{G(M)}(s) = \langle x\rangle, \\ \langle s,t\rangle \cap C_{G(M)}(s) = \langle x,y\rangle, & \langle s,u\rangle \cap C_{G(M)}(s) = \langle w,x\rangle, & G(M) \cap C_{G(M)}(s) = C_{G(M)}(s). \end{array}$$

And the $\mathcal{T}_\Gamma$-parabolic subgroups of $G(M)$ are the conjugates of the $\mathcal{T}_\Gamma$-standard parabolic subgroups.

Moreover, note that $C_{G(M)}(s)$ is also a Garside group for the Garside monoid $\langle a,b,c \mid ab = ba, ac = ca, bcbc = cbcb\rangle^+$ (Artin-Tits monoid of type $A_1 \times B_2$). In this monoid, both b and c are parabolic Garside elements, and both $\langle b\rangle$ and $\langle c\rangle$ are parabolic subgroups. However, in $C_{G(M)}(s)$, $\langle x\rangle$ is a $\mathcal{T}_\Gamma$-parabolic subgroup, while $\langle y\rangle$ is not a $\mathcal{T}_\Gamma$-parabolic subgroup (since y is not conjugate to either x or w).

Part II

Complex braid groups

Chapter 6

Complex reflection groups and complex braid groups

In this introductory chapter, we present the basic notions and results in the theory of complex reflection groups and their braid groups. This chapter contains no new results, and can certainly be skipped by people who are already familiar with this theory.

Contents

Complex reflection groups appear as a natural generalization of finite Coxeter groups, which play an important role in many modern areas of algebra. This well-understood class of groups enjoy many interesting properties. For instance, they have been characterized by Shephard-Todd and Chevalley as the finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ whose algebra of invariants in the associated symmetric algebra is a polynomial algebra. Moreover, certain particular centralizers (regular centralizers) in complex reflection groups are again complex reflection groups. This last property is not shared by finite Coxeter groups: a regular centralizer in a finite Coxeter group may be a (strictly) complex reflection groups.

More recently, complex reflection groups were found to play a key role in the structure and representation theory of finite reductive groups. Several objects defined for Coxeter groups were generalized to complex reflection groups by Broué, Malle and Rouquier in [BMR98]. One of the most important of these objects are the generalized braid groups, which will be the core object of study in this second part of the thesis.

In this first section we state several results about complex reflection group which will be used throughout the sequel. Mainly we recall the Shephard-Todd classification of irreducible complex reflection groups, along with the first results of the theory of Springer regular elements. Most of the results are taken from [LT09] and the proofs are skipped. One important distinction will be made between well-generated and badly-generated complex reflection groups. This seemingly arbitrary distinction has in fact important consequences on the behavior of the associated complex braid groups.

In the second section, we turn our attention to complex braid groups attached to complex reflection groups. These groups are defined as fundamental groups of the regular orbit space attached to a complex reflection group. In the case of a complexified real reflection group, this definition coincides with the more classical Artin group. By construction, the regular orbit space attached to a complex reflection group is the complement of some algebraic hypersurface (the discriminant hypersurface). The geometry of this hypersurface can be used to give general results about complex braid groups. In particular, well-generatedness can be read on the geometry of the discriminant hypersurface.

6.1 Complex reflection groups

6.1.1 Definition and first properties

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V.

Definition 6.1.1 (Complex reflection group). An element $s \in \mathrm{GL}(V)$ is called a *reflexion* if it has finite order and if $\mathrm{Ker}(s-1)$ has codimension 1. The hyperplane $\mathrm{Ker}(s-1)$ is called the *reflecting hyperplane* of s.
A finite subgroup W of $\mathrm{GL}(V)$ is called a *complex reflection group* if it is generated by the set $\mathrm{Ref}(W)$ of reflections it contains.

Obvious examples of complex reflection groups include the group $\mu_k = \mu_k(\mathbb{C})$ of k-th roots of unity in $\mathbb{C}^*$ (for $k \geqslant 1$ an integer), and the symmetric group $\mathfrak{S}_n$, acting on $\mathbb{C}^n$ by permutation matrices (the reflections are the transpositions of $\mathfrak{S}_n$).

By replacing $\mathbb{C}$ with an arbitrary field k, one obtains the concept of reflection group over k. If $k = \mathbb{R}$, then any reflection group admits a Coxeter presentation, and is isomorphic to a finite Coxeter group. If $k = \mathbb{Q}$, then one actually recovers finite Weyl groups.

Note that, if k is a subfield of $\mathbb{C}$, then any reflection group over k can be seen as a complex reflection group by embedding $\mathrm{GL}(V)$ into $\mathrm{GL}(V \otimes_k \mathbb{C})$.

Let $G \subset \mathrm{GL}(V)$ be a finite subgroup. The action of G on V extends to an action of G on the symmetric algebra S attached to V, and we can consider the invariant algebra S^G under this action. By [LT09, Corollary 3.8], S^G is finitely generated as a $\mathbb{C}$-algebra. Moreover, one can show ([LT09, Theorem 3.12]) that the extension $S^G \subset S$ is finite. Thus one can consider families of n algebraically independent elements of S^G. Furthermore, since G acts linearly on V, the action of G on S preserves the graduation, we can then consider families of n homogeneous algebraically independent elements of S^G.

Moreover, we can consider the contragredient representation of G on V^*. Since an element of G acts on V as a reflection if and only if it acts on V^* as a reflection, we obtain that $G \subset \mathrm{GL}(V)$ is a complex reflection group if and only if $G \subset \mathrm{GL}(V^*)$ is a complex reflection group. We consider this because the symmetric algebra S^* on V^* identifies with the algebra $\mathbb{C}[V]$ of polynomial functions on V.

The classical theorem of Chevalley, Shephard, and Todd allows us to characterize complex reflection groups in terms of invariant algebras. We can state it using either S^G or $\mathbb{C}[V]^G$. Since the point of view of polynomial functions will be useful to us later, we state it using $\mathbb{C}[V]$.

Theorem 6.1.2 (Chevalley, Shephard, Todd). *[LT09, Theorem 4.19]*
Let V be a complex vector space of finite dimension n, and let $G \subset \mathrm{GL}(V)$ be a finite group. Let also $f_1, \dots, f_n \in \mathbb{C}[V]^G$ be homogeneous algebraically independent elements, of respective degrees $d_1, \dots, d_n$. Then

(a) $|G| \leqslant d_1 \cdots d_n$.

(b) If $|G| = d_1 \cdots d_n$, then G is a complex reflection group, and $\mathbb{C}[V]^G = \mathbb{C}[f_1, \dots, f_n]$.

(c) If $\mathbb{C}[V]^G = \mathbb{C}[f_1, \dots, f_n]$, then $|G| = d_1 \dots d_n$ and G is a complex reflection group.

Example 6.1.3.

- Consider the subgroup W of $\mathrm{GL}_2(\mathbb{C})$ generated by $-I_2$. It is a cyclic group of order 2, which is not a complex reflection group. The associated invariant algebra $\mathbb{C}[X, Y]^W$ is generated by X^2, Y^2, XY. A family of homogeneous algebraically independent elements is given by $f_1 = X^2$, $f_2 = Y^2$, and their degrees are $2, 2$.
- Consider the group $W = \mu_k \subset \mathbb{C}^*$ of k-th roots of unity. The invariant algebra $\mathbb{C}[X]^W$ is given by $\mathbb{C}[X^k]$.
- Consider the subgroup $W = \mathfrak{S}_n \subset \mathrm{GL}_n(\mathbb{C})$ (by the permutation matrices). The associated action of W on $\mathbb{C}[X_1, \dots, X_n]$ is by permutation of the variables, and the invariant algebra $\mathbb{C}[X_1, \dots, X_n]^W$ is the algebra of symmetric polynomials. It is well-known that $\mathbb{C}[X_1, \dots, X_n]^W$ is a polynomial algebra, generated for instance by elementary symmetric polynomials $\omega_1, \dots, \omega_n$. Since ω_i is homogeneous of degree n, we recover that $|\mathfrak{S}_n| = n!$.

Definition 6.1.4 (System of basic invariants, degrees). Let $W \subset \mathrm{GL}(V)$ be a complex reflection group. A sequence $f_1, \dots, f_n$ of homogeneous algebraically independent elements of $\mathbb{C}[V]^W$ such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$ is called a *system of basic invariants* for W. By [LT09, Proposition 3.25], the multiset of degrees of the f_i depends only on W, they are called the *degrees* of W.

Let $W \subset \mathrm{GL}(V)$ be a complex reflection group. Note that, if $(f_1, \dots, f_n)$ is a system of basic invariants for W, then, up to reordering the f_i, we can assume that the sequence of the degrees of f_i is nondecreasing. When we talk about the degrees of W, we always assume that they are ordered in nondecreasing order.

On top of the algebra $\mathbb{C}[V]^W$ of W-invariant functions on V, one can consider the $\mathbb{C}[V]^W$ module $(\mathbb{C}[V] \otimes V)^W$ of W-invariant vector fields on V. By [OT92, Lemma 6.48], it is a free homogeneous $\mathbb{C}[V]^W$-module of rank n. A homogeneous $\mathbb{C}[V]^W$-basis $\xi = (\xi_1, \dots, \xi_n)$ is a *system of basic derivations* for W [Bes15, Definition 1.2], the multisets of degrees of the ξ_i (in nonincreasing order) does not depend on the choice of ξ. It is called the sequence of *codegrees* of W. Since there are invariant vector fields of degree 0, we always have $d_n^* = 0$.

In general, if f is a system of basic invariants for W, then $(df_1, \dots, df_n)$ is a $\mathbb{C}[V]^W$-basis for the module $(\mathbb{C}[V] \otimes V^*)^W$ of invariant differential forms. In the case where W is a complexified real reflection group, the vector spaces V and V^* are isomorphic as representations of W. In particular, the $\mathbb{C}[V]$-modules $(\mathbb{C}[V]\otimes V)^W$ and $(\mathbb{C}[V]\otimes V^*)^W$ are isomorphic. We then obtain a system of basic derivations, and we have $d_i^* = d_{n-i+1}-2$.

One of the important objects one can attach to a complex reflection group $W\subset \mathrm{GL}(V)$ is the *reflection arrangement*

$$\mathcal{A}(W)=\mathcal{A}:=\{\mathrm{Ker}(s-1)\mid s\in \mathrm{Ref}(W)\}.$$

If W contains only involutive reflections, then a reflecting hyperplane is attached to exactly one reflection, and thus $|\mathrm{Ref}(W)|=|\mathcal{A}|$ (this is for instance the case when W is a complexified real reflection group). On the other hand, if W contains reflections of order >2, then we have $|\mathrm{Ref}(W)|>|\mathcal{A}|$. For instance, the group $W=\mu_k\subset \mathrm{GL}_1(\mathbb{C})$ admits only one reflecting hyperplane, namely $\{0\}$, while it has $k-1$ reflections. A first use of degrees and codegrees is that they allow us to count reflections in W along with reflecting hyperplanes.

Lemma 6.1.5. *[Bes01, Lemma 1.1] Let $W\subset \mathrm{GL}(V)$ be a complex reflection group. We have $|\mathrm{Ref}(W)|=d_1+d_2+\cdots+d_n-n$, and $|\mathcal{A}|=d_1^*+d_2^*+\cdots+d_n^*+n$.*

Let $W\subset \mathrm{GL}(V)$ be a complex reflection group. To the reflection arrangement $\mathcal{A}=\mathcal{A}(W)$ is attached the *intersection lattice*

$$L(\mathcal{A})=\left\{\bigcap_{H\in A} H\mid A\subset\mathcal{A}\right\},$$

whose elements are called *flats*. The set $L(\mathcal{A})$ is ordered by reverse inclusion [OT92, Definition 2.1]

$$\forall F,L, F\leqslant L\Leftrightarrow L\subset F.$$

Endowed with the partial order $\leqslant$, $L(\mathcal{A})$ becomes a (geometric) lattice, where joins and meets are defined by

$$F\wedge L=\bigcap\{E\in L\mid F\cup L\subset E\} \text{ and } F\vee L=F\cap L.$$

The function sending $F\in L(\mathcal{A})$ to its codimension in V is a rank function on $L(\mathcal{A})$ [OT92, Definition 2.2].

Let $W\subset \mathrm{GL}(V)$ be a complex reflection group. The flats of the intersection lattice $L(\mathcal{A}(W))$ are related to a particular family of subgroups of W called parabolic subgroups. These subgroups are useful tools in the study of complex reflection groups, and one of the purposes of Chapter 7 will be to generalize their definition to complex braid groups.

Definition 6.1.6 (Parabolic subgroup). [LT09, Definition 9.1]
A *parabolic subgroup* of a complex reflection group $W\subset \mathrm{GL}(V)$ is a subgroup of W of the form

$$W_0=\{w\in W\mid \forall x\in X, w.x=x\},$$

where X is a subset of V.

Let $W\subset \mathrm{GL}(V)$ be a complex reflection group. Since the action of W on V is linear, we can always replace X by $\mathrm{span}(X)$ in the above definition. Furthermore, since W is finite, a parabolic subgroup W_0 can always be written as the stabilizer of a single element of V. A classical theorem of Steinberg ensures that parabolic subgroups are again complex reflection groups.

Theorem 6.1.7 (Steinberg). *[LT09, Theorem 9.44] Let V be a finite dimensional complex vector space, and let $W\subset \mathrm{GL}(V)$ be a complex reflection group. Any parabolic subgroup $W_0\subset W$ is generated by the set $\mathrm{Ref}(W_0)=\mathrm{Ref}(W)\cap W_0$, and is in particular a complex reflection group acting on V.*

Remark 6.1.8. Let $W \subset \mathrm{GL}(V)$ be a complex reflection group. Although it is true that any parabolic subgroup of W is generated by a subset of $\mathrm{Ref}(W)$ (i.e. is a *reflection subgroup*), not every subgroup of W generated by reflections is a parabolic subgroup. For instance, the subgroup $\mu_2 \subset \mu_4 \subset \mathrm{GL}_1(C)$ is generated by reflections, but it is not a parabolic subgroup of μ_4.

Let $W \subset \mathrm{GL}(V)$ be a complex reflection group. A first easy consequence of Steinberg's Theorem is that parabolic subgroups of W form a lattice for inclusion, which is isomorphic to the intersection lattice $L(\mathcal{A})$.

Lemma 6.1.9 (Lattice of parabolic subgroups). *Let $W \subset \mathrm{GL}(V)$ be a complex reflection group, and let $\mathcal{A}$ be the associated reflection arrangement.*

(a) If $W_0 \subset W$ is a parabolic subgroup, then $F(W_0) := \{v \in V \mid \forall w \in W_0, w.v = v\} \in L(\mathcal{A})$.

(b) If $F \in L(\mathcal{A})$ is a flat, then $W_F := \{w \in W \mid \forall v \in F, w.v = v\}$ is a parabolic subgroup of W.

(c) The maps $W_0 \mapsto F(W_0)$ and $F \mapsto W_F$ are inverse isomorphisms of posets between $(L(\mathcal{A}), \leqslant)$ and the set of parabolic subgroups of W, ordered by inclusion.

Proof. (a) Since W_0 is generated by $R_0 := \mathrm{Ref}(W) \cap W_0$, we have $v \in F(W_0)$ if and only if $r.v = v$ for all $r \in R_0$. We then have

$$F(W_0) = \bigcap_{r \in R_0} \mathrm{Ker}(r-1) \in L(\mathcal{A}).$$

(b) Is by definition of a parabolic subgroup.
(c) Let $W_0 = \mathrm{Stab}_W(v)$ be a parabolic subgroup of W. Since all elements of $R_0 = \mathrm{Ref}(W) \cap W_0$ fix $F(W_0)$, we have $R_0 \subset W_{F(W_0)}$ and $W_0 \subset W_{F(W_0)}$ since W_0 is generated by R_0. Conversely, the elements of R_0 are exactly the reflections which fix v. Since $v \in F(W_0)$, an element of W which fixes $F(W_0)$ pointwise fixes v, and thus $W_{F(W_0)} \subset W_0$ and $W_{F(W_0)} = W_0$.

Since the set of parabolic subgroups of W is finite, as well as $L(\mathcal{A})$ (since $\mathcal{A}$ is finite), we deduce that the two considered maps are inverse bijections. It remains to show that both of these bijections are increasing.

If W_0, W_1 are two parabolic subgroups such that $W_0 \subset W_1$, then for $v \in V$, we have $v \in F(W_0)$ if $v \in F(W_1)$ and thus $F(W_1) \subset F(W_0)$ $F(W_0) \leqslant F(W_1)$. Likewise, if $F, L \in L(\mathcal{A})$ are such that $L \subset F$, then $w \in W$ fixes L pointwise if it fixes F pointwise, thus $W_F \subset W_L$. □

In particular, if W_0 and W_1 are two parabolic subgroups of a complex reflection group $W \subset \mathrm{GL}(V)$, then we have that $W_0 \cap W_1 = W_{F(W_0) \wedge F(W_1)}$ is a parabolic subgroup. The parabolic subgroup $W_{F(W_0) \vee F(W_1)}$ is the smallest parabolic subgroup of W containing both W_0 and W_1.

6.1.2 Irreducible complex reflection groups and classification

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V. We also fix a complex reflection group $W \subset \mathrm{GL}(V)$, and we keep the notation from the last section.

Definition 6.1.10 (Irreducible complex reflection group). The group $W \subset \mathrm{GL}(V)$ is called *irreducible* if the only globally W-invariant subspaces of V are $\{0\}$ and V itself.

The inclusion $W \subset \mathrm{GL}(V)$ can be seen as a representation of an abstract group isomorphic to W, the *reflection representation*. Stating that W is irreducible amounts to saying that the reflection representation is an irreducible representation of W. In particular, if W is irreducible, then we know by Schur's Lemma that the center of W is cyclic. Furthermore, if $(d_1, \dots, d_n)$ denotes the degrees of W, then the order of $Z(W)$ is the gcd $d_1 \wedge d_2 \cdots \wedge d_n$ by [LT09, Corollary 3.24].

Remark 6.1.11. As we said, "being a complex reflection group" is not a group theoretic property, and refers to an additional structure on an abstract group W (namely, a faithful reflection representation). However when the context is clear, we will sometimes abuse the notation and consider complex reflection groups without an explicit reference to the space V.

The group $1_V := \{\mathrm{Id}_V\} \subset \mathrm{GL}(V)$ is a complex reflection group (generated by the empty set). It is irreducible if and only if V has dimension 1. More generally, if $W \subset \mathrm{GL}(V)$ is a complex reflection group, then we can consider

$$V^W = \{v \in W \mid \forall w \in W,\ w.v = v\},$$

which is clearly a fixed subspace of V. Notice that $V^W = F(W)$ in the notation of Lemma 6.1.9 (lattice of parabolic subgroups). It is the maximal element of the intersection lattice $L(\mathcal{A})$.

If W is irreducible, then we either have $V^W = V$, in which case $W = 1_V$ and V has dimension 1, or $V^W = \{0\}$, in which case W is called *essential*. The codimension of V^W in V is called the *rank* of W. In particular, the rank of W is the dimension of V if and only if W is essential.

Remark 6.1.12. Let $W_0 \subsetneq W$ be a parabolic subgroup. Seen as a reflection group acting on V, the group W_0 is not irreducible since V^{W_0} is nontrivial by definition. We will say that W_0 is an irreducible parabolic subgroup of W if the action of W_0 on $V/(V^{W_0})$ is irreducible.

If $W_1, \dots, W_k$ is a family of complex reflection groups acting on complex vector spaces $V_1, \dots, V_k$, then the direct product $W := W_1 \times \cdots \times W_k$ naturally acts on $V := V_1 \oplus \cdots \oplus V_k$, again as a complex reflection group. All the spaces V_i are globally W-invariant, and thus W is not irreducible.

Conversely, if $W \subset \mathrm{GL}(V)$ is an arbitrary complex reflection group, then, since W is finite, we can by Maschke's Theorem consider a hermitian scalar product on V which is W-invariant. In other words, we can always assume (up to conjugacy in $\mathrm{GL}(V)$) that W is a unitary reflection group. We then have the following decomposition result:

Proposition 6.1.13. *[LT09, Theorem 1.27] Let V be endowed with a W-invariant scalar product. There is an orthogonal sum $V = V^W \perp V_1 \perp \cdots \perp V_k$ such that the restriction W_i of W to V_i acts as an irreducible complex reflection group, and that $W = W_1 \times \cdots \times W_k$. Furthermore, this decomposition is unique up to permutations of the factors.*

Example 6.1.14. In the case of $W := \mathfrak{S}_n$ acting on $V := \mathbb{C}^n$ by permuting a basis, the space V^W is a line generated by $v_0 := (1, \dots, 1)$. We have $V = \mathbb{C}v_0 \oplus H$, where $H = \{(x_1, \dots, x_n) \mid x_1 + \cdots + x_n = 0\}$. The group W acts on H, again as a reflection group, and this action is irreducible.

This proposition allows us to restrict our attention to irreducible complex reflection groups, which we will often do in later sections. This is very useful since the complete classification of irreducible complex reflection groups has been given by Shephard and Todd in [ST54]. We give a very quick outline of the classification here (one can consult [LT09, Chapter 8] for a more complete proof).

Monomial irreducible complex reflection groups

Many irreducible complex reflection groups are isomorphic (as complex reflection groups) to groups of monomial matrices in $\mathrm{GL}_n(\mathbb{C})$ (i.e. matrices with exactly one nonzero entry per row/per column). We define these monomial reflection groups here, mostly in order to fix notation which will be useful later on. We fix an integer $m \geqslant 1$.

First, we can easily identify what are the monomial matrices which are also reflections.

Definition 6.1.15. Let $m \geqslant 1$ be an integer. Let also $\zeta \in \mathbb{C}^*$.

- For $k \in [\![1, m]\!]$, we define $r_{\zeta,k} \in \mathrm{GL}_m(\mathbb{C})$ by

$$r_{\zeta,k} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, \zeta x_k, x_{k+1}, \ldots, x_n).$$

  The order of $r_{\zeta,k}$ is equal to the order of ζ in $\mathbb{C}^*$.

- For $1 \leqslant i < j \leqslant n$, we define $s_{\zeta,i,j} \in \mathrm{GL}_m(\mathbb{C})$ by

$$s_{\zeta,i,j} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \zeta x_j, x_{i+1}, \ldots, x_{j-1}, \zeta^{-1} x_i, x_{j+1}, \ldots, x_n).$$

  The order of $s_{\zeta,i,j}$ is always 2.

For $k \in [\![1, m]\!]$ and $\zeta \in \mathbb{C}$ of finite order, the element $r_{\zeta,k}$ is a reflection, whose reflecting hyperplane is defined by the equation $x_k = 0$. Likewise, for $1 \leqslant i < j \leqslant n$ and an arbitrary $\zeta \in \mathbb{C}^*$, the element $s_{\zeta,i,j}$ is a reflection, whose reflecting hyperplane is defined by the equation $x_i = \zeta x_j$. Moreover, by [LT09, Lemma 2.8], these are all the monomial reflections in $\mathrm{GL}_m(\mathbb{C})$.

Let now $r \geqslant 1$ be an integer. We denote by $G(r, 1, m) \subset \mathrm{GL}_m(\mathbb{C})$ the set of monomial matrices whose nonzero coefficients belong to the set μ_r of r-th roots of unity in $\mathbb{C}^*$. This set is a group of order $n!r^n$, and the reflection it contains are the $r_{\zeta,k}$ for $\zeta \in \mu_r$ and $k \in [\![1, m]\!]$, along with the $s_{\zeta,i,j}$ for $\zeta \in \mu_r$ and $1 \leqslant i < j \leqslant m$.

The map $G(r, 1, n) \to \mu_r$ sending an element to the product of its nonzero entries is a group morphism. If $r = de$ for integers d, e, then the group $G(de, e, n) \lhd G(r, 1, n)$ is defined as the preimage of $\mu_d \subset \mu_r$ under this morphism. In particular, $G(de, e, n)$ is a subgroup of $G(de, 1, n)$ of index e (and of order $n!d^n e^{n-1}$). The reflections contained in $G(de, e, n)$ are $r_{\zeta,k}$ for $\zeta \in \mu_d \setminus \{1\}$ and $k \in [\![1, n]\!]$ and all the $s_{\zeta,i,j}$. Note that if $d = 1$, then only the second type of reflection remains.

Proposition 6.1.16. *For all $d, e, m \geqslant 1$, the group $G(de, e, m) \subset \mathrm{GL}_m(\mathbb{C})$ is a complex reflection group. It is also irreducible, except for $d = e = 1$ or $d = 1, e = 2 = n$.*

The group $G(2, 2, 2)$ is isomorphic to $\mu_2 \times \mu_2$ acting on $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. The group $G(1, 1, m)$, which is simply $\mathfrak{S}_m$ acting on $\mathbb{C}^m$ by permuting coordinates, is not irreducible. However, we saw in Example 6.1.14 that restricting the action to a convenient hyperplane gives an irreducible reflection representation of $\mathfrak{S}_m$. We denote this complex reflection group by $\widetilde{G}(1, 1, m)$ from now on.

Since we know the reflecting hyperplanes associated to all monomial reflections, we deduce that $\mathcal{A}(G(de, e, n)) = \mathcal{A}(G(de, 1, n))$ if $d > 1$. In particular we deduce the following result by Lemma 6.1.9 (lattice of parabolic subgroups).

Lemma 6.1.17. *Let d, e, m be integers such that $d, e, m > 1$. The intersection lattices of the groups $G(de, e, m)$ and $G(de, 1, m)$ are equal. Furthermore, the map sending a parabolic subgroup*

W_0 of $G(de,1,m)$ to $W_0 \cap G(de,e,nm)$ induces an isomorphism between the lattices of parabolic subgroups of $G(de,1,m)$ and of $G(de,e,m)$.

This result fails when d is equal to 1. In other words, there is no bijection between parabolic subgroups of $G(e,1,m)$ and parabolic subgroups of $G(e,e,m)$. For instance, $G(4,1,2)$ has 6 reflecting hyperplanes by Lemma 6.1.5, and thus $L(\mathcal{A}(G(4,1,2))$ contains 8 elements, whereas $G(4,4,2)$ has 4 reflecting hyperplanes, and thus $L(\mathcal{A}(G(4,4,2))$ contains 6 elements.

Corollary 6.1.18. *Let d,e,m be integers such that $d,e,m > 1$, and let $W_0 \subset G(de,1,m)$ be a parabolic subgroup. The parabolic subgroup $W_0 \cap G(de,e,m)$ of $G(de,e,m)$ is irreducible if and only if W_0 is irreducible.*

Proof. First, if $W_0 \cap G(de,e,m)$ is irreducible, then so is W_0 as it contains $W_0 \cap G(de,e,m)$ and acts on the same space. Conversely, assume that W_0 is irreducible, i.e. we cannot partition the reflections of W_0 in two sets commuting with one another. We know that the reflections of W_0 which are not in $W_0 \cap G(de,e,m)$ are exactly the $r_{\zeta,i}$ with $\zeta \in \mu_{de}$ such that there is some $r_{\lambda,i} \in W_0 \cap G(de,e,m)$ with $\lambda \in \mu_d$. Since $r_{\zeta,i}$ and $r_{\lambda,i}$ then have the same centralizers in $\mathrm{GL}_m(\mathbb{C})$, we obtain that the reflections of $W_0 \cap G(de,e,m)$ cannot be partitioned in two sets commuting with one another, and $W_0 \cap G(de,e,m)$ is irreducible. □

The classification

Not every irreducible complex reflection group is isomorphic to a monomial matrix group: Along with groups of the form $G(de,e,n)$ or $\widetilde{G}(1,1,n)$, there is a family of so-called exceptional irreducible complex reflection groups, labeled G_4 to G_{37}, which are not isomorphic to groups of the form $G(de,e,n)$. Among those, $G_4,\dots,G_{22}$ have rank 2, $G_{23},\dots,G_{27}$ have rank 3, $G_{28},\dots,G_{32}$ have rank 4, G_{33} has rank 5, G_{34},G_{35} have rank 6, and G_{36},G_{37} have rank 7 and 8 respectively.

Theorem 6.1.19 (Shephard-Todd). *[LT09, Theorem 8.29] Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group of rank n. There is an isomorphism $V \simeq \mathbb{C}^n$ under which we either have*

- *$n = 1$ and W is identified with a cyclic group.*
- *$n \geqslant 2$ and W is identified with $G(de,e,n)$, with $de > 1$.*
- *$n \geqslant 2$ and W is identified with one of the 34 exceptional groups $G_4,\dots,G_{37}$.*
- *$n \geqslant 4$ and W is identified with $\widetilde{G}(1,1,n)$.*

Notice that cyclic groups can be described as the groups $G(r,1,1)$ for $r \geqslant 1$. Thus, in the Shephard-Todd classification, three out of the four cases consist of groups related to monomial complex reflection groups. This set of cases will be called the *infinite series*, in order to distinguish it from the case of exceptional irreducible complex reflection groups.

Complexified irreducible real reflection groups, i.e. finite irreducible Coxeter groups, appear in this classification. The correspondence is given below:

$$\begin{array}{llll} \widetilde{G}(1,1,n) \leftrightarrow A_{n-1}, & G(2,1,n) \leftrightarrow B_n, & G(2,2,n) \leftrightarrow D_n, & G(e,e,2) \leftrightarrow I_2(e), \\ G_{23} \leftrightarrow H_3, & G_{28} \leftrightarrow F_4, & G_{30} \leftrightarrow H_4, & G_{35} \leftrightarrow E_6, \\ G_{36} \leftrightarrow E_7, & G_{37} \leftrightarrow E_8. & & \end{array}$$

The degrees and codegrees of irreducible complex reflection group have already been computed (see [BMR98, Table 1, 2, 3, 4, 5]). We will often use these results without proof in the sequel.

Well-generated complex reflection groups

A notion which will play a key role in the sequel is the notion of well-generated complex reflection group. This definition, while seemingly *ad hoc*, is actually equivalent to strong results on the behavior of the associated braid group (see for instance Proposition 6.2.19 and Theorem 6.1.22).

Definition 6.1.20 (Well-generated complex reflection group). [Bes15, Section 2]
The complex reflection group W is said to be *well-generated* if it can be generated by a set of $\mathrm{rk}(W)$ reflections (where $\mathrm{rk}(W)$ is the rank of W). Otherwise, W is said to be *badly-generated*.

By [Bou81, Theorem V.3.1], complexified real reflection groups are always well-generated. Likewise, complex reflection group of rank 1 are also well-generated (they are cyclic groups). However, not all complex reflection groups are well-generated, as seen in the example below:

Example 6.1.21. The complex reflection group $W := G(4, 2, 2)$ is the group of 2×2 monomial matrices whose nonzero entries in $\mu_4 = \{\pm 1, \pm i\}$, and the product of those entries belongs to $\{\pm 1\}$. It contains 6 reflections:

$$r_{-1,1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_{-1,2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s_{1,1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$s_{i,1,2} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad s_{-1,1,2} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s_{-i,1,2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Since W is irreducible, it is well-generated if and only if it can be generated by a pair of reflections. Since all reflections of the form $s_{\zeta,i,j}$ belong to $G(4, 4, 2)$, no pair of reflections of this form can generate W. One can check that every other pair of reflections generate a subgroup of W of order 8, whereas W has order 16. Thus W is badly-generated, and it admits a set of generating reflections of size 3, namely $\{r_{-1,1}, s_{1,1,2}, s_{i,1,2}\}$.

Among the irreducible groups in the infinite series, the one that are badly-generated are precisely the $G(de, e, n)$ with $e, d > 1$ (and $n \geqslant 2$). Conversely, the groups of the forms $G(r, 1, n)$ and $G(e, e, n)$ are well-generated, as well as $\widetilde{G}(1, 1, n)$ which is a complexified real reflection group. The exceptional irreducible complex reflection groups which are badly generated are $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}$ and G_{31}. Among those, only G_{31} has rank bigger than 2 (it has rank 4).

By observing the classification of irreducible complex reflection groups, along with their respective degrees and codegrees, we obtain the following characterization of well-generated complex reflection groups:

Theorem 6.1.22 (Characterization of well-generated groups). *[Bes15, Theorem 2.4]*
Let W be an irreducible complex reflection group with degrees $d_1 \leqslant \ldots \leqslant d_n$ and codegrees $d_n^* \leqslant \ldots \leqslant d_1^*$. The following assertions are equivalent:

(i) W is well-generated.

(ii) For all $i \in [\![1, n]\!]$, we have $d_i + d_i^* = d_n$.

(iii) For all $i \in [\![1, n]\!]$, we have $d_i + d_i^* \leqslant d_n$.

This theorem is the first of many results we will see for which we only know a case-by-case proof, done by inspecting each case of the Shephard-Todd classification. The following result also relies on a case-by-case proof:

Proposition 6.1.23 (Parabolic subgroup of a well-generated group). *[Bes15, Lemma 2.7] A parabolic subgroup of a well-generated complex reflection group is again well-generated.*

The converse of this statement is false if we consider only proper parabolic subgroups. For instance the group $G(4,2,2)$ of Example 6.1.21 is badly-generated and it has rank 2, thus all of its proper parabolic subgroups have rank 1, and thus they are well-generated.

6.1.3 Springer theory of regular elements

The theory of regular elements in complex reflection groups was introduced by Springer in [Spr74]. One of its strength is that centralizers of regular elements in complex reflection groups are again complex reflection groups (acting on some subspace). This point of view then allows us to study some complex reflection groups by embedding them in better behaved complex reflection groups. The most important example would be the embedding $G_{31} \to G_{37}$ as a centralizer of an i-regular element (see Example 6.1.31 below).

Later, in Section 9.1, this theory will be adapted to complex braid groups, allowing us for instance to see the complex braid group $B(G_{31})$ as the centralizer of a "regular braid" in the complex braid group $B(G_{37})$.

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V. We also fix a complex reflection group $W \subset \mathrm{GL}(V)$ and we keep the notation from Section 6.1.1. For an integer d, we denote by ζ_d the complex root of unity $\exp\left(\frac{2i\pi}{d}\right)$.

Definition 6.1.24 (Regular elements). [LT09, Definition 11.21]
An element $g \in W$ is *ζ-regular* for some $\zeta \in \mathbb{C}^*$ if the eigenspace $V(g,\zeta)$ contains points lying on no reflecting hyperplane of a reflection in $\mathrm{Ref}(W)$. For $d \geqslant 1$, an element of W is called *d-regular* if it is ζ_d-regular. If such elements exist, then the integer d is called *regular* for W.

Let $w \in W$, and let ζ be an eigenvalue of w. The pointwise stabilizer of $V(w,\zeta)$ in W is a parabolic subgroup of W, and thus it is generated by the reflections which pointwise fix $V(w,\zeta)$ by Steinberg's Theorem. Thus w is ζ-regular if and only if the pointwise stabilizer of $V(w,\zeta)$ in W is trivial, which is equivalent to stating that $V(g,\zeta)$ is not contained in any element of the reflection arrangement $\mathcal{A}$.

A 1-regular element is an element of W which fixes points outside of any hyperplane. By Steinberg's Theorem, we obtain that $\mathrm{Id}_V \in W$ is the only 1-regular element. Likewise, if W is irreducible, then a central element of W has the form $\zeta\,\mathrm{Id}_V$, and it is always a ζ-regular element.

A first consequence of Definition 6.1.24 is that the order of a regular element is easily computable.

Lemma 6.1.25. *Let $g \in W$ be a regular element for some $\zeta \in \mathbb{C}^*$. The order of g in W is equal to the order of ζ in $\mathbb{C}^*$.*

Proof. Let $k \geqslant 1$ be such that $g^k = 1$. Since g admits a nonzero eigenspace for the eigenvalue ζ, and since ζ has order d, we have that d divides k. Conversely, g^d fixes $V(g,\zeta)$ by construction. In particular, g^d lies in the pointwise stabilizer of a point outside of any reflecting hyperplane of W. Since any such subgroup of W is trivial after Steinberg's Theorem, we obtain that g^d is trivial, and g has order d. □

We actually know by [Spr74, Proposition 4.5] that the eigenvalues of a ζ-regular element in W are given by ζ^{1-d_i} for $i \in [\![1,n]\!]$.

Let $g \in W$ be ζ-regular for some $\zeta \in \mathbb{C}^*$. Since $V(g,\zeta) \subset V(g^k,\zeta^k)$ for $k \geqslant 1$, we obtain that the g^k is ζ^k-regular. In particular, an integer d is regular for W if and only if there is a ζ-regular element in W, where ζ is a primitive d-th root of unity.

By construction, the centralizer $C_W(g)$ acts on the eigenspace $V(g,\zeta)$. The following theorem, due to Springer [Spr74] (except the part about codegrees, which comes from [Bro10, Proposition 5.19]) proves that $C_W(g)$ acts again as a complex reflection group. We will denote this complex reflection group by W_g in the sequel.

Theorem 6.1.26 (Springer theory in complex reflection groups). *[LT09, Theorem 11.24] Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be a complex reflection group. Let also $g \in W$ be a ζ-regular element for some $\zeta \in \mathbb{C}^*$.*
The centralizer $W_g := C_W(g)$ acts on $V(g,\zeta)$ as a complex reflection group, which we call a regular centralizer *in W. The degrees (resp. codegrees) of W_g are precisely the degrees (resp. codegrees) of W which are divisible by d, where d is the order of ζ.*

Let $g \in W$ be ζ-regular for some $\zeta \in \mathbb{C}^*$. Note that, since g acts on $V(g,\zeta)$ by multiplication by ζ, we have that W_g is essential if $\zeta \neq 1$. In particular, a regular number must divide as much degrees of W as codegrees of W. A remarkable result is that the converse also holds.

Theorem 6.1.27 (Characterization of regular numbers). *[LT09, Theorem 11.28] Let W be a complex reflection group. Let also $d_1,\dots,d_n$ (resp. $d_1^*,\dots,d_n^*$) be the degrees (resp. codegrees) of W. Given an integer d, let $a(d) = |\{i \mid d \text{ divides } d_i\}|$, and let $b(d) = |\{i \mid d \text{ divides } d_i^*\}|$. Then $a(d) \leqslant b(d)$, and d is a regular number for W if and only if $a(d) = b(d)$.*

Let g, g' be two regular elements for two eigenvalues ζ, ζ' of the same order $d \geqslant 1$. The centralizers W_g and $W_{g'}$ act on the spaces $V(g,\zeta), V(g',\zeta')$ as complex reflection groups. A priori, these groups could be non isomorphic complex reflection groups[1]. However, the two following results show that this is never the case:

Corollary 6.1.28. *[LT09, Corollary 11.25] Let $\zeta \in \mathbb{C}^*$. Any two ζ-regular elements W are conjugates in W.*

Corollary 6.1.29. *Let $g, g' \in W$ be two regular elements for two eigenvalues ζ, ζ' of the same order $d \geqslant 1$. There is some $w \in W$ such that $wW_gw^{-1} = W_{g'}$ and such that $w.V(g,\zeta) = V(g',\zeta')$. In particular, W_g and $W_{g'}$ are isomorphic complex reflection groups.*

Proof. Since ζ, ζ' have the same order, there is some integer k such that $\zeta^k = \zeta'$, and in particular g^k is a ζ'-regular element. By Corollary 6.1.28, there is some $w \in W$ such that $wg^kw^{-1} = g'$. We then have $w.V(g^k,\zeta') = V(g',\zeta')$ and $wW_{g^k}w^{-1} = W_{g'}$.

We claim that $W_{g^k} = W_g$ and that $V(g^k,\zeta') = V(g,\zeta)$. First, we obviously have $W_g \subset W_{g^k}$. Then, the order of W_{g^k} is the product of its degrees, and the same goes for W_g, thus $|W_{g^k}| = |W_g|$ since their degrees as complex reflection groups are the same after Theorem 6.1.26. Likewise, we have $V(g,\zeta) \subset V(g^k,\zeta^k)$, and equality comes from the facts that these spaces share the same dimension (which is the number of degrees of $W_g = W_{g^k}$). ☐

[1] although they have the same degrees and codegrees by Theorem 6.1.26, and we will see later (Lemma 9.1.24) that there are very few pairs of non isomorphic complex reflection groups sharing the same degrees and codegrees.

Theorem 6.1.30. *[LT09, Theorem 11.38] A regular centralizer in an irreducible complex reflection groups acts as an irreducible complex reflection group on the associated eigenspace.*

Example 6.1.31. Consider the complex reflection group G_{37}, its degrees (resp. codegrees) are

$$2, 8, 12, 14, 18, 20, 24, 30 \text{ (resp. } 0, 6, 10, 12, 16, 18, 22, 28).$$

In particular, the degrees divisible by 4 are $12, 20, 24, 30$, while the codegrees divisible by 4 are $0, 12, 16, 28$. By Theorem 6.1.27, 4 is a regular number for G_{37}. An inspection of the classification of irreducible complex braid groups proves that the centralizer of a 4-regular element in G_{37} is isomorphic to G_{31}.

Theorem 6.1.32 (Reflection arrangement of regular centralizer). *[LT09, Theorem 11.33] Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be a complex reflection group. Let also $g \in W$ be a ζ-regular element for some $\zeta \in \mathbb{C}^*$. The reflecting hyperplanes of the group W_g acting on $V(g, \zeta)$ are the intersections with E of the reflecting hyperplanes of W with $V(g, \zeta)$. In other words, we have*

$$\mathcal{A}_g = \mathcal{A}(W_g) = \{H \cap V(g, \zeta) \mid H \in \mathcal{A}\}.$$

Although the above theorem gives an explicit description of the hyperplane arrangement of a regular centralizer, it does not give a description of the reflections of a regular centralizer, seen as elements of the ambient complex reflection group. We finish this section with a more technical lemma, which will allow us to give such a description in particular for G_{31} seen as a regular centralizer in G_{37}.

Lemma 6.1.33. *Assume that W is a complexified real reflection group which contains $-\mathrm{Id}$ and for which 4 is regular. If $g \in W$ is an i-regular element and r is a reflection of W, then r and r^g commute.*

Proof. First, the element $-\mathrm{Id}$ is a -1-regular element in W. Since $-\mathrm{Id}$ is central in W, and since all -1-regular elements in W are conjugate to $-\mathrm{Id}$, we get that $-\mathrm{Id}$ is in fact the only -1-regular element in W. Now, since g is i-regular, g^2 is $i^2 = -1$-regular, hence equal to $-\mathrm{Id}$.

We assume that $r \neq r^g$ (otherwise our claim is immediate). Let $V_{\mathbb{R}}$ be a real vector space such that $V = V \otimes_{\mathbb{R}} \mathbb{C}$ and that $W \subset \mathrm{GL}(V_{\mathbb{R}})$ is a real reflection group. We fix $\langle -, - \rangle$ a W-invariant scalar product on $V_{\mathbb{R}}$. Let α be a root for r (that is, a generator of $\mathrm{Ker}(r - \mathrm{Id})^{\perp}$). We have

$$g^2 r g^{-2} = r \Rightarrow g r g^{-1} = g^{-1} r g.$$

Thus $g^{-1}(\alpha)$ and $g(\alpha)$ are two roots of the same reflection r^g. Let $\alpha \in \mathbb{R}$ such that $g^{-1}(\alpha) = \lambda g(\alpha)$. We have $\alpha = \lambda g^2(\alpha) = -\lambda\alpha$ and $\lambda = -1$,thus

$$\langle g(\alpha), \alpha \rangle = \left\langle \alpha, g^{-1}(\alpha) \right\rangle = \langle \alpha, -g(\alpha) \rangle = -\langle g(\alpha), \alpha \rangle .$$

The two roots α and $g(\alpha)$ are then orthogonal for $\langle -, - \rangle$ and r and r^g commute. □

With the notation of the lemma, we see in particular that $rr^g \in W_g$, and we can show that the reflections of W_g are exactly the rr^g for $r \in \mathrm{Ref}(W)$.

6.2 Complex braid groups

6.2.1 Discriminant hypersurface, isodiscriminantality

As we will see, complex braid groups are defined as fundamental groups of complements of some open subset of the orbit space of a complex reflection group. Here we describe topological and geometrical properties of this orbit space.

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V. We also fix a complex reflection group $W \subset \mathrm{GL}(V)$ and we keep the notation from Section 6.1.

The inclusion $\mathbb{C}[V]^W \to \mathbb{C}[V]$ induces a map $\mathrm{Specmax}(\mathbb{C}[V]) = V \to \mathrm{Specmax}(\mathbb{C}[V]^W)$, where $\mathrm{Specmax}(R)$ designates the set of all maximal ideals of a ring R. By [LT09, Theorem 3.5], two points have the same image under this map if and only if they belong to the same W-orbit. This yields an identification $V/W \simeq \mathrm{Specmax}(\mathbb{C}[V]^W)$. Furthermore, the Chevalley-Shephard-Todd Theorem proves that the choice of a system of basic invariants $f = (f_1, \ldots, f_n)$ yields an identification $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$. This identification in turn yields an isomorphism of algebraic varieties $V/W \simeq \mathbb{C}^n$. Note that the isomorphism $\mathbb{C}[V]^W \simeq \mathbb{C}[X_1, \ldots, X_n]$ induced by f is a graded algebra isomorphism if we declare the indeterminate X_i to be weighted homogeneous with degree $d_i = \deg(f_i)$. If need be, we will distinguish between the (usual) degree of an element of $\mathbb{C}[X_1, \ldots, X_n]$ and the weighted degree, given with the identification $\mathbb{C}[X_1, \ldots, X_n] \simeq \mathbb{C}[f_1, \ldots, f_n]$.

Moreover, the identification $V/W \simeq \mathbb{C}^n$ obtained through the choice of a system of basic invariants is not only algebraic: Let $f : V \to \mathbb{C}^n$ denote the map defined by $f(v) = (f_1(v), \ldots, f_n(v))$, and let $\pi : V \to V/W$ denote the natural quotient map. Since the f_i are W-invariant, the map f factors through π into a continuous map $\overline{f} : V/W \to \mathbb{C}^n$ as in the following diagram:

$$\begin{tikzcd} V \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ V/W \arrow[r, "\overline{f}"'] & \mathbb{C}^n \end{tikzcd}$$

By [LT09, Proposition 9.3], the map $\overline{f}$ is a bijection, and it can actually be shown to be a homeomorphism for the usual topology ($\overline{f}$ is proper since f is proper, which can be shown using the integrality of $\mathbb{C}[V]$ as a $\mathbb{C}[V]^W$-algebra).

Now, let us consider the union $\Sigma := \bigcup \mathcal{A}$ of the reflecting hyperplanes attached to W. By Steinberg's Theorem, the set $X := V \setminus \bigcup \mathcal{A}$ is the set of elements of V whose stabilizer in W is trivial. In particular, the action of W on X is free, and we actually have the following result:

Proposition 6.2.1 (Discriminant hypersurface). *[OT92, Proposition 6.106]*
The projection map $\pi : V \to V/W$ is a branched covering, whose branch locus is $\Sigma := \bigcup \mathcal{A}$. The image $\mathcal{H}$ of the branch locus in V/W is called the discriminant locus *or* discriminant hypersurface*. The restriction of π to X is a covering map $X \to X/W$ with fiber W.*

Since the branch locus Σ is a union of hyperplanes in V, it is in particular an algebraic hypersurface of V (seen as an affine space). Likewise, $\mathcal{H}$ is a hypersurface in V/W, and we are interested in describing equations defining Σ in V, and $\mathcal{H}$ in V/W.

First, let $H \in \mathcal{A}$. Since H is a hyperplane, an element of W which fixes H pointwise must be a reflection of W. Since reflections around the same hyperplane commute with one another, we obtain that the pointwise stabilizer W_H of H in W is abelian. Actually, the determinant induces an injective morphism $W_H \to \mathbb{C}^*$, which proves that W_H is a cyclic group. We define e_H to be the cardinality of W_H. Note that for $w \in W$, we have $wW_Hw^{-1} = W_{w.H}$, and thus $e_{w.H} = e_H$.

Lemma 6.2.2. *For all $H \in \mathcal{A}$, chose some $\alpha_H \in V^*$ such that $\operatorname{Ker}\alpha = H$. The function*

$$D(W) = D = \prod_{H\in\mathcal{A}} (\alpha_H)^{e_H} \in \mathbb{C}[V]$$

is W-invariant, and it is an equation for $\bigcup\mathcal{A}$ in V. It is also a reduced equation for the discriminant locus in $\mathbb{C}[V/W]$.

Proof. For $w \in W$, and $H \in \mathcal{A}$, we have $\operatorname{Ker}(w.\alpha_H) = w.H$ and thus $w.\alpha_H = \lambda_{w,H}\alpha_{w.H}$ for some nonzero scalar $\lambda_{w,H}$. One easily deduce several relations on the scalars $\lambda_{w,H}$, for instance $\lambda_{w^k,H} = \lambda_{w,H}\lambda_{w,w.H}\cdots\lambda_{w,w^{k-1}.H}$. Similarly, if $w.H = H$, then $w \in W_H$ is a reflection of reflecting hyperplane H, and $\lambda_{w,w.H}$ is a e_H-th root of unity.

Let $\{H_1,\dots,H_{k-1}\}$ be an w-orbit in $\mathcal{A}$, written so that $w.H_i = H_{i+1}$ for $i \in [\![1,k-2]\!]$ and $w.H_{k-1} = H_1$. We have $w^n.H_1 = H_1$, and thus $\lambda_{w^k,H_1} = \lambda_{w,H}\lambda_{w,H_2}\lambda_{w,H_{k-1}}$ is an e_H-th root of unity. In particular, we get that

$$\prod_{i=1}^{k}(\alpha_{H_i})^{e_{H_i}} \in \mathbb{C}[V]$$

is w-invariant. Since this holds for all orbits of w acting on $\mathcal{A}$, we obtain that

$$D(W) = D = \prod_{H\in\mathcal{A}} (\alpha_H)^{e_H} \in \mathbb{C}[V]$$

is w-invariant. Since this holds for all $w \in W$, we obtain that $D \in \mathbb{C}[V]^W$. Of course D is well-defined up to scalar multiplication in $\mathbb{C}[V]$. Nonetheless, we see that, for $v \in V$, we have $D(v) = 0$ is and only if v belongs to the union $\bigcup\mathcal{A}$ of all the reflecting hyperplanes of W. The fact that D is reduced in $\mathbb{C}[V]^W$ comes from [Bes01, Definition 1.2]. $\square$

More specifically, this lemma also gives us that, if $\mathcal{O}$ is a W-orbit in $\mathcal{A}$, then

$$\prod_{H\in\mathcal{O}} (\alpha_H)^{e_H} \in \mathbb{C}[V]$$

is W-invariant, and it is the reduced equation of $\pi(H)$ in V/W for $H \in \mathcal{O}$.

By construction, for $H \in \mathcal{A}$, the integer e_H is equal to 1 plus the number of reflections of W with reflecting hyperplane H. The degree of $D(W)$ in $\mathbb{C}[V]$ is then equal to $|\mathcal{A}| + |\operatorname{Ref}(W)|$. By Lemma 6.1.5, this is also equal to the sum of the degrees and the codegrees of W.

We saw that D belongs to $\mathbb{C}[V/W]$. However, if we wish to understand the discriminant locus $\mathcal{H}$ less abstractly, we would like to use the Chevalley-Shephard-Todd to see $\mathcal{H}$ as an explicit hypersurface in $\mathbb{C}^n$. Of course this depends on the choice of a system of basic invariants.

Definition 6.2.3 (Discriminant). If $f = (f_1,\dots,f_n)$ is a system of basic invariants for W, then the identification $\mathbb{C}[V]^W \simeq \mathbb{C}[X_1,\dots,X_n]$ allows us to write $D(W)$ as a polynomial $\Delta(W,f) \in \mathbb{C}[X_1,\dots,X_n]$, that we call the *discriminant* of W (attached to the system f).

The first thing we must keep in mind is that the polynomial $\Delta(W,f)$ depends on the choice of f, as seen in the example below:

Example 6.2.4. Consider the group $W = G(4,2,2)$. By [LT09, Section 2.8], a system of basic invariants for W is given by $P(X,Y) = X^2Y^2, Q(X,Y) = X^4 + Y^4$. The arrangement $\mathcal{A}(W)$ contains 6 hyperplanes, given by the following equations:

$$x = 0, \quad y = 0, \quad x = y, \quad x = iy, \quad x = -y, \quad x = -iy.$$

We then have

$$\begin{aligned} D(G(4,2,2)) &= X^2Y^2(X-Y)^2(X+Y)^2(X-iY)^2(X+iY)^2 \\ &= X^2Y^2(X^2-Y^2)^2(X^2+Y^2)^2 \\ &= X^2Y^2(X^4-Y^4)^2 \\ &= X^2Y^2((X^4+Y^4)^2-4X^4Y^4). \end{aligned}$$

And thus $\Delta(W,(P,Q)) = P(Q^2-4P^2) = PQ^2-4P^3$. If we consider the system of basic invariants $P(X,Y) = X^2Y^2$ and $Q_1(X,Y) = (X^2+Y^2)^2$, then we have $\Delta(W,(P,Q_1)) = P(Q_1-4P)Q_1 = PQ_1^2 - 4P^2Q_1$.

By construction, the polynomial $\Delta(W,f)$ is weighted homogeneous of weight $|\mathcal{A}| + |\operatorname{Ref}(W)|$, but it needs not be homogeneous for the usual graduation of $\mathbb{C}[X_1,\ldots,X_n]$, as seen in the example below:

Example 6.2.5. Consider the group $W = G(6,3,2)$. By [LT09, Section 2.8], a system of basic invariants for W is given by $P(X,Y) = X^2Y^2, Q(X,Y) = X^6 + Y^6$. The arrangement $\mathcal{A}(W)$ contains 8 hyperplanes, given by the following equations:

$$x = 0, \quad y = 0, \quad x = y, \quad x = -j^2y, \quad x = jy, \quad x = -y, \quad x = j^2y, \quad x = -jy.$$

We then have

$$\begin{aligned} D(G(6,3,2)) &= X^2Y^2(X-Y)^2(X+Y)^2(X-jY)^2(X+jY)^2(X-j^2Y)^2(X+j^2Y)^2 \\ &= X^2Y^2(X^2-Y^2)^2(X^2-j^2Y^2)^2(X^2-jY^2)^2 \\ &= X^2Y^2(X^6-Y^6)^2 \\ &= X^2Y^2((X^6+Y^6)^2-4X^6Y^6), \end{aligned}$$

and thus $\Delta(W,(P,Q)) = P(Q^2-4P^3) = PQ^2-4P^4$ is not homogeneous for the usual graduation of $\mathbb{C}[P,Q]$.

Moreover, two distinct groups may have the same discriminant polynomial (for convenient choices of systems of basic invariants). This situation will play an important role later on when considering braid group. The term “isodiscriminantal” was coined by Bessis to describe this situation.

Definition 6.2.6 (Isodiscriminantality). [Bes15, Definition 2.1]
Let V, V' be two finite dimensional complex vector spaces, and let $W \subset \mathrm{GL}(V), W' \subset \mathrm{GL}(V')$ be two complex reflection groups. The groups W and W' are called *isodicriminantal* if there are respective systems of basic invariants f, f' for W and W' such that $\Delta(W,f) = \Delta(W',f)$.

Let W, W' be two complex reflection groups, and let f, f' be two systems of basic invariants such that $\Delta(W,f) = \Delta(W',f')$. The homeomorphism between X/W and the set $\{x \in \mathbb{C}^n \mid \Delta(W,f)(x) \neq 0\}$ induces a homeomorphism $X/W \simeq X'/W'$.

A surprising theorem is that every complex reflection group is isodiscriminantal to a complex reflection group generated by involutive reflections [Bes15, Theorem 2.2]. This result is obtained by directly observing the Chevalley-Shephard-Todd classification, and no general proof is in sight. Using the Chevalley-Shephard-Todd classification, we will obtain a complete classification of irreducible complex reflection groups in Theorem 6.2.30.

The intersection lattice $L(\mathcal{A})$ endows V with a stratification, which in turn induces a stratification of V/W. This stratification will be useful to us when considering parabolic subgroups of braid groups, and we introduce it now.

For $F \in L(\mathcal{A})$, consider F^0 to be the complement in F of the flats strictly included in F. By construction, we have $V^0 = X$, and $(V^W)^0 = V^W$. We also have that F is the closure of F^0 in V. Note that, for $v \in V$, the parabolic subgroup $W_0 := \{w \in W \mid w.v = v\}$ is attached to the unique flat F such that $v \in F^0$. Indeed, let H be a hyperplane containing v, $F \cap H$ is a flat contained in F, and which contains v. Since $v \in F^0$, $F \cap H$ cannot be a flat strictly included in F, and thus $F \cap H = F$ and $F \subset H$. Thus a hyperplane $H \in \mathcal{A}$ contains F if and only if it contains v.

The set $(F^0)_{F \in L(\mathcal{A})}$ is then in bijection with the lattice of parabolic subgroups of W. Furthermore, we can endow this set with the following relation:

$$\forall F, L \in L(\mathcal{A}),\ F^0 \leqslant L^0 \Leftrightarrow F \leqslant L \Leftrightarrow L^0 \subset \overline{F^0},$$

which endows it with a lattice structure, again isomorphic with the lattice of parabolic subgroups of W by Lemma 6.1.9 (lattice of parabolic subgroups).

Lemma 6.2.7. *[OT92, Lemma 6.107] Let $F \in L(\mathcal{A})$, and let W_F be the parabolic subgroup of W fixing F pointwise. The restriction of π to F^0 is a $|W/W_F|$-fold covering $F^0 \to \pi(F^0)$.*

Proof. The only part not proven in [OT92, Lemma 6.107] is the cardinality of the fiber. Let $x \in \pi(F^0)$, the fiber $\pi^{-1}(x)$ is the W-orbit of some point $v \in F^0$. The cardinality of such an orbit is the index in W of the stabilizer of v in W, which we know is equal to W_F. □

Definition 6.2.8 (Discriminant stratification). By [OT92, Definition 5.16], the set $(F^0)_{F \in L(\mathcal{A})}$ is a stratification of V. Since W acts on $L(\mathcal{A})$, this stratification induces a stratification $(\pi(F^0))_{\pi(F) \in L(\mathcal{A})/W}$ of V/W, that we call the *discriminant stratification*.

The set $(\pi(F^0))_{\pi(F) \in L(\mathcal{A})/W}$ is endowed with a partial ordering inherited from that of $(F^0)_{F \in L(\mathcal{A})}$:

$$\forall F, L \in L(\mathcal{A}),\ \pi(F^0) \leqslant \pi(L^0) \Leftrightarrow \exists w \in W \mid F^0 \leqslant w.L^0 \Leftrightarrow \exists w' \in W \mid w'.F^0 \leqslant L^0.$$

Since the isomorphism between $(L(\mathcal{A}), \leqslant)$ and the lattice of parabolic subgroups of W given in Lemma 6.1.9 is also an isomorphism of W-sets, we easily obtain the following result:

Lemma 6.2.9. *The isomorphism of lattices between $((F^0)_{F \in L(\mathcal{A})}, \leqslant)$ and the lattice of parabolic subgroups of W induces a lattice isomorphism between $((\pi(F^0))_{\pi(F) \in L(\mathcal{A})/W}, \leqslant)$ and the lattice of conjugacy classes of parabolic subgroups of W.*

Let now $P \in \mathbb{C}[V]^W$. The identification $\mathbb{C}[V]^W \simeq \mathbb{C}[X_1, \dots, X_n]$ given by a system of basic invariants allows us to define the valuation of P as the degree of its lowest degree monomial (in the usual graduation of $\mathbb{C}[X_1, \dots, X_n]$). An argument of [Bes01, Section 1.2] shows that the valuation of P defined this way does not depend on the choice of a system of basic invariants.

Definition 6.2.10 (Multiplicity). Let f be a system of basic invariants for W. The *multiplicity* of some $x \in V/W$ in the discriminant stratification of V/W is defined as the valuation $\mathrm{mul}_{\mathcal{H}}$ of the polynomial $\Delta(W,f)_x$ defined by

$$\Delta(W,f)_x(X_1,\ldots,X_n) = \Delta(W,f)(X_1+\overline{f}(x)_1,\ldots,X_n+\overline{f}(x)_n),$$

where $\overline{f} : V/W \to \mathbb{C}^n$ is the homeomorphism induced by f.

By [Dou17, Definition 32 and Proposition 33], this definition does not depend on the choice of a system of basic invariants. By construction, $\mathrm{mul}_{\mathcal{H}}(x) = 0$ if and only if $D(v) \neq 0$ for $v \in \pi^{-1}(x)$, i.e. if $x \in X/W$. Likewise, $\mathrm{mul}_{\mathcal{H}}(0)$ is simply the valuation of $\Delta(W,f)$ for some (any) system of basic invariants f.

The multiplicity of points in the discriminant hypersurfaces is closely related to well-generatedness, as we will see in Proposition 6.2.19 below. Moreover, we will see in Proposition 7.3.9 that this multiplicity is also related to parabolic subgroups of complex braid groups.

Example 6.2.11. Consider the group $W = G(4,2,2)$. We saw in Example 6.2.4 that a system of basic invariants for W is given by $P(X,Y) = X^2Y^2$ and $Q(X,Y) = X^4+Y^4$, and we have $\Delta(W,(P,Q)) = PQ^2 - 4P^3$. For $a,b \in \mathbb{C}$, we have

$$(P+a)(Q+b)^2-4(P+a)^3 = PQ^2-4P^3+2bPQ+aQ^2-12P^2a+P(b^2+2a)+2abQ+ab^2-4a^3.$$

Let $x \in V/W$ be the orbit associated to $(a,b) \in \mathbb{C}^2$ under the homeomorphism $V/W \simeq \mathbb{C}^2$ induced by (P,Q). The multiplicity of x in the discriminant stratification is zero if $ab^2-4a^3 \neq 0$ (i.e. $\Delta(W,(P,Q))(a,b) \neq 0$, and $x \notin \mathcal{H}$). Furthermore, we have

$$\begin{cases} b^2+2a = 0, \\ 2ab = 0. \end{cases} \Leftrightarrow a = b = 0.$$

Thus the multiplicity of x in the discriminant stratification cannot be 2, since $b^2+2a = 2ab = 0$ implies $b = a = 0$. The multiplicity of x in the discriminant stratification is then equal to

- 0 if $x \notin \mathcal{H}$,
- 1 if $x \in \mathcal{H}$ and $x \neq 0$.
- 3 if $x = 0$.

6.2.2 Braid groups and braided reflections

We are now ready to define complex braid groups and pure complex braid groups, as the respective fundamental groups of the spaces X/W and X attached to a complex reflection group W as before. We will then use the geometry of these spaces to provide a set of generators, called braided reflections.

In this section, we fix a finite dimensional complex vector space V, and we fix n to be the dimension of V. We also fix a complex reflection group $W \subset \mathrm{GL}(V)$ and we keep the above notation. In particular X designates the complement in V of the union Σ of the reflecting hyperplanes of W.

Definition 6.2.12 (Braid groups). Let $v \in X$, the *pure braid group* attached to W is defined as $P(W) := \pi_1(X,v)$. The *braid group* attached to W is defined as $B(W) := \pi_1(X/W, W.v)$.

Since the spaces X and X/W are path connected, the isomorphism type of $P(W)$ and $B(W)$ does not depend on the choice of a basepoint in V. Recall from Proposition 6.2.1 (discriminant hypersurface) that the quotient map $\pi : V \to V/W$ induces a covering map $X \to X/W$ with fiber W. This yields a short exact sequence

$$1 \longrightarrow P(W) \longrightarrow B(W) \xrightarrow{p} W \longrightarrow 1.$$

The map $P(W) \to B(W)$ is simply obtained by functoriality of the fundamental group. The map $p : B(W) \to W$ is obtained in the following way: let $b \in B(W)$ be represented by a path $\gamma : W.v \to W.v$ in X/W. Since the projection map $X \to X/W$ is a covering map, there is a unique path $\widetilde{\gamma}$ in X which lifts γ and which starts from v. The target of $\widetilde{\gamma}$ is an element of the orbit $W.v$, and thus can be uniquely written $w.v$ for some $w \in W$. This $w \in W$ is the image of b under p.

Example 6.2.13. The situation is very easily described if $W = \mu_d \subset \mathbb{C}^*$ is a cyclic group. A system of basic invariants is clearly given by $f_1 = X^d$. And the map $z \mapsto z^d$ from $\mathbb{C}$ to itself induces a homeomorphism $\mathbb{C}/W \simeq \mathbb{C}$. Under this homeomorphism, the discriminant locus $\{0\}$ is sent to $\{0\}$. We obtain that $B(W)$ is the fundamental group of $\mathbb{C}^*$, i.e. an infinite cyclic group.

The first thing we can do with braid groups is giving a generating system. This is actually an application of a general method giving generating systems of fundamental groups of complements of algebraic hypersurfaces detailed in [BMR98, Appendix 1].

We assume that V is endowed with a W-invariant hermitian scalar product. Let $*$ be our chosen basepoint in V for defining $P(W)$ and $B(W)$. Let $s \in \mathrm{Ref}(W)$ be a reflection of reflecting hyperplane $H \in \mathcal{A}$. Let also $v_0 \in H^0$, and let U be an open ball containing v_0 and such that $U \cap H = U \cap \Sigma$ (i.e. H is the only hyperplane in $\mathcal{A}$ which intersects U nontrivially). Let $v_1 \in (H^\perp + v_0) \cap U$. By construction, we have $\pi_1(U \cap X, v_1) \simeq \mathbb{Z}$ since $U \cap X$ is homotopically equivalent to $\mathbb{C}^*$.

If we write $v_1 = v_0 + \alpha$ with $\alpha \in H^\perp$, then we have $s.v_1 = s.v_0 + s.\alpha = v_0 + \zeta\alpha$, where $\zeta = e^{i\theta}$ is the nontrivial eigenvalue of s acting on V. We can then consider $r_\zeta : [0,1] \to U$ defined as $r_\zeta(t) = v_0 + e^{it\theta}\alpha$. It is a path from v_1 to $s.v_1$ in U.

Definition 6.2.14 (Braided reflections). With the above notation, the class in $B(W)$ of a path in X of the form $\gamma * r_\zeta * (s.\gamma)^{-1}$, where γ is a path from v to v_1 in X, is called a *braided reflection* (attached to s).

By construction, the image under p of a braided reflection attached to $s \in \mathrm{Ref}(W)$ is s itself. By construction again, a conjugate in $B(W)$ of a braided reflection is again a braided reflection. The first result we have on braided reflections is that their conjugacy is determined by conjugacy of reflections in W.

Proposition 6.2.15. *[Bro01, Lemma 2.12] Two braided reflections $\sigma, \sigma' \in B(W)$ are conjugate in $B(W)$ if and only if $p(\sigma)$ and $p(\sigma')$ are conjugate in W. Furthermore, σ and σ' are conjugate in $B(W)$ by an element of $P(W)$ if and only if $p(\sigma) = p(\sigma')$.*

Now, if $s, s' \in \mathrm{Ref}(W)$ are such that $s^n = s'$ for some $n \geqslant 1$ inferior to the order of s, then the n-th power of a braided reflection attached to s is a braided reflection attached to s'. Likewise, if n is the order of s in W, then the n-th power of a braided reflection attached to $B(W)$ actually lies in $P(W)$. We then want to consider braided reflections which are minimal in the sense that they cannot be written as a power of another braided reflection.

Let $H \in \mathcal{A}$, and let e_H be the cardinality of the parabolic subgroup W_H of W. We know that det is an injective morphism $W_H \to \mathbb{C}^*$. There is then a unique reflection in W_H of nontrivial eigenvalue $e^{\frac{2i\pi}{e_H}}$. This reflection generates W_H, and we call it the *distinguished reflection* attached to H.

Note that the set of distinguished reflections generate $\mathrm{Ref}(W)$ and thus W. Note also that the set of distinguished reflections of W is equal to $\mathrm{Ref}(W)$ if and only if W contains only involutive reflections.

We call *distinguished braided reflection* a braided reflection in $B(W)$ attached to a distinguished reflection in W.

Proposition 6.2.16. *[BMR98, Theorem 2.17, Proposition 2.18] The group $B(W)$ is generated by (distinguished) braided reflections, and the set*

$$\{\sigma^n \mid \sigma \text{ is a braided reflection attached to a reflection of order } n\}$$

generates $P(W)$.

Lemma 6.2.17. *[BMR98, Proposition 2.16] The morphism $\ell : B(W) \to \mathbb{Z}$ induced by $D : X/W \to \mathbb{C}^*$ will be called the* length function*. It is the unique morphism $B(W) \to \mathbb{Z}$ which sends distinguished braided reflections to* 1 *in $\mathbb{Z}$.*

In particular, since $\ell(x) = 1$ implies that x is rootless in $B(W)$, we obtain that distinguished braided reflections are exactly the rootless braided reflections in $B(W)$, or the braided reflections of length 1 in $B(W)$. Combining Proposition 6.2.16 with Proposition 6.2.15 can be used to compute the abelianization of $B(W)$.

Proposition 6.2.18. *[BMR98, Theorem 2.17] The abelianization of $B(W)$ is a free abelian group of rank $|\mathcal{A}/W|$, generated by images of distinguished braided reflections.*

Now, as we stated in the last section, well-generatedness of the group W can be read on both its braid group $B(W)$, and on the associated discriminant stratification by the following proposition:

Proposition 6.2.19. *[Bes15, Proposition 4.2] The following integers are equal:*

- *The minimum number of reflections needed to generate W.*
- *The minimum number of braided reflections needed to generate $B(W)$.*
- *The multiplicity of* 0 *in the discriminant stratification of V/W.*

In particular, the minimum number of reflections needed to generate W is equal to the minimum number of braided reflections needed to generate $B(W)$.

Proof. The original proof by Bessis only covers the case where W is irreducible. The general case is easily deduced from Lemma 6.2.26. ☐

Remark 6.2.20. Proposition 6.2.19 also proves that well-generatedness of W can be detected by only looking at $B(W)$ and its braided reflections. We can then talk about “well-generated complex braid groups” instead of “complex braid group associated to a well-generated complex reflection group”.

Proposition 6.2.19 gives us a first geometric characterization of well-generated complex reflection groups. The following result gives a stronger characterization in terms of the geometry of the discriminant. It is a consequence of [Bes15, Theorem 2.4].

Theorem 6.2.21. *Let W be an irreducible complex reflection group of rank n. The following assertions are equivalent:*

(i) W is well-generated.

(ii) For any system of basic invariants f, we have $\frac{\partial^n \Delta(W,f)}{(\partial X_n)^n} \in \mathbb{C}^*$. In other words, $\Delta(W,f)$, viewed as a polynomial in X_n, with coefficients in $\mathbb{C}[X_1,\dots,X_{n-1}]$, is monic of degree n.

(iii) For one system of basic invariants f, we have $\frac{\partial^n \Delta(W,f)}{(\partial X_n)^n} \in \mathbb{C}^*$.

Proof. The equivalence between (i) and (ii) is [Bes15, Theorem 2.4], and $(ii) \Rightarrow (iii)$ is trivial. Now, assume that (iii) holds, and let f be a system of basic invariants such that $\Delta(W,f)$ is monic of degree n in $(\mathbb{C}[X_1,\dots,X_{n-1}])[X_n]$. The polynomial $\Delta(W,f)$ then contains a monomial of the form aX_n^n, and it has valuation at most n. By Proposition 6.2.19, the valuation of $\Delta(W,f)$ is the number of reflections needed to generate W. The group W can then be generated by a family of reflections of cardinality $\leqslant n$, and it is well-generated. □

If W is well-generated, then the sum of degrees and codegrees of W is equal to nd_n by Theorem 6.1.22 (characterization of well-generated groups), we then know that $\Delta(W,f)$ is weighted homogeneous of weighted degree nd_n. By Theorem 6.2.21, we can write

$$\Delta(W,f) = X_n^n + \alpha_1(X_1,\dots,X_{n-1})X_n^{n-1} + \cdots + \alpha_n(X_1,\dots,X_{n-1}).$$

By weighted homogeneity, we have that α_i is weighted homogeneous of weighted degree id_n.

In particular, $\alpha_i(f_1,\dots,f_n) \in \mathbb{C}[V]^W$ has degree d_n, and $f_n' := f_n - \frac{\alpha_1(f_1,\dots,f_n)}{n}$ is a homogeneous element of $\mathbb{C}[V]^W$ of degree d_n. Furthermore, $f' := (f_1,\dots,f_{n-1},f_n')$ is a system of basic invariants for W such that $\Delta(W,f')$ can be written

$$\begin{aligned}\Delta(W,f') &= \left(X_n' + \frac{\alpha_1(X_1,\dots,X_n)}{n}\right)^n + \cdots + \alpha_n(X_1,\dots,X_{n-1}) \\ &= X_n'^n + \alpha_2'(X_1,\dots,X_{n-1})X_n'^{n-2} + \cdots + \alpha_n'(X_1,\dots,X_{n-1}).\end{aligned}$$

These kind of decompositions of the discriminant will be the starting point of Section 8.1, where we study Lyashko-Looijenga morphisms.

Remark 6.2.22. Let W, W' be two complex reflection groups that are isodiscriminantal. We have a homeomorphism $X/W \simeq X'/W$, which induces a group isomorphism $B(W) \simeq B(W')$. Moreover, since the homeomorphism $X/W \simeq X'/W'$ comes from an isomorphism between $\mathbb{C}[V/W]$ and $\mathbb{C}[V'/W']$, this isomorphism preserves all "geometric information", like braided reflection, or multiplicity in the discriminant stratification.

Moreover, if W is well-generated, then W' is also well-generated by Proposition 6.2.19. If f, g are respective systems of basic invariants for W and W' such that $\Delta(W,f) = \Delta(W',g)$, then taking f', g' as above gives us two systems of basic invariants which are also such that $\Delta(W,f') = \Delta(W',g')$.

6.2.3 Interlude: the rank 2 case

In the last section, we skipped the proof of Theorem 6.2.21, which plays a fundamental role in the work of Bessis on understanding the geometry of the space X/W in the case where W is well-generated. Indeed, it is crucial in the construction of the Lyashko-Looijenga morphism of Section 8.1. In this section, we propose a proof of Theorem 6.2.21 in the case where W has rank 2. This proof is seemingly case-free, but it uses Theorem 6.1.22 (characterization of well-generated groups), which itself is proven by a case-by-case analysis. As a byproduct, we obtain that well-generated irreducible complex reflection groups of rank 2 are all isodiscriminantal to a group $G(e,e,2)$ with $e>1$.

This short section can be skipped without logical harm, it should be seen more as a guiding example for the results we are interested in on complex reflection groups and their discriminant.

In this section, we fix a 2-dimensional complex vector space V, along with an irreducible complex reflection group $W \subset \mathrm{GL}(V)$. We denote its degrees by $d \leqslant h$, and its codegrees by $0 \leqslant d^*$.

We know that the polynomial function $D(W)$ is homogeneous of degree $d+h+d^*$ in $\mathbb{C}[V]^W \subset \mathbb{C}[V]$.

Let (P,Q) be a system of basic invariants for W. If P^iQ^j is a monomial appearing with nonzero coefficient in $\Delta(W,(P,Q))$, then $di+jh = d+h+d^*$ by weighted homogeneity, i.e. we have $d(i-1)+h(j-1)=d^*$. Obviously, there cannot be a valuation 1 monomial (i.e. $i+j=1$) for which $d(i-1)+h(j-1)=d^*$ can hold.

Lemma 6.2.23. *The group W is well-generated if and only if $\Delta(W,(P,Q))$ has valuation 2. In this case, d divides $2h$ and the discriminant is written*

$$\Delta(W,(P,Q)) = \alpha Q^2 + \beta Q P^{\frac{h}{d}} + \gamma P^{\frac{2h}{d}},$$

where $\alpha \neq 0$ and possibly $\beta = 0$ (for instance if d does not divide h).

Proof. Assume that $\Delta(W,(P,Q))$ has valuation 2, and let $P^{2-j}Q^j$ be a monomial of valuation 2 in $\mathbb{C}[P,Q]$ which appears in $\Delta(W,(P,Q))$. By weighted homogeneity, we have $d(1-j)+h(j-1)=d^*$. If $j=0$, then we have $d-h=d^* \leqslant 0$, and $d^*=0$ is impossible since W is irreducible. If $j=1$, then we also have $0=d^*$. We must then have $j=2$, and we obtain that $h-d=d^*$, and W is well-generated by Theorem 6.1.22.

Conversely, assume that W is well-generated. Let P^iQ^j be a monomial appearing in $\Delta(W,(P,Q))$ with nonzero coefficient. By weighted homogeneity, we have $di+h(j-2)=0$ since $d+d^*=h$. Since both i and j are nonnegative, we either have $j=2, i=0$, $j=1, i=\frac{h}{d}$ (in which case, d must divide h), or $j=0$ and $i=\frac{2h}{d}$ (in which case, d must divide $2h$). The discriminant can then be written as

$$\Delta(W,(P,Q)) = \alpha Q^2 + \beta Q P^{\frac{h}{d}} + \gamma P^{\frac{2h}{d}}.$$

We claim that d divides $2h$. If this is not the case, then d does not divide h either, and we must have $\Delta(W,(P,Q)) = \alpha Q^2$, with $\alpha \neq 0$ since $\Delta(W,(P,Q))$ is nonzero. In this case, we have

$$X/W \simeq \{(p,q) \in \mathbb{C}^2 \mid q^2 \neq 0\} = \mathbb{C} \times \mathbb{C}^*,$$

and thus $B(W) \simeq \mathbb{Z}$, and W is cyclic as a quotient of $\mathbb{Z}$. However, W cannot be abelian as $|Z(W)| = d \wedge h \neq dh = |W|$. ‘

It remains to show that $\alpha \neq 0$. Since $d^* \neq 0$, we have $d, d^* < h$, and thus h divides exactly one degree. By Theorem 6.1.27 (characterization of regular numbers), we have that h is a regular number for W. Let $g \in W$ be ζ_h-regular, and let $v \in X$ be a ζ_h-eigenvector for g. Since v and $\zeta_h v = g.v$ lie in the same W-orbit, we have $P(v) = P(\zeta_h v) = \zeta_h^d P(v)$, and thus $P(v) = 0$ since $\zeta_h^d \neq 1$. Since $v \in X$, we have $\Delta(W,(P,Q))(P(v),Q(v)) = \alpha Q(v)^2 \neq 0$, and thus $\alpha \neq 0$. □

As a corollary, we obtain a proof of Theorem 6.2.21 for irreducible complex reflection groups of rank 2. Indeed, if W is well-generated, then $\frac{\partial^2 \Delta(W,(P,Q))}{\partial^2 Q} = \alpha \neq 0$ by Lemma 6.2.23, which is $(i) \Rightarrow (ii)$ of Theorem 6.2.21. The implication $(ii) \Rightarrow (iii)$ is always immediate. Lastly, the implication $(iii) \Rightarrow (i)$ comes from the fact that, if $\Delta(W,(P,Q))$ is monic of degree 2, then it has valuation 1 or 2. Since $\Delta(W,(P,Q))$ having valuation 1 is impossible, we obtain that W is well-generated by Lemma 6.2.23.

Remark 6.2.24. If W is badly-generated, then $\Delta(W,(P,Q))$ must have valuation at least 3. If we apply Proposition 6.2.19, then we obtain that $\Delta(W,(P,Q))$ has valuation exactly 3. For $W = G(4,2,2)$, all the monomials of degree 3 can appear in $\Delta(W,(P,Q))$ for convenient choices of (P,Q):

- For $P(X,Y) = X^2Y^2, Q(X,Y) = X^4 + Y^4$, we showed in Example 6.2.4 that $\Delta(W,(P,Q)) = PQ^2 - 4P^3$.
- For $P(X,Y) = X^4 + Y^4, Q(X,Y) = X^2Y^2$, we have $\Delta(W,(P,Q)) = QP^2 - 4Q^3$.

Corollary 6.2.25. *Assume that W is well-generated. There is a system of basic invariants (P,Q) for W such that the discriminant is written $\Delta(W,(P,Q)) = Q^2 - P^{\frac{2h}{d}}$. In particular, the braid group of W depends only on the value of $\frac{2h}{d}$. It is isomorphic to the Artin group of type $I_2(\frac{2h}{d})$.*

Proof. Let (P,Q) be a system of basic invariants for W. By Lemma 6.2.23, the discriminant is written

$$\Delta(W,(P,Q)) = \alpha Q^2 + \beta Q P^{\frac{h}{d}} + \gamma P^{\frac{2h}{d}},$$

with $\alpha \neq 0$, and β, γ are not both 0. First, let a be a square root of α in $\mathbb{C}$, and let $Q_1 = aQ + \frac{\beta P^{\frac{h}{d}}}{2a}$. The couple (P,Q_1) is a system of basic invariants for W, and we have

$$\Delta(W,(P,Q_1)) = Q_1^2 + P^{\frac{2h}{d}} \frac{4\alpha\gamma - \beta^2}{4\alpha}.$$

Furthermore, since we saw in the proof of Lemma 6.2.23 that $\Delta(W,(P,Q_1))$ cannot be equal to Q_1^2, we have that $\frac{4\alpha\gamma-\beta^2}{4\alpha} \neq 0$. Let then $r \neq 0$ be a $\frac{2h}{d}$-th root of $\frac{\beta^2-4\alpha\gamma}{4\alpha}$, and let $P_1 = rP$. The couple (P_1,Q_1) is a system of basic invariants for W, and we have

$$\Delta(W,(P_1,Q_1)) = Q_1^2 - P_1^{\frac{2h}{d}}.$$

In particular, we obtain that W is isodiscriminantal to $G(\frac{2h}{d}, \frac{2h}{d}, 2)$. Let now $e > 1$. by [Bri71], it is known that the fundamental group of the space $\{(x,y) \in \mathbb{C}^2 \mid x^2 \neq y^e\}$ is the Artin group of type $I_2(e)$. We obtain the result as X/W is homeomorphic to $\{(x,y) \in \mathbb{C}^2 \mid x^2 \neq y^{\frac{2h}{d}}\}$ by isodiscriminantality. □

We finish this section by giving the value of $\frac{2h}{d}$ for all irreducible well-generated complex reflection groups of rank 2.

- For $r > 1$, the degrees of $G(r,1,2)$ are $d = r, h = 2r$, and we have $\frac{2h}{d} = 4$.
- For $e > 1$, the degrees of $G(e,e,2)$ are $d = 2, h = e$, and we have $\frac{2h}{d} = e$.
- For $i \in \{4,8,16\}$, the number $\frac{2h}{d}$ attached to the exceptional group G_i is 3.
- For $i \in \{5,10,18\}$, the number $\frac{2h}{d}$ attached to the exceptional group G_i is 4.
- For $i \in \{6,9,17\}$, the number $\frac{2h}{d}$ attached to the exceptional group G_i is 6.
- The degrees of G_{14} are $d = 6, h = 24$, and we have $\frac{2h}{d} = 8$.
- The degrees of G_{20} are $d = 12, h = 30$, and we have $\frac{2h}{d} = 5$.
- The degrees of G_{21} are $d = 12, h = 60$, and we have $\frac{2h}{d} = 10$.

6.2.4 Braid groups of irreducible complex reflection groups

In the sequel, several theorems on complex braid groups will be shown through a case-by-case analysis on braid groups associated to irreducible complex reflection groups (this is already the case of a few theorems in the last section). Thus we are interested in studying braid groups of irreducible complex reflection groups more precisely. We begin by showing that studying braid groups of irreducible complex reflection groups is enough to understand all braid groups of complex reflection groups.

Let $W \subset \mathrm{GL}(V)$ be a complex reflection group, and let V be endowed with a W-invariant hermitian scalar product. Let $V = V^W \perp V_1 \perp \cdots \perp V_r$ and $W = W_1 \times \cdots \times W_r$ be the decomposition of W given by Proposition 6.1.13. We can, for $i \in [\![1, r]\!]$, consider the complement X_i of the reflecting hyperplanes of W_i acting on V_i. We obtain the following result:

Lemma 6.2.26. *The natural homeomorphism $V \simeq V^W \times V_1 \times \cdots \times V_k$ restricts to the following homeomorphisms:*

$$\begin{aligned} X &\simeq V^W \times X_1 \times \cdots \times X_r, \\ V/W &\simeq V^W \times V_1/W_1 \times \cdots \times V_r/W_r, \\ X/W &\simeq V^W \times X_1/W_1 \times \cdots \times X_r/W_r. \end{aligned}$$

In particular, we have $P(W) \simeq P(W_1) \times \cdots \times P(W_r)$ and $B(W) \simeq B(W_1) \times \ldots \times B(W_r)$.

Proof. An element $v \in V$ is associated with a unique tuple $(v_0, \ldots, v_k)$ under the identification $V \simeq V^W \times V_1 \times \ldots \times V_k$. An element $w = (w_1, \ldots, w_r)$ of $W \simeq W_1 \times \cdots \times W_r$ acts on $v = (v_0, \ldots, v_1)$ by

$$w.v = (v_0, w_1.v_1, \ldots, w_r.v_r).$$

From this we easily deduce that the W-orbit $W.v$ of v is identified $\{v_0\} \times W_1.v_1 \times \cdots \times W_r.v_r$, whence the second homeomorphism.

The stabilizer of v is then the product of the respective stabilizers of v_i inside W_i for $i \in [\![1, r]\!]$. The stabilizer of v is then trivial if and only if these respective stabilizers are all trivial, i.e. if $v \in V^W \times X_1 \times \cdots \times X_r$, whence the first homeomorphism. The third homeomorphism is an immediate consequence of the first two. □

Using this lemma, we will allow us to talk about irreducible complex braid groups (instead of complex braid groups associated to an irreducible complex reflection group). We are now interested in describing irreducible complex braid groups. Later on (see for instance Sections 8.2, 8.5 and 9.2) we will obtain finer details about Garside structures on irreducible complex braid groups, but for now on, we just give presentations of several irreducible complex braid groups, following the Shephard-Todd classification (the easy case of cyclic groups was done in Example 6.2.13).

Let $W := G(1,1,n)$. In Example 6.1.14, we saw that $\mathbb{C}^n$ decomposes as $\mathbb{C}^n = D \perp H$ (for the usual hermitian scalar product on $\mathbb{C}^n$), where D is the line generated by $(1,\dots,1)$ and H is the hyperplane defined by the equation $x_1+\cdots+x_n=0$. The group W acts as an irreducible complex reflection group on H. We denoted this group by $\widetilde{G}(1,1,n)$. A first application of Lemma 6.2.26 is that $\mathbb{C}^n/G(1,1,n) \simeq \mathbb{C}\times H/\widetilde{G}(1,1,n)$, and that $B(W)$ is isomorphic to $B_{n-1} := B(\widetilde{G}(1,1,n))$.

Moreover, since $G(1,1,n)$ acts on $\mathbb{C}^n$ by permuting the coordinates, we easily get that X is the set of vectors in $\mathbb{C}^n$ whose coordinates are pairwise distinct, and that X/W is the set of *configurations* of n unordered points in $\mathbb{C}$. In particular $B(G(1,1,n)) \simeq B_{n-1}$ is the *usual braid group* on n strands. A presentation of this group was given in [Art25], and we have

$$B_n = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall i \in [\![1,n-2]\!], \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \forall i,j \in [\![1,n-1]\!], |i-j|>1. \end{array} \right\rangle.$$

The generators are braided reflections, and σ_i is sent to the transposition $(i\ i+1)$ in $\mathfrak{S}_n$. More generally, Brieskorn showed in [Bri71] that if W is a complexified real reflection group, and if (W,S) is a Coxeter system for W, then $B(W)$ admits a presentation where generators are formal copies of elements of S, and where the relations have the form $\mathbf{sts}\cdots = \mathbf{tst}\cdots$ for $s,t\in S$ (each word has $m(s,t)$ letters, where $m(s,t)$ is the order of the product st in W). This way, the braid group $B(W)$ becomes a topological model for the more algebraic *Artin group* attached to W.

The next groups we are interested in are groups of the form $G(r,1,n)$ for integers $n,r>1$. The braid groups of these complex reflection groups can actually be studied "all at once" thanks to the following lemma:

Proposition 6.2.27. *Let $r,r',n>1$ be integers. The complex reflection groups $G(r,1,n)$ and $G(r',1,n)$ are isodiscriminantal.*

Proof. Let $\omega_1,\dots,\omega_n$ be the elementary symmetric polynomials. By [LT09, Section 2.8], a system of basic invariants for $G(r,1,n)$ is given by $f_i := \omega_i(X_1^r,\dots,X_n^r)$. The reflecting hyperplanes of $G(r,1,n)$ have respective equations $X_i=0$ for $i\in[\![1,n]\!]$, and $X_i-\zeta X_j$ for $1\leqslant i<j\leqslant n$ and $\zeta\in\mu_r$. Computing in the field of fractions $\mathbb{C}(X_1,\dots,X_n)$, we have

$$\prod_{\zeta\in\mu_r} X_i-\zeta X_j = X_j^r \prod_{\zeta\in\mu_r}\left(\frac{X_i}{X_j}-\zeta\right) = X_j^r\left(\frac{X_i^r}{X_j^r}-1\right) = (X_i^r-X_j^r).$$

We then obtain that

$$D(G(r,1,n)) = X_1^r\cdots X_n^r \prod_{i<j}\prod_{\zeta\in\mu_r}(X_i-\zeta X_j)^2 = (X_1\dots X_n)^r\prod_{i<j}(X_i^r-X_i^r)^2.$$

In the case of $G(1,1,n)$, we know that $D(G(1,1,n)) = \prod_{i<j}(X_i-X_j)^2$, we can then consider $P(\omega_1,\dots,\omega_n) = \Delta(G(1,1,n),\omega)$. Since the X_i^r are algebraically independent, we deduce that

$$\prod_{i<j}(X_i^r-X_j^r)^2 = P(f_1,\dots,f_n),$$

and thus $\Delta(G(r,1,n),f) = f_n P(f_1,\ldots,f_n)$. We obtain the desired result since this expression does not depend on r. □

In particular, we deduce that $B_n^* = B(G(r,1,n))$ depends only on n as soon as $r > 1$. Moreover, since $G(r,1,n)$ is isodiscriminantal to $B(2,1,n)$, which is a complexified real reflection group of type B, we obtain a presentation of this group by [Bri71]:

$$B_n^* = \left\langle \tau, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall i \in [\![2, n-1]\!], \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \forall i,j \in [\![2,n]\!],\ |i-j| > 1, \\ \tau\sigma_i = \sigma_i\tau & \forall i \in [\![3,n]\!], \\ \tau\sigma_2\tau\sigma_2 = \sigma_2\tau\sigma_2\tau, & \end{array} \right\rangle.$$

The generators are distinguished braided reflections, and the morphism $B_n^* \to G(r,1,n)$ sends τ to r_{1,ζ_r}, and σ_i to $s_{1,i-1,i}$. One can also show that B_n^* is the surface braid group on n strands attached to $\mathbb{C}^*$ (whereas B_n is the surface braid group on n strands attached to $\mathbb{C}$).

Consider now integers $d, e, n > 1$. The group $G(de,e,n)$ is a normal subgroup of $G(r,1,n)$, where $r = de$. We saw in Lemma 6.1.17 that $\mathcal{A}(G(de,e,n)) = \mathcal{A}(G(r,1,n))$, and thus the space $X(G(de,e,n)) = X(G(r,1,n))$ does not depend on e. We denote this space by $X_n(r)$. Since $G(de,e,n)$ is a normal subgroup of $G(r,1,n)$, the covering map $X_n(r) \to X_n(r)/G(r,1,n)$ factors through the covering map $X_n(r) \to X_n(r)/G(de,e,n)$, as in the following diagram:

$$\begin{tikzcd} X_n(r) \arrow[d, two heads] \arrow[dr, two heads] & \\ X_n(r)/G(de,e,n) \arrow[r, dashed, two heads] & X_n(r)/G(r,1,n) \end{tikzcd}$$

The bottom map $X_n(r)/G(de,e,n) \to X_n(r)/G(r,1,n)$ is a covering map corresponding to the action of $G(r,1,n)/G(de,e,n) \simeq \mathbb{Z}/e\mathbb{Z}$ on $X_n(r)/G(de,e,n)$, whence the following short exact sequence:

$$1 \longrightarrow B(G(de,e,n)) \longrightarrow B(G(r,1,n)) \longrightarrow \mathbb{Z}/e\mathbb{Z} \longrightarrow 1.$$

Now, recall that we have a presentation of $B(G(r,1,n)) \simeq B_n^*$. This presentation allows us to define a morphism $\mathrm{wd} : B_n^* \to \mathbb{Z}$ sending τ to 1 and σ_i to 0 that we call the *winding number* [CLL15, Definition 2.2].

Since the image of $r_{\zeta_d,1}$ in $G(r,1,n)/G(de,e,n)$ is a generator of $G(r,1,n)/G(de,e,n) \simeq \mathbb{Z}/e\mathbb{Z}$, and since the $s_{\zeta,i,j}$ all belong to $G(de,e,n)$, we obtain that the morphism $B_n^* \simeq B(G(r,1,n)) \to \mathbb{Z}/e\mathbb{Z}$ in the above short exact sequence sends τ to $1 \in \mathbb{Z}/e\mathbb{Z}$, and σ_i to $0 \in \mathbb{Z}/e\mathbb{Z}$ for $i \in [\![2,n]\!]$. It is equal to the winding number taken modulo e. The isomorphism $B_n^* \simeq B(G(r,1,n))$ then identifies $B(G(de,e,n))$ with the kernel of the winding number taken modulo e. The above short exact sequence then becomes

$$\begin{tikzcd} & & & \mathbb{Z} \arrow[d, two heads] & \\ 1 \arrow[r] & B_n^*(e) \arrow[r] & B_n^* \arrow[ur, "\mathrm{wd}"] \arrow[r] & \mathbb{Z}/e\mathbb{Z} \arrow[r] & 1 \end{tikzcd}$$

Notice again that $B_n^*(e) \simeq B(G(de,e,n))$ does not depend on d, as long as $d > 1$. In order to compute a presentation of $B_n^*(e)$ starting from our presentation of B_n^*, we can use the Reidemeister-Schreier method (see Section 1.5.2). Since by definition, all the σ_i for $i \in [\![2,n]\!]$

belong to $B_n^*(e)$, a Schreier transversal is given by $T = \{1, \tau, \ldots, \tau^{e-1}\}$. We obtain the following set of generators:

$$\widetilde{\tau} = \tau^e,\ \sigma_{2,k} = \tau^k \sigma_2 \tau^{-k}\ \forall k \in [\![0, e-1]\!],\ \sigma_3, \ldots, \sigma_n.$$

And the relations are given by

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall i \in [\![3, n-1]\!], \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \forall i,j \in [\![3, n]\!],\ |i-j| > 1, \\ \sigma_{2,k}\sigma_3\sigma_{2,k} = \sigma_3\sigma_{2,k}\sigma_3 & \forall k \in [\![0, e-1]\!], \\ \sigma_{2,k}\sigma_i = \sigma_i\sigma_{2,k} & \forall k \in [\![0, e-1]\!], i \in [\![4, n]\!], \\ \widetilde{\tau}\sigma_i = \sigma_i\widetilde{\tau} & \forall i \in [\![3, n]\!], \\ \sigma_{2,k}\sigma_{2,k+1} = \sigma_{2,0}\sigma_{2,1} & \forall k \in [\![1, e-2]\!], \\ \sigma_{2,e-2}\sigma_{2,e-1}\widetilde{\tau} = \sigma_{2,e-1}\widetilde{\tau}\sigma_{2,0} = \widetilde{\tau}\sigma_{2,0}\sigma_{2,1}. \end{cases}$$

The generators are braided reflections, and the morphism $B_n^*(e) \to G(de, e, n)$ sends $\widetilde{\tau}$ to $r_{\zeta_d,1}$, $\sigma_{2,k}$ to $s_{\zeta_{de}^k,1,2}$ and σ_i to $s_{1,i-1,i}$ for $i \in [\![3, n]\!]$.

Example 6.2.28. In the case where $d = e = n = 2$, the above presentation becomes a presentation of $B_2^*(2) = B(G(4, 2, 2))$ with generators $a = \sigma_{2,0}, b = \sigma_{2,1}$ and $c = \widetilde{\tau}$ and relations

$$ab = bcac^{-1} = c^{-1}bca \Leftrightarrow abc = bca = cab.$$

Lastly, if $e, n > 1$ are integers, we want to describe the braid group $B_n(e) := B(G(e, e, n))$. We cannot use the group $G(e, 1, n)$ as before, since the spaces $X(G(e, e, n))$ and $X(G(e, 1, n)) = X(e, n)$ are no longer equal. However, we have $X(e, n) = X(G(e, e, n)) \cap (\mathbb{C}^*)^n \subset X(G(e, e, n))$, and it is possible [Mar21, Chapter 7] to deduce that $B_n(e)$ is the quotient of $B_n^*(e)$ by the normal subgroup generated by $\widetilde{\tau}$. We deduce the following presentation of $B_n(e)$: The generators are given by $\sigma_{2,k}$ for $k \in [\![1, e-1]\!]$ along with $\sigma_3, \ldots, \sigma_n$. The relations are given by

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall i \in [\![3, n-1]\!], \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \forall i,j \in [\![3, n]\!],\ |i-j| > 1, \\ \sigma_{2,k}\sigma_3\sigma_{2,k} = \sigma_3\sigma_{2,k}\sigma_3 & \forall k \in [\![0, e-1]\!], \\ \sigma_{2,k}\sigma_i = \sigma_i\sigma_{2,k} & \forall k \in [\![0, e-1]\!], i \in [\![4, n]\!], \\ \sigma_{2,k}\sigma_{2,k+1} = \sigma_{2,0}\sigma_{2,1} & \forall k \in \mathbb{Z}/e\mathbb{Z}. \end{cases}$$

The generators are braided reflections, and the morphism $B_n(e) \to G(e, e, n)$ sends $\sigma_{2,k}$ to $s_{\zeta_{de}^k,1,2}$ and σ_i to $s_{1,i-1,i}$ for $i \in [\![3, n]\!]$. This presentation was first introduced by Corran and Picantin in [CP11, Section 2.2], and we will sometimes refer to it as the "Corran-Picantin presentation" of $B_n(e)$.

Example 6.2.29. In the case where $e = 2$, the above presentation becomes a presentation of $B_n(2)$ with generators $a := \sigma_{2,0}, b := \sigma_{2,1}, \sigma_3, \ldots, \sigma_n$ and relations

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall i \in [\![3, n-1]\!], \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \forall i,j \in [\![3, n]\!],\ |i-j| > 1, \\ a\sigma_i = \sigma_i a,\ b\sigma_i = \sigma_i b & \forall i \in [\![4, n]\!], \\ a\sigma_2 a = \sigma_2 a\sigma_2, b\sigma_2 b = \sigma_2 b\sigma_2,\ ab = ba. \end{cases}$$

This is the usual presentation of the Artin group of type D_n.

Since we now know presentations for all braid groups associated to irreducible complex reflection groups of the infinite series, we can give the complete classification of irreducible complex reflection groups up to isodiscriminantality. The case of well-generated groups of rank 2 comes from Corollary 6.2.25. More generally, the classification of irreducible complex reflection groups of rank 2 up to isodiscriminantality can be achieved using the computations of discriminant done in [Ban76, Section 2]. For groups of higher rank, we can use the classification of braid groups of exceptional irreducible complex reflection groups of rank > 2 up to abstract group isomorphism done in [CM14, Theorem 1.2] (note however that $B(G_{35}), B(G_{36})$ and $B(G_{37})$ are omitted by mistake from [CM14, Theorem 1.2]).

Theorem 6.2.30 (Classification of complex braid groups up to isodiscriminantality). *Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be an irreducible complex reflection group. The group W is isodiscriminantal to exactly one of the following groups:*

- *μ_2 and $B(W) \simeq \mathbb{Z}$.*
- *$\widetilde{G}(1,1,n)$ with $n > 3$, and $B(W) \simeq B_n$.*
- *$G(2,1,n)$ with $n > 2$, and $B(W) \simeq B_n^*$.*
- *$G(2e,e,n)$ with $n, e > 1$ and $B(W) \simeq B_n^*(e)$.*
- *$G(e,e,n)$ with $n, e > 1$, and $B(W) \simeq B_n(e)$.*
- *One of the exceptional groups G_i with $i \in [\![12,13]\!] \sqcup [\![22,23,24]\!] \sqcup [\![27,31]\!] \sqcup [\![33,37]\!]$.*

At this stage, it remains to find presentations of exceptional irreducible complex braid groups (which are not complexified real reflection groups). That is, all the braid groups of exceptional groups in Theorem 6.2.30 except $B(G_{23}), B(G_{28}), B(G_{30}), B(G_{35}), B(G_{36})$ and $B(G_{37})$. This is a hard task relying on heavy computations often done case-by-case (see for instance [Ban76], [BMR98], [BM04], [Bes15]).

6.2.5 A word on Garside structures

So far we have given generating systems for complex braid groups (braided reflections), and presentation for braid groups attached to irreducible complex reflection groups belonging to the infinite series. In the sequel, we are going to study many finer aspects of complex braid groups, like the center, centralizers of particular elements (related to regular elements in the attached reflection group)... In order to do so, we will make great use of Garside structures on complex braid groups, especially ones coming from the so-called "dual braid monoid" (see Section 8.2).

In this short section, we review other Garside structures that exist on complex braid groups. We do not go into many details about these structures, as they will not be our point of focus in the sequel.

The first Garside monoids attached to some complex braid groups were attached to braid groups of real reflection groups (i.e. Artin groups of spherical type) long before the term "Garside group" was introduced by Dehornoy and Paris in [DP99]. Actually, the first Garside monoid to ever be considered [Gar69] is the monoid defined by the monoid presentation attached to the presentation we gave of the usual braid group B_n on n strands in the last section. It was later generalized for all real reflection groups:

Let (W, S) be a Coxeter system [Bou81, Definition IV.1.3], where W is a finite group. The set S is by definition a positive generating system of W, and we can consider the length ℓ_S respective to this system, which is the usual length of elements in a Coxeter group (the length of a reduced expression). By [Bou81, Exercice 1.22], there is a unique well defined element $w_0 \in W$ of longest length, and the associated interval $I(w_0)_S$ is the whole group W. By [DDGKM, Section IX.1.3], the interval germ attached to the data (W, S, w_0) is a Garside germ, and the associated interval monoid is the Artin-Tits monoid attached to (W, S). It contains $|W|$ simple elements.

Moreover, W can be seen as a real reflection group with the Tietze representation [Bou81, Proposition IV.4.3]. It can then be complexified into a complex reflection group, and we know by [Bri71] that the braid group $B(W)$ is isomorphic to the Artin group attached to W. Thus the interval monoid attached to (W, S, w_0) is a Garside monoid whose group of fractions is isomorphic to $B(W)$. Since every complexified real reflection group can be obtained this way, we have a family of Garside monoids whose groups of fractions are the braid groups of complexified real reflection groups.

The situation is quite similar for groups of the form $G(r, 1, n)$. Indeed we showed in Proposition 6.2.27 that, for $r > 1$, the group $G(r, 1, n)$ is isodiscriminantal to $G(2, 1, n)$, which is a complexified real reflection group. This implies that the Artin-Tits monoid attached to $B(G(2, 1, n)) \simeq B_n^*$ is also a Garside monoid with group of fractions $B(G(r, 1, n)) \simeq B_n^*$. It contains $|G(2, 1, n)| = n!2^n$ simple elements. Moreover, it is possible to show [Mar21, Chapter 12] that this monoid can also be recovered as an interval monoid attached to the data $(G(r, 1, n), S, w)$ where

$$S = \{r_{1,\zeta_d}, s_{1,1,2} \dots, s_{1,n-1,n}\} \text{ and } w = \mathrm{diag}(\zeta_d, \dots, \zeta_d).$$

For the group $G(de, e, n)$, the presentation of $B_n^*(e)$ given in the last section is positive. We can then consider the monoid $B_n^*(e)^+$ given by the same presentation. However, this is monoid is not a Garside monoid in general by the following lemma:

Lemma 6.2.31. *For $n > 2$ and $e > 2$, the monoid $B_n^*(e)^+$ does not admit right-lcms. In particular it cannot be endowed with a Garside element.*

Proof. We know that lcms exist in a Garside monoid. When lcms exist in a monoid M, we can then consider the right-complement operation (see Section 2.1.4), and we have

$$\forall u, v, w \in M,\ (u \vee v)((u\backslash v)\backslash(u\backslash w)) = u \vee v \vee w = (u \vee v)((v\backslash u)\backslash(v\backslash w)). \tag{6.2.1}$$

In our case, the presentation of $B_n^*(e)^+$ is homogeneous, and a length function on $B_n^*(e)^+$ is given by the length of any word representing a given element. By construction of a length function, if two elements of $B_n^*(e)^+$ have a right-lcm, then it must be the unique shortest common right-multiple of these elements.

Assume that $B_n^*(e)^+$ admits right-lcms. Let $k \in [\![0, e-1]\!]$, the element $\sigma_{2,k}\sigma_3\sigma_{2,k} = \sigma_3\sigma_{2,k}\sigma_3$ is the shortest common right-multiple of σ_3 and $\sigma_{2,k}$. It must then be the right-lcm of $\sigma_{2,k}$ and σ_3, and we have $\sigma_{2,k}\backslash\sigma_3 = \sigma_3\sigma_{2,k}$ and $\sigma_3\backslash\sigma_{2,k} = \sigma_{2,k}\sigma_3$. Similarly, we can compute the right-lcm of any two given generators of $B_n^*(e)^+$, along with their right complement. Using Lemma 2.1.31 (iteration of complement) then allows us to compute the right-complement of any couple of elements of $B_n^*(e)^+$.

However, if we take $u = \sigma_{2,e-3}$, $v = \sigma_{2,e-2}$ and $w = \sigma_3$, then after computations, we obtain

$$(u\backslash v)\backslash(u\backslash w) = \sigma_{2,e-2}\backslash(\sigma_3\sigma_{2,e-1}) = \sigma_3\sigma_{2,0}\sigma_{2,1}\sigma_3.$$

$$(v\backslash u)\backslash(v\backslash w) = \sigma_{2,e-1}\backslash(\sigma_3\sigma_{2,e-2}) = \sigma_3\sigma_{2,e-1}\tau\sigma_{2,0}\sigma_3.$$

As these two elements are not equal in $B_n^*(e)^+$ (they do not have the same length as words in the generators), this contradicts Equation (6.2.1) and the monoid $B_n^*(e)^+$ cannot be a Garside monoid. □

Remark 6.2.32. It was recently pointed out to us by I. Haladjian that the monoid $B_2^*(e)$ is a Garside monoid for $e \geqslant 3$ at least when e is even (see [Hal24, Theorem 1.12 and Table 1]).

This being said, we know that $B_n^*(e)$ is a subgroup of B_n^* of index e. Thus we can use the methods of Section 4.2 (groupoid of cosets) in order to construct a Garside groupoid equivalent to $B_n^*(e)$. By construction, the underlying Garside category of this groupoid contains $n!2^n e$ simple morphisms.

For the group $G(e,e,n)$, the presentation of $B_n(e)$ given in the last section is positive. We can then consider the monoid $B_n(e)^+$ given by the same presentation (the so-called *parachute monoid*). By a theorem of Neaime [Nea18, Corollary 3.3.13, Definition 3.4.1, Theorem 3.4.15], the parachute monoid is the interval Garside monoid attached to $(G(e,e,n),S,c)$, where

$$S := \{s_{\zeta_{de}^k,1,2} \mid k \in [\![1,e-1]\!]\} \sqcup \{s_{1,i-1,i} \mid i \in [\![3,n]\!]\} \text{ and } c = \operatorname{diag}(\zeta_e^{1-n},\zeta_e,\ldots,\zeta_e).$$

In particular it is a Garside monoid, which was already showed by Corran and Picantin [CP11, Theorem 3.2]. The parachute monoid admits $B_n(e)$ as its group of fractions by construction. By [CP11, Corollary 3.9], it admits $\prod_{k=2}^n(2(k-1)+e)$ simple elements.

Braid groups of complex reflection groups of rank 2 have been extensively studied. Bannai provided presentations for all of these groups in [Ban76], and they often give rise to Garside monoids. However, for the exceptional groups $B(G_{12}), B(G_{13}), B(G_{22})$, even though the presentations given in [Ban76] give rise to Garside monoids, they do not have braided reflections as generators. It is nonetheless possible to find presentations of these groups with braided reflections as generators, and which give rise to Garside monoids.

$$\begin{aligned}
B(G_{12}) &= \langle s,t,u \mid stus = tust = ustu\rangle,\\
B(G_{13}) &= \langle s,t,u \mid ustus = stust, sustu = tusts, tust = ustu\rangle,\\
B(G_{22}) &= \langle s,t,u \mid stust = tustu, ustus\rangle.
\end{aligned}$$

The monoids given by the same presentations were proven to be Garside monoids in [Pic00, Example 11, 13]. Note that we already encountered the second monoid in Section 5.2.3, where we proved that the shoal of all its standard parabolic subgroups is support-preserving. Later on, in Section B.1 we will consider a family of Garside groups which will contain in particular all braid groups of complex reflection groups of rank 2.

At this stage, we know by Theorem 6.2.30 that all irreducible complex braid groups are weak Garside groups, except maybe $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{31}), B(G_{33})$ and $B(G_{34})$. Bessis showed in [Bes15] that all these groups were Garside groups (using the dual braid monoid), except $B(G_{31})$, which is only a weak Garside group. We will review these structures in later chapters, and finally state (Corollary 9.1.16) that every complex braid group is a weak Garside group.

Chapter 7

Local fundamental groups and parabolic subgroups of complex braid groups

In [GM22], parabolic subgroups of complex braid groups are defined as a particular case of a general concept of local fundamental group for a topological pair. In Section 7.1 and 7.2 we provide a slight extension of the concept of local fundamental group defined in [GM22], and we prove that this extension behaves well with respect to certain covering maps.

In Section 7.3, we show that our extended definition actually coincides with the one given in [GM22] in the case of complex braid groups, and we recall the first results of [GM22] on parabolic subgroups of complex braid groups.

Contents

7.1 Local fundamental groups

In this section, we fix a topological pair (X, Y). By this we mean a topological space Y, along with a subset $X \subset Y$ (endowed with the induced topology). We write $\mathcal{H} := Y \setminus X$ (one should think of $\mathcal{H}$ as a singular locus in the space Y). We also fix a basepoint $* \in X$, and we consider some point $v \in Y$.

For $x \in Y$, we write $V(x, Y)$ the set of neighborhoods of x in Y. It is a lattice under inclusion, and we identify it with its associated category.

If $\gamma : [0,1] \to X$ is a continuous path, then we write c_γ for the isomorphism $\pi_1(X, \gamma(0)) \to \pi_1(X, \gamma(1))$ induced by γ. For $a, b \in [0,1]$, we write $\gamma_{[a;b]}$ for the path $t \mapsto \gamma(tb + (t-1)a)$.

We want to define a topological concept of local fundamental group of X around the point v. Naively, one would like to consider the intersection of the fundamental groups of a system of neighborhoods of v in Y (intersected with X). This approach presents two problems:

- For each neighborhood U of v in Y, one needs to chose a basepoint $y \in U \cap X$ to define $\pi_1(U \cap X, y)$.
- If we want to work with subgroups of $\pi_1(X, *)$, we need a way to transfer a local situation around v to a global situation, attached to the basepoint $*$.

We can solve both of these problems by considering a convenient path joining $*$ and v. In [GM22], the authors use the concept of (closed) normal ray as convenient paths (see Section 7.3 for more details). Here we introduce the weaker notion of capillary path, which relaxes the simple connectedness condition which exists on normal rays.

Definition 7.1.1 (Capillary path). A *capillary path* based at $*$ and terminating at v is a path $\eta : [0,1] \to Y$ such that $\eta(0) = *$, $\eta(1) = v$, and such that $\eta(t) \in X$ for all $t \in [0,1[$.

Note that, if v belongs to the interior of $\mathcal{H}$, then there is no capillary path terminating at v. If $v \in X$, then a capillary path based at $*$ and terminating at v is simply a path from $*$ to v in X.

In order to properly define a notion of local fundamental group, we will require a few notation and intermediate results.

Lemma 7.1.2. *Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. Let also $(t_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence in $[0,1[$ such that $t_0 = 0$ and $\lim_{n\to\infty} t_n = 1$. For $U \in V(Y,v)$, we define*

$$k_U := \min\left\{ i \in \mathbb{N} \mid [t_i, 1] \subset \eta^{-1}(U) \right\}.$$

This is always a finite integer, and we can set $t_U := t_{k_U}$. If $U, U' \in V(Y,v)$ are such that $U' \subset U$, then we have $t_{U'} \geqslant t_U$.

Proof. By continuity of η, $\eta^{-1}(U)$ is a neighborhood of 1 in $[0,1]$. In particular it contains an interval of the form $]\alpha, 1]$ for some $\alpha \in [0,1[$, and there is an integer k such that $\eta(t) \in U$ for all $t \geqslant t_k$. The set

$$\{i \in \mathbb{N} \mid [t_i, 1] \subset \eta^{-1}(U)\}$$

is then a nonempty subset of $\mathbb{N}$, and it admits a minimum k_U. If $U', U \in V(Y,v)$ are such that $U' \subset U$, then we have

$$\{i \in \mathbb{N} \mid [t_i, 1] \subset \eta^{-1}(U')\} \subset \{i \in \mathbb{N} \mid [t_i, 1] \subset \eta^{-1}(U)\},$$

and $k'_U \geqslant k_U$. Since the sequence $(t_n)_{n\in\mathbb{N}}$ is nondecreasing, we have $t_{U'} \geqslant t_U$. □

Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. Let also $(t_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2. For $U \in V(Y,v)$, we can define $P_\eta(U) := \pi_1(U \cap X, \eta(t_U))$. For $U', U \in V(Y,v)$ such that $U' \subset U$, we define $f_{U,U'}$ as the morphism $P_\eta(U') \to \pi_1(U \cap X, t_{U'})$ induced by the inclusion map $U' \cap X \hookrightarrow U \cap X$, and $g_{U',U} := c_{\gamma_{[t_{U'};t_U]}} : \pi_1(U \cap X, \gamma(t_{U'})) \to \pi_1(U' \cap \gamma_{T_u})$. We define $\iota_{U',U} := g_{U',U} \circ f_{U',U}$, so that we have the following commutative diagram

$$\begin{tikzcd} P_\eta(U') \arrow[rr, "\iota_{U',U}"] \arrow[dr, "f_{U',U}"'] & & P_\eta(U) \\ & \pi_1(U' \cap X, \eta(t_{U'})) \arrow[ur, "\sim", "g_{U',U}"'] & \end{tikzcd}$$

Lemma 7.1.3 (Functor). *Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. Let also $(t_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2. The map which sends $U \in V(Y,v)$ to $P_\eta(U) := \pi_1(U \cap X, \eta(t_U))$ and $U' \subset U$ to $\iota_{U',U}$ as defined above induces a functor P_η from $(V(Y,v), \subset)$ to the category of groups.*

Proof. Let $U \in V(Y,v)$. Since both $f_{U,U}$ and $g_{U,U}$ are the identity morphism on $P_\gamma(U)$, we have $\iota_{U,U} = 1_{P_\gamma(U)}$ by definition.

Let then $U'' \subset U' \subset U$ be three neighborhoods of v in Y. We have to show that $\iota_{U',U} \circ \iota_{U'',U'} = \iota_{U'',U}$. By definition, we have

$$\iota_{U',U} \circ \iota_{U'',U'} = g_{U',U} \circ f_{U',U} \circ g_{U'',U'} \circ f_{U'',U'}.$$

We also have the following commutative square

$$\begin{tikzcd} \pi_1(U' \cap X, \gamma(t_{U''})) \arrow[r, "g_{U'',U'}"] \arrow[d, "h"'] & \pi_1(U' \cap X, \gamma(t_{U'})) \arrow[d, "f_{U',U}"] \\ \pi_1(U \cap X, \gamma(t_{U''})) \arrow[r, "c_{\gamma_{[t_{U''};t_{U'}]}}"'] & \pi_1(U \cap X, \gamma(t_{U'})) \end{tikzcd}$$

which shows that $\iota_{U',U} \circ \iota_{U'',U'} = g_{U',U} \circ c_{\gamma_{[t_{U''};t_{U'}]}} \circ h \circ f_{U'',U'}$. By definition, we have $g_{U',U} = c_{\gamma_{[t_{U'};t_U]}}$, thus $g_{U',U} \circ c_{\gamma_{[t_{U''};t_{U'}]}} = c_{\gamma_{[t_{U''};t_U]}} = g_{U'',U}$. Likewise, $f_{U'',U'}$ is the morphism $\pi_1(U'' \cap X, \gamma(t_{U''})) \to \pi_1(U' \cap X, \gamma(t_{U''}))$ induced by the inclusion map $U'' \cap X \to U' \cap X$. Thus $h \circ f_{U'',U'}$ is the morphism $\pi_1(U'' \cap X, \gamma(t_{U''})) \to \pi_1(U \cap X, \gamma(t_{U''}))$ induced by the inclusion map $U'' \cap X \to U \cap X$. This is the definition of $f_{U'',U}$. We obtain that $\iota_{U',U} \circ \iota_{U'',U'} = g_{U'',U} \circ f_{U'',U} = \iota_{U'',U}$ as claimed. ☐

Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. Let also $(t_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2. At this stage, the functor P_η depends both on η and on $(t_n)_{n\in\mathbb{N}}$. We claim that choosing another nondecreasing sequence as in Lemma 7.1.2 yields a functor P'_η which is naturally isomorphic to P_η.

Let $(t'_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2. We denote by P'_η the functor from $(V(Y,v), \subset)$ to **Grp** induced by η and $(t'_n)_{n\in\mathbb{N}}$. For $U \in V(Y,v)$, we write

$$k'_U := \min\{i \in \mathbb{N} \mid [t'_i, 1] \subset \eta^{-1}(U)\},$$

and $t'_U := t_{k'_U}$, so that $P'_\eta(U) = \pi_1(X \cap U, \gamma(t'_U))$. By construction, the path $\eta_{[t_U;t'_U]}$ is always a path in $U \cap X$.

Proposition 7.1.4 (Isomorphism of functors). *Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. Let also $(t_n)_{n\in\mathbb{N}}$,$(t'_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2.*
With the above notation, we define, for $U \in V(Y,y)$, $\varepsilon_U := c_{\eta_{[t'_U;t_U]}} : P'_\eta(U) \to P_\eta(U)$. The family $(\varepsilon_U)_{U\in V(Y,v)}$ defines a natural isomorphism from P'_η to P_η.

Proof. The morphism ε_U is always an isomorphism by definition (it is a mere change of basepoint). It just remains to show that $(\varepsilon_U)_{U\in V(Y,v)}$ is a natural transformation. Let $U', U \in V(Y,v)$ be such that $U' \subset U$. Recall that $\iota_{U',U} = g_{U',U} \circ f_{U',U}$, we write $\iota'_{U',U}$, $f'_{U',U}$ and $g'_{U',U}$ for the associated morphisms defining P'_η. We have the following diagram of groups

$$\begin{tikzcd}
P'_{U'} \arrow[r, "f'_{U',U}"'] \arrow[d, "\varepsilon_{U'}"'] \arrow[rr, bend left, "\iota'_{U',U}"] & \pi_1(U\cap X, \eta(t'_{U'})) \arrow[r, "g'_{U',U}"'] \arrow[d, "c_{\eta_{\left[t'_{U'};t_{U'}\right]}}"'] & P'_U \arrow[d, "\varepsilon_U"] \\
P_{U'} \arrow[r, "f_{U',U}"] \arrow[rr, bend right, "\iota_{U',U}"'] & \pi_1(U\cap X, \eta(t_{U'})) \arrow[r, "g_{U',U}"] & P_U
\end{tikzcd}$$

The right-hand square of this diagram is commutative. Indeed, we have

$$\begin{aligned}
\varepsilon_U \circ g'_{U',U} &= c_{\eta_{\left[t'_U;t_U\right]}} \circ c_{\eta_{\left[t'_{U'};t'_U\right]}} \\
&= c_{\eta_{\left[t'_{U'};t_U\right]}} \\
&= c_{\eta_{\left[t_{U'};t_U\right]}} \circ c_{\eta_{\left[t'_{U'};t_{U'}\right]}} \\
&= g_{U',U} \circ c_{\eta_{\left[t'_{U'};t_{U'}\right]}}.
\end{aligned}$$

Then, we show that the left-hand square of the diagram is also commutative. Let $g \in P'_{U'}$ be represented by a path γ from $\eta(t'_{U'})$ to itself. Both $f_{U',U}(\varepsilon_{U'}(g))$ and $c_{\eta_{\left[t'_{U'};t_{U'}\right]}}(f'_{U',U}(g))$ are equal to the homotopy class in $\pi_1(U\cap X, \gamma(t_{U'}))$ of the path

$$\eta_{[t_{U'};t'_{U'}]} * \gamma * \eta_{[t'_{U'};t_{U'}]}.$$

We showed that the whole diagram is commutative. This implies that $(\varepsilon_U)_{U\in V(Y,v)}$ is a natural transformation, as claimed. □

Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. From now on, we will often drop the reference to the sequence $(t_n)_{n\in\mathbb{N}}$, and simply refer to P_η as the functor $(V(Y,v),\subset) \to \mathbf{Grp}$ induced by η. Now, if we consider $Y \in V(Y,v)$, we have $t_Y = t_0 = 0$ and $P_\eta(Y) = \pi_1(Y\cap X, \eta(0)) = \pi_1(X,*)$ by definition.

Definition 7.1.5 (Local fundamental group). Let $\eta : [0,1] \to Y$ be a capillary path based at $*$ and terminating at v. The *local fundamental group* of X attached to η, denoted by $\pi_1^{\mathrm{loc}}(X,\eta)$, is the limit of the codirected system of groups induced by the functor P_η. The image of the natural morphism $\pi_1^{\mathrm{loc}}(X,\eta) \to P_\eta(Y) = \pi_1(X,*)$ is called a *pseudoparabolic subgroup* of $\pi_1(X,*)$.

Let η be a capillary path in Y based at $*$ and terminating at v, and let $(t_n)_{n\in\mathbb{N}}, (t'_n)_{n\in\mathbb{N}}$ be as in Lemma 7.1.2. Let also $P_\eta, P_{\eta'}$ be the respective functors induced by η and η'. The isomorphism $P_\eta \simeq P'_\eta$ of Proposition 7.1.4 induces an isomorphism between the associated limits. Thus $\pi_1^{\mathrm{loc}}(X,\eta)$ does not depend, up to natural isomorphism, on the choice of a nondecreasing sequence $(t_n)_{n\in\mathbb{N}}$. Furthermore, as the isomorphism $P_\eta \simeq P'_\eta$ is natural, the images of the natural morphism from the associated limits to $\pi_1(X,*)$ are also preserved by this isomorphism. The concept of pseudoparabolic subgroup of $\pi_1(X,*)$ is thus well-defined and depends only on the choice of the capillary path η.

Of course, it is unclear at this stage how to actually compute local fundamental groups, and how they depend on the choice of the capillary path η. The following proposition shows that we can compute local fundamental groups by restricting the diagram to a neighborhood basis of v.

Proposition 7.1.6 (Local fundamental group on neighborhood basis). *Let η be a capillary path in Y based at $* \in X$ and terminating at $v \in Y$, and let $\mathcal{B}$ be a neighborhood basis for v in Y. We write $P_\eta^{\mathcal{B}}$ for the restriction of the functor P_η to $\mathcal{B} \subset V(Y,v)$. The natural transformation $P_\eta^{\mathcal{B}} \to P_\eta$ induced by the inclusion $\mathcal{B} \subset V(Y,v)$ induces an isomorphism between the corresponding limits.*

Proof. Let L denote the limit of the functor $P_\eta^{\mathcal{B}}$ in the category of groups, and let p_U denote the natural morphism $\pi_1^{\mathrm{loc}}(X,\eta) \to P_\eta(U)$ for $U \in V(Y,v)$. The family $(p_U)_{U\in V(Y,v)}$ restricts to a family $(p_U)_{U\in\mathcal{B}}$. By definition, we then have that $\pi_1^{\mathrm{loc}}(X,\eta)$ is a cone over the functor $P_\eta^{\mathcal{B}}$, and we obtain a unique factorizing morphism $\varphi : \pi_1^{\mathrm{loc}}(X,\eta) \to L$. We claim that φ is an isomorphism.

In order to construct the inverse of φ, we show that L is a cone over the functor P_η. For $\Omega \in \mathcal{B}$, we denote by f_Ω the natural morphism $L \to P_\eta^{\mathcal{B}}(\Omega) = P_\eta(\Omega)$. Let $U \in V(Y,y)$. By definition, there is some $\Omega \subset U$ such that $U' \in \mathcal{B}$. We then define $f_U : \iota_{\Omega,U} \circ f_\Omega$. We show that f_U does not depend on the choice of $\Omega \subset U$ in $\mathcal{B}$. Let $\Omega, \Omega' \subset U$ be two elements in $\mathcal{B}$. Since $\mathcal{B}$ is a neighborhood basis, we can consider $\Omega'' \subset \Omega \cap \Omega'$. We then have

$$\begin{aligned}\iota_{\Omega,U} \circ f_\Omega &= \iota_{\Omega,U} \circ \iota_{\Omega'',\Omega} \circ f_{\Omega''}\\ &= \iota_{\Omega'',U} \circ f_{\Omega''}\\ &= \iota_{\Omega',U} \circ \iota_{\Omega'',\Omega'} \circ f_{\Omega''}\\ &= \iota_{\Omega',U} \circ f_{\Omega'},\end{aligned}$$

and f_U does not depend on the choice of Ω. Now, we show that $(f_U)_{U\in V(Y,v)}$ makes L into a cone over P_η. Let $U' \subset U$ be two neighborhoods of v in Y. Let $\Omega \subset U'$ be an element of $\mathcal{B}$. By definition, we have $f_U = \iota_{\Omega,U} \circ f_\Omega$ and $f_{U'} = \iota_{\Omega,U'} \circ f_\Omega$. We then obtain

$$\iota_{U',U} \circ f_{U'} = \iota_{U',U} \circ \iota_{\Omega,U'} f_\Omega = \iota_{\Omega,U} \circ f_\Omega = f_U,$$

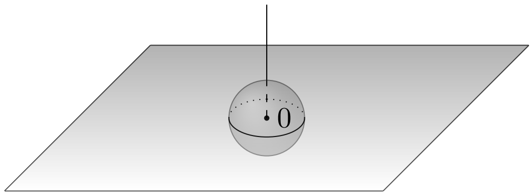
which proves that L is a cone over P_η. We then obtain a unique morphism $\psi : L \to \pi_1^{\mathrm{loc}}(X,\eta)$ such that $\psi \circ p_U = f_U$.

By construction, we see that, for $U \in V(Y,v)$, we have $\varphi \circ \psi \circ p_U = \varphi \circ f_U = p_U$ and $\psi \circ \varphi \circ f_U = \psi \circ p_U = f_U$. Thus $\psi \circ \psi$ and $\psi \circ \psi$ are both identity morphisms and the result follows. $\square$

With the notation of the proposition, we also obtain that the image of $\pi_1^{\mathrm{loc}}(X,\eta)$ in $\pi_1(X,*)$ is equal to the image of the limit of the functor $P_\eta^{\mathcal{B}}$. We can thus compute pseudoparabolic subgroups of $\pi_1(X,*)$ by restricting to a neighborhood basis.

At first, capillary paths can be seen as a simple way to transport a local situation happening close to v to a more global situation in $\pi_1(X,*)$. However, without further assumptions on the pair (X,Y), the concept of local fundamental group truly depends on the choice of the capillary path η, and not only on its endpoint, as in the following example:

Example 7.1.7. Consider the topological pair $(X,\mathbb{R}^3)$, where X consists of $\mathbb{R}^3$ minus the sets $[-1,1]^2\times\{0\}$ and $\{0\}^2\times\mathbb{R}_+$. If U is a small enough open ball of center 0 in $\mathbb{R}^3$, then $U\cap X$ consists of two path connected components: $U\cap(\mathbb{R}^2\times\mathbb{R}_+^*)$ and $U\cap(\mathbb{R}^2\times\mathbb{R}_-)$ as in the picture below.



The first connected component is homotopically equivalent to $\mathbb{R}^2\setminus\{0\}$, and the second is contractible. If η is a capillary path based at some point in X and terminating at 0, then we have

$$\pi_1^{\mathrm{loc}}(X,\eta)\simeq\begin{cases}\mathbb{Z} & \text{if } \eta(t)\in\mathbb{R}^2\times\mathbb{R}_+ \text{ for } t \text{ big enough,}\\ \{1\} & \text{if } \eta(t)\in\mathbb{R}^2\times\mathbb{R}_- \text{ for } t \text{ big enough.}\end{cases}$$

Note however, that the fundamental group of X is trivial, and thus the pseudoparabolic subgroups associated to η is always trivial.

The definition of local fundamental group given in [GM22] does not rely on projective limits. Instead, it requires the fundamental group of $U\cap X$ to be always the same for U small enough. More precisely (in our notation), the image of $P_\eta(U)\to\pi_1(X,*)$ should always be the same for U small enough. This situation is formalized in our context by the following definition:

Definition 7.1.8 (Stabilizing local fundamental group). Let η be a capillary path in Y based at $*\in X$ and terminating at $v\in Y$. We say that P_η *stabilizes* at $U_0\in V(Y,v)$ if ι_{U,U_0} is an isomorphism for all $U\in V(Y,v)$ with $U\subset U_0$.

Let η be a capillary path in Y based at $*\in X$ and terminating at $v\in Y$. Note that, if $(t_n)_{n\in\mathbb{N}}$ and $(t'_n)_{n\in\mathbb{N}}$ are two sequences as in Lemma 7.1.2, then the functors P_η and P'_η defined using $(t_n)_{n\in\mathbb{N}}$ and $(t'_n)_{n\in\mathbb{N}}$ are naturally isomorphic after Proposition 7.1.4. Thus, saying that P_η stabilizes at $U_0\in V(Y,v)$ is equivalent to saying that P'_η stabilizes at U_0.

An elementary category theory result ensures that, if P_η stabilizes at some $U_0\in V(Y,v)$, then the natural morphism $\pi_1^{\mathrm{loc}}(X,\eta)\to P_\eta(U_0)$ is an isomorphism. In particular $\pi_1^{\mathrm{loc}}(X,\eta)$ can be realized as an actual fundamental group in (a subset of) X, instead of an abstract projective limits of groups. Under the assumption that systems of the form P_η stabilize, we can show that the local fundamental group $\pi_1^{\mathrm{loc}}(X,\eta)$ (and the associated pseudoparabolic subgroup) does not depend too much on the path η, under suitable assumptions.

First, we show that the pseudoparabolic subgroup of $\pi_1(X,*)$ induced by a capillary path η depends only on its homotopy class.

Lemma 7.1.9. *Let η be a capillary path in Y based at $*\in X$ and terminating at $v\in Y$. Assume that there is a homotopy $H:[0,1]\times[0,1]\to Y$ from η to γ with fixed endpoints, and such that*

$$\forall s\in[0,1],t\in[0,1[,\ \ H(s,t)\in X.$$

If P_η and P_γ both stabilize, then the pseudoparabolic subgroups of $\pi_1(X,*)$ induced by η and γ are equal.

Proof. Let U_0 (resp. U_1) be a neighborhood of v at which P_η (resp. P_γ) stabilizes. Both P_η and P_γ stabilize at $U_2 := U_0 \cap U_1$. The morphisms $\pi_1^{\mathrm{loc}}(X,\eta) \to P_\eta(U_2)$ and $\pi_1^{\mathrm{loc}}(X,\gamma) \to P_\gamma(U_2)$ are then isomorphisms.

Now, for a fixed $t < 1$, the path $\delta_t : s \mapsto H(s,t)$ is a path from $\eta(t)$ to $\gamma(t)$ in X. The set

$$t \in [0,1] \mid [t,1] \times [0,1] \subset H^{-1}(U_2)$$

is an open subset of $[0,1]$ which contains 1, it then contains some $\tau < 1$. In particular, the path δ_τ is a path in U_2. Let us then consider the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_0 := 0$, and $t_n := (1-\tau)\frac{n-1}{n} + \tau$ for $n > 0$. We have $t_{U_2} = \tau$ for both η and γ, and the path δ_τ induces an isomorphism between $P_\eta(U_2)$ and $P_\gamma(U_2)$.

By construction, the paths $\eta_{[0,\tau]} * \delta_\tau$ and $\gamma_{[0,\tau]}$ are homotopic in X, and thus the following diagram is commutative:

$$\begin{tikzcd}
P_\eta(U_2) \arrow[r] \arrow[d] & \pi_1(X,*) \\
P_\gamma(U_2) \arrow[ur] &
\end{tikzcd}$$

which proves that the pseudoparabolic subgroups of $\pi_1(X,*)$ induced by γ and by η are equal. $\square$

Lastly, we show that, under suitable assumptions, the conjugacy class of a pseudoparabolic subgroup associated to a capillary path depends only on its endpoint.

Lemma 7.1.10. *Let η, γ be capillary paths in Y based at $* \in X$ and terminating at $v \in Y$. If P_η and P_γ stabilize both at some $U_0 \in V(Y,v)$ such that $U_0 \cap X$ is path connected, then the pseudoparabolic subgroups of $\pi_1(X,*)$ induced by η and γ are conjugate in $\pi_1(X,*)$.*

Proof. By definition, there is $t_1 \in [0,1[$ (resp. $t_2 \in [0,1[$) such that the pseudoparabolic subgroup of $\pi_1(X,*)$ induced by η (resp. by γ) is the image in $\pi_1(X,*)$ of $\pi_1(X \cap U_0, \eta(t_1))$ under $\iota_{U_0,Y}$ (resp. of $\pi_1(X \cap U_0, \gamma(t_2)$ under $\iota'_{U_0,Y}$). Since $U_0 \cap X$ is path connected, we can consider a path $\delta : [0,1] \to U_0 \cap X$ from $\eta(t_1)$ to $\gamma(t_2)$. The homotopy class of the path $\eta_{[0,t_1]} * \delta * \gamma_{[t_2,0]}$ gives an element of $\pi_1(X,*)$ which conjugates $\iota_{U_0,Y}(P_\eta(U_0))$ to $\iota'_{U_0,Y}(P_\gamma(U_0))$. $\square$

In the context we will be interested in in Section 7.3, systems of the form P_η will always stabilize, and thus the assumption of stabilization in the two results above will always be satisfied.

7.2 Interactions with covering maps

One important feature of our definition of capillary paths is that we made almost no assumption on the path η. This means that capillary paths can easily be lifted under covering maps, and that the image of a capillary path under a continuous map will often again be a capillary path. In theory, given a morphism of topological pairs $(E,A) \to (X,Y)$ whose restriction to E is a covering map $E \to X$, we should be able to relate the pseudoparabolic subgroups of E to those of X. For now, we will only be able to study the case where A is a metric space and the map $A \to Y$ is a quotient under an isometric action.

7.2.1 Covering maps and groupoids of cosets

Before getting back to local fundamental groups and pseudoparabolic subgroups, we show that a finite covering map gives a topological interpretation of the groupoid of cosets we defined in Section 1.4.

In this subsection, we fix two path connected topological spaces E, X, along with a covering map $p : E \to X$. We also fix a discrete set of points $Y \subset X$.

The set Y can be used as a groupoid basepoint for X (the connected components of Y are points, in particular they are simply connected). We then set $\mathcal{G} := \pi_1(X, Y)$. More explicitly, we have $\mathrm{Ob}(\mathcal{G}) = Y$ and, for $y, y' \in Y$, $\mathcal{G}(y, y')$ is made of homotopy classes of paths from y to y' in X (relative to the endpoints).

We fix $y \in Y$ and $a_0 \in p^{-1}(y)$, and we let $H = \pi_1(E, a_0)$. The long exact sequence associated to p gives a bijection between $p^{-1}(y)$ and the right-cosets of H in $\pi_1(X, y) = \mathcal{G}(y, y)$. We adapt this argument to construct a bijection between $A := \bigsqcup_{y \in Y} p^{-1}(y)$ and the set $H\backslash\mathcal{G}$ of right-cosets of H in $\mathcal{G}$ in the sense of Definition 1.4.1.

Let $y' \in Y$, and let $g \in \mathcal{G}(y, -)$ be represented by a path γ. Since p is a covering map there is a unique lift $\widetilde{\gamma}$ of γ in E which starts from a_0. The homotopy class (relative to the endpoints) of $\widetilde{\gamma}$ depends only on g. In particular we can associate to g the endpoint of the path $\widetilde{\gamma}$, that we denote $\pi(g)$. This point belongs to A by construction, and we have the following lemma:

Lemma 7.2.1. *The map $\pi : \mathcal{G}(y, -) \to A$ constructed above induces a bijection between $H\backslash\mathcal{G}$ and A.*

Proof. First, we show that π is constant on right-cosets. Let $g, g' \in \mathcal{G}(y, y')$. Let γ, γ' be paths in X representing g, g', respectively, and let θ be a path representing gg'^{-1}. We consider the lifts $\widetilde{\gamma}, \widetilde{\gamma'}, \widetilde{\theta}$ of γ, γ', θ in E starting from a_0. If $Hg = Hg'$, then $gg'^{-1} \in H$ and a_0 is the endpoint of $\widetilde{\theta}$. Thus $\widetilde{\theta} * \widetilde{\gamma'}$ is a lift of $\theta * \gamma'$ which starts at a_0. Since $\theta * \gamma'$ is homotopic to γ, we get that $\widetilde{\theta} * \widetilde{\gamma'}$ is homotopic (with fixed endpoints) to $\widetilde{\gamma}$. In particular the endpoints of $\widetilde{\gamma}$ and $\widetilde{\gamma'}$ are equal and $\pi(g) = \pi(g')$. This shows that $\pi : H\backslash\mathcal{G} \to A$ is well-defined.

Conversely, if $\pi(g) = \pi(g')$, then the endpoints of $\widetilde{\gamma}$ and $\widetilde{\gamma'}$ are equal, then the concatenation of $\widetilde{\gamma}$ with the opposite of $\widetilde{\gamma'}$ is a lift in E of a path in X representing gg'^{-1}. Since this path ends at a_0, we get that $gg'^{-1} \in H$ and $Hg = Hg'$. This shows that $\pi : H\backslash\mathcal{G} \to A$ is injective.

Lastly, for $a \in A$, since E is path connected, we can consider a path γ from a_0 to a in E. The homotopy class g of $p \circ \gamma$ is then an element of $\mathcal{G}(y, -)$ such that $\pi(g) = a$. This shows that $\pi : H\backslash\mathcal{G} \to A$ is surjective. □

From now on, we identify $H\backslash\mathcal{G}$ and A using the bijection π of Lemma 7.2.1. Since $A = \bigsqcup_{y \in Y} p^{-1}(y)$ is discrete, it can be used as a fat groupoid basepoint in E, and we can define $\mathcal{H} := \pi_1(E, A)$.

We consider the groupoid $\mathcal{G}_H$ of right-cosets of H in $\mathcal{G}$. We have $\mathrm{Ob}(\mathcal{G}_H) \simeq \mathrm{Ob}(\mathcal{H})$ by Lemma 7.2.1. By Definition 1.4.5 (groupoid of cosets), a morphism in $\mathcal{G}_H$ has the form g_a, where $a \in A \simeq H\backslash\mathcal{G}$ and $g \in \mathcal{G}(p(a), -)$. If g_a is such a morphism, and if γ ifs a path in X starting from $p(a)$ and representing g. There is a unique lift $\widetilde{\gamma}$ of γ in E which starts from a. The homotopy class of $\widetilde{\gamma}$ in E depends only on g. It is a well-defined element $\psi(g_a) \in \mathcal{H}(a, -)$.

Proposition 7.2.2 (Topological groupoid of cosets). *Let $p : E \to X$ be a covering map between two path connected spaces, and let $Y \subset X$ be a discrete set. With the above notation, the map $\psi : \mathcal{G}_H \to \mathcal{H}$ is an isomorphism of groupoids.*

Proof. First, we show that ψ is a functor. For $a \in \mathrm{Ob}(\mathcal{H}) = \mathrm{Ob}(\mathcal{G}_H)$, $\psi(1_a) = 1_a$ is immediate. Let $g_a \in \mathcal{G}_H(a, a')$ and $g'_{a'} \in \mathcal{G}_H(a', a'')$. Let γ (resp. γ') be a path from $p(a)$ to $p(a')$ (resp. from $p(a')$ to $p(a''))$ in X representing g (resp. g') in $\mathcal{G}$. Let $\widetilde{\gamma}$ (resp. $\widetilde{\gamma'}$) be the unique lift in E of γ starting at a (resp. of γ' starting at a'). By construction, the composition $\psi(g_a)\psi(g'_{a'})$ is represented by the concatenation $\widetilde{\gamma} * \widetilde{\gamma'}$, which is the unique lift in E starting at a of the path $\gamma * \gamma'$, which represents gg' in G. Thus $\psi(g_a)\psi(g'_{a'}) = \psi((gg')_a)$ as we wanted to show.

By construction, ψ is bijective on objects, thus it only remains to show that it is full and faithful. Let $f \in \mathcal{H}(a, a')$ be represented by a path δ. The path $p \circ \delta$ represents an element g of $\mathcal{G}$, and δ is a lift of g starting at a. Thus $f = \psi(g_a)$ and ψ is full.

Lastly, we show that ψ is faithful. Let $a, a' \in A$, and let $g_a, g'_a \in G_H(a, a')$ be such that $\psi(g_a) = \psi(g_{\prime})$. Let γ (resp γ') be a path from $p(a)$ to $p(a')$ in X representing g (resp. g') in $\mathcal{G}$. Let $\widetilde{\gamma}$ (resp. $\widetilde{\gamma'}$) be the unique lift in E of γ (resp. of γ') starting at a. By assumption, there is a homotopy $H := [0, 1] \times [0, 1] \to E$ such that

$$\forall s \in [0,1], \begin{cases} H(0,s) = a, \\ H(1,s) = a'. \end{cases}, \ \forall t \in [0,1], \begin{cases} H(t,0) = \widetilde{\gamma}(t), \\ H(t,1) = \widetilde{\gamma'}(t). \end{cases}$$

The homotopy $p \circ H$ then gives an homotopy between γ and γ'. Thus $g = g'$ and $g_a = g'_a$, as we wanted to show. □

Remark 7.2.3. More generally, if the path connected components of A are simply connected, but not reduced to points, we can, for each $u \in \pi_0(A)$, choose a point $a \in u$. The groupoids $\pi_1(X, A)$ and $\pi_1(X, \bigsqcup_{u \in \pi_0(A)} a)$ are identified, and one can apply the above argument.

Remark 7.2.4. If $A = \{x\}$ is reduced to a point, then $\mathcal{G} = \pi_1(X, x)$ is a fundamental group in the usual sense, and H is a subgroup of $\mathcal{G}$. The groupoid $\pi_1(E, p^{-1}(x))$ is then isomorphic to the coset groupoid $\mathcal{G}_H$.

We saw in Section 1.4 (groupoid of cosets) that the groupoid $\mathcal{G}_H$ comes equipped with a natural functor $P_1 : \mathcal{G}_H \to \mathcal{G}$ (denoted π there). On the other hand, the map $p : E \to X$ induces a functor $P_2 : \mathcal{H} \to \mathcal{G}$. We see that we have a commutative triangle of functors

$$\begin{tikzcd} \mathcal{G}_H \arrow[rr, "\psi", "\sim"'] \arrow[dr, "P_1"'] & & \mathcal{H} \arrow[dl, "P_2"] \\ & \mathcal{G} & \end{tikzcd}$$

7.2.2 Branched covering maps and pseudoparabolic subgroups

Let (E, A) and (X, Y) be two topological pairs. As in Section 7.1, we can consider the respective collections of pseudoparabolic subgroups of the fundamental group of E and X. If $p : E \to X$ is a covering map, then the fundamental group of E can be seen as a subgroup of the fundamental group of X, and we expect to have some connection between the associated collections of pseudoparabolic subgroups. However, we also need some connection between the topological pairs (E, A) and (X, Y), and not only between E and X. If $p : A \twoheadrightarrow Y$ is a surjective continuous open map, then we say that p is a branched covering with branch set $A \setminus E$ if $p_{|E} : E \to X$ is a covering map, and if $A \setminus E$ is nowhere dense (i.e. its closure has empty interior in A).

In this section, we fix two path connected topological spaces A and Y, along with a branched covering map $p : A \twoheadrightarrow Y$. We also fix $E \subset A$ the complement of the branch locus of p, which we assume to be path connected, and we fix $X := p(E)$ so that we have a covering map $p : E \to X$. We also fix $* \in E$ and $a \in A$, along with a capillary path $\eta : [0,1] \to A$ from $*$ to a.

The composition $p \circ \eta$ is a capillary path from $p(*)$ to $p(a)$. We denote by H_η (resp. by $H_{p\circ\eta}$) the pseudoparabolic subgroup of $\pi_1(E,*)$ induced by η (resp. of $\pi_1(X, p(*))$ induced by $p \circ \eta$). We want to compare H_η with $H_{p\circ\eta}$ using the map p.

First, let $\mathcal{B}$ be a neighborhood basis for a in A. The set $p(\mathcal{B}) := \{p(U) \mid U \in \mathcal{B}\}$ is a neighborhood basis for $p(a)$ in Y. Indeed, all the $p(U)$ for $U \in p(\mathcal{B})$ are neighborhoods since p is an open map and, if $V \subset Y$ is a neighborhood of $p(a)$, then the set $p^{-1}(V)$ is a neighborhood of a since p is continuous. Thus it contains some $U \in \mathcal{B}$ and $p(p^{-1}(V)) = V$ contains $p(U)$.

Thanks to Proposition 7.1.6 (local fundamental group on neighborhood basis), we can compute the group $\pi_1^{\mathrm{loc}}(E,\eta)$ (resp. $\pi_1^{\mathrm{loc}}(X, p\circ\eta)$) using the neighborhood basis $\mathcal{B}$ (resp. $p(\mathcal{B})$), and the associated functor $P_\eta^{\mathcal{B}}$ (resp. $P_{p\circ\eta}^{p(\mathcal{B})}$).

We begin by showing that the map p induces a group morphism from $\pi_1^{\mathrm{loc}}(E,\eta)$ to $\pi_1^{\mathrm{loc}}(X, p\circ\eta)$.

Lemma 7.2.5. *The map p induces a natural transformation $P_\eta^{\mathcal{B}} \Rightarrow P_{p\circ\eta}^{p(\mathcal{B})}$ and a morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X, p\circ\eta)$.*

Proof. First, let $(t_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of $[0,1[$, starting at 0 and converging towards 1. In order to define $P_\eta^{\mathcal{B}}(U)$ and $P_{p\circ\eta}^{p(\mathcal{B})}(p(U))$, we need to compute the integers

$$k_U := \min\{i \in \mathbb{N} \mid [t_i, 1] \subset \eta^{-1}(U)\} \text{ and } k_{p(U)} := \min\{i \in \mathbb{N} \mid [t_i, 1] \subset (p\circ\eta)^{-1}(p(U))\}.$$

By definition, we have $U \subset p^{-1}(p(U))$, and thus $\eta^{-1}(U) \subset \eta^{-1}(p^{-1}(p(U)))$. We deduce that $k_{P(U)} \leqslant k_U$, and that $t_{P(U)} = t_{k_{P(U)}} \leqslant t_U = t_{k_U}$.

Now, the projection p provides a natural group morphism $P_\eta^{\mathcal{B}}(U) \to \pi_1(X, p(\eta(t_U)))$, which we can compose with the isomorphism $\pi_1(X, p(\eta(t_U))) \simeq \pi_1(X, p(\eta(t_{p(U)}))) = P_{p\circ\eta}^{p(\mathcal{B})}(p(U))$ induced by the path $(p\circ\eta)_{[t_U; t_{P(U)}]}$. Like in the proof of Proposition 7.1.4 (isomorphism of functors), the induced morphism $P_\eta^{\mathcal{B}}(U) \to P_{p\circ\eta}^{p(\mathcal{B})}(p(U))$ fits into a natural transformation $P_\eta^{\mathcal{B}} \Rightarrow P_{p\circ\eta}^{p(\mathcal{B})}$. By functoriality of the projective limit, we obtain a well-defined morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X, p\circ\eta)$. □

We can be more explicit concerning the morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X, p\circ\eta)$. It is standard that $\pi_1^{\mathrm{loc}}(E,\eta)$, as a projective limit of groups, is given by

$$\pi_1^{\mathrm{loc}}(E,\eta) = \left\{ (s_U)_{U\in\mathcal{B}} \in \prod_{U\in\mathcal{B}} P_\eta^{\mathcal{B}}(U) \;\middle|\; \forall U, U' \in \mathcal{B},\ \iota_{U,U'}(s_U) = s_{U'} \right\}.$$

If we assume that $t_U = t_{p(U)}$ for all $U \in \mathcal{B}$ (to avoid heavy expressions), then the image of $(s_U)_{U\in\mathcal{B}}$ in $\pi_1^{\mathrm{loc}}(X, p\circ\eta)$ is simply $(p(s_U))_{U\in\mathcal{B}}$, where p (abusively) denotes the map $\pi_1(E\cap U, \eta(t_U)) \to \pi_1(X \cap p(U), (p\circ\eta)(t_U))$ induced by p. This morphism is also the morphism obtained by functoriality of π_1.

From now on, we assume that $\mathcal{B}$ contains A. In this case, the natural transformation $P_\eta^{\mathcal{B}} \Rightarrow P_{p\circ\eta}^{p(\mathcal{B})}$ induced by p induces in turn a morphism $P_\eta^{\mathcal{B}}(A) = \pi_1(E,*) \to \pi_1(X,p(*)) = P_{p\circ\eta}^{p(\mathcal{B})}(Y)$. We then have a commutative square:

$$\begin{tikzcd} \pi_1^{\mathrm{loc}}(E,\eta) \arrow[r] \arrow[d] & \pi_1(E,*) \arrow[d, hook] \\ \pi_1^{\mathrm{loc}}(X,p\circ\eta) \arrow[r] & \pi_1(X,p(*)) \end{tikzcd}$$

In particular, the injective morphism $\pi_1(E,*) \to \pi_1(X,p(*))$ restricts to an injective morphism $H_\eta \to H_{p\circ\eta} \cap \pi_1(E,*) \subset H_{p\circ\eta}$. We are now interested in describing the image of this last morphism.

We conjecture that the image of the morphism $H_\eta \to H_{p\circ\eta}$ is exactly $H_{p\circ\eta} \cap \pi_1(E,*)$. In general, we have the following conjecture about the image of the morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X,p\circ\eta)$:

Conjecture 7.2.6 (Local fundamental groups and covering maps). *The image of the morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X,p\circ\eta)$ is the subgroup*

$$\pi_1^{\mathrm{loc}}(X,p\circ\eta)_E := \left\{(\sigma_{p(U)})_{U\in\mathcal{B}} \in \pi_1^{\mathrm{loc}}(X,p\circ\eta) \mid \sigma_Y \in \pi_1(E,p(*)) \subset \pi_1(X,*)\right\}.$$

By definition, the group $\pi_1^{\mathrm{loc}}(X,p\circ\eta)_E$ is the inverse image of $H_{p\circ\eta} \cap \pi_1(E,*)$ under the natural map $\pi_1^{\mathrm{loc}}(X,p\circ\eta) \to \pi_1(X,p(*))$. We then have a pullback square of groups

$$\begin{tikzcd} \pi_1^{\mathrm{loc}}(X,p\circ\eta)_E \arrow[r] \arrow[d] & \pi_1(E,*) \arrow[d] \\ \pi_1^{\mathrm{loc}}(X,p\circ\eta) \arrow[r] & \pi_1(X,p(*)) \end{tikzcd}$$

which shows that the image of $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X,p\circ\eta)$ is included in $\pi_1^{\mathrm{loc}}(X,p\circ\eta)_E$. It remains to show that the considered morphism is surjective with values in the subgroup $\pi_1^{\mathrm{loc}}(X,p\circ\eta)_E$.

If Conjecture 7.2.6 holds, then the image of the morphism $H_\eta \to H_{p\circ\eta}$ is clearly equal to $H_{p\circ\eta} \cap \pi_1(E,*)$ (which is the actual result we are interested in).

We will only be able to prove Conjecture 7.2.6 in a restrictive case, which is however sufficient for handling parabolic subgroups of complex braid groups.

The case of a finite group acting by isometries

In this subsection, we fix a metric space (A,d), and a finite group G acting on A by isometries (i.e. $d(g.a,g.b) = d(a,b)$ for all $g \in G$, $x,y \in A$). We denote by $p : A \twoheadrightarrow Y := A/G$ the projection map. The orbit space $Y := A/G$ is metrizable, for instance with the distance δ defined by

$$\forall a,b \in A,\ \delta(p(a),p(b)) := \inf_{g\in G} d(a,g.b) = \inf_{g,h\in G} d(g.a,h.b).$$

Lemma 7.2.7. *Let $a \in A$, and let $r > 0$. If $B_d(x,r)$ denotes the open ball in A with center x and radius r, then we have $p(B_d(a,r)) = B_\delta(p(a),r)$.*

Proof. Let $b \in A$. If $d(a,b) < r$, then by construction, we have $\delta(p(a), p(b)) \leqslant d(a,b) < r$ and thus $p(b) \in B_\delta(p(a), r)$. Conversely, assume that $p(b) \in B_\delta(p(a), r)$, we have

$$\delta(p(a), p(b)) = \inf_{g \in G} d(a, g.b) < r.$$

Since G is finite, this infimum is actually a minimum, and thus there is some $g \in G$ such that $d(a, g.b) < r$. We then have $g.b \in B_d(a,r)$ and $p(b) = p(g.b) \in p(B_d(a,r))$, as we wanted to show. □

Let us consider the set

$$E := \{a \in A \mid \mathrm{Stab}_G(a) = \{1\}\}.$$

By construction, the action of G on E is free, and $E = p^{-1}(p(E))$ since $p(E)$ is exactly the set of orbits of cardinality $|G|$. Since G is finite and acts freely on E by isometries, the projection map p restricts to a covering map $E \to E/G$, whose fibers are identified with G. If we write $X := E/G$, then p becomes a branched covering map as in the beginning of the section, and we are in the same situation as above.

Let $a \in A$, and let $\mathcal{B}$ be the set of open balls in A centered in a. By Lemma 7.2.7, the neighborhood basis $p(\mathcal{B})$ is simply the set of open balls (for the distance δ) in Y centered in $p(a)$. Since the orbit $G.a$ is finite, there is a radius $r > 0$ such that, for all $g \in G$, we either have $g.a = a$ or $B_d(g.a, r) \cap B_d(a,r) = \varnothing$. In this case $p^{-1}(p(B_d(a,r)))$ is a disjoint union of open balls $B_d(a', r)$, for a' ranging among the orbit $G.a$. Using this particular neighborhood basis, we are able to show that Conjecture 7.2.6 holds in this context.

Theorem 7.2.8. *Let A be a metric space, and let G be a finite group acting by isometries on A. Let also E by the set of points of A with trivial stabilizer in G, and let $Y := A/G$, $X := E/G$. If η is a capillary path from some $* \in E$ to some $a \in A$, then the image of the natural morphism $\pi_1^{\mathrm{loc}}(E, \eta) \to \pi_1^{\mathrm{loc}}(X, p \circ \eta)$ is $\pi_1^{\mathrm{loc}}(X, p \circ \eta)_E$.*

Proof. As stated earlier, we only have to show that any element of $\pi_1^{\mathrm{loc}}(X, p \circ \eta)_E$ is the image of some element of $\pi_1^{\mathrm{loc}}(E, \eta)$.

First, let $r_0 > 0$ be such that for all $g \in G$, we either have $g.a = a$ or $B_d(g.a, r_0) \cap B_d(a, r_0) = \varnothing$. For $r < r_0$, we then have

$$p^{-1}(p(B_d(a,r))) = p^{-1}(B_\delta(p(a), r)) = \bigsqcup_{a' \in G.a} B_d(a', r).$$

Let us then consider the set $\mathcal{B}$ of open balls in A centered in a and with radius $r < r_0$.

For $r < r_0$ and $t < 1$, we already know that, if $\eta([t,1]) \subset B_d(a,r)$, then $p(\eta([t,1])) \subset p(B_d(a,r)) = B_\delta(p(a), r)$. Conversely, if $p(\eta([t,1])) \subset B_\delta(p(a), r)$, then we have

$$\eta([t,1]) \subset p^{-1}(B_\delta(p(a), r)) = \bigsqcup_{a' \in G.a} B_d(a', r).$$

Since η is continuous, and since $\eta(s) \in B_d(a,r)$ for s big enough, this implies that $\eta([t,1]) \subset B_d(a,r)$.

Let now $(t_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence in $[0,1[$, starting at 0 and converging towards 1. The argument above proves that $k_{B_d(a,r)}$ and $k_{B_\delta(p(a),r)}$ are equal for $r < r_0$, and thus $t_r := t_{B_d(a,r)} = t_{B_\delta(p(a),r)}$.

Since the elements of $\mathcal{B}$ and $p(\mathcal{B})$ are just indexed by a real number $r_0 < r$, we we can rewrite $\pi_1^{\mathrm{loc}}(E,\eta)$ and $\pi_1^{\mathrm{loc}}(X,p\circ\eta)$ to alleviate notations:

$$\pi_1^{\mathrm{loc}}(E,\eta) = \left\{ (s_r)_{r<r_0} \in \prod_{r<r_0} P_\eta^{\mathcal{B}}(B_d(a,r)) \;\middle|\; \forall r,r' < r_0,\ \iota_{r,r'}(s_r) = s_{r'} \right\},$$

$$\pi_1^{\mathrm{loc}}(X,p\circ\eta) = \left\{ (\sigma_r)_{r<r_0} \in \prod_{r<r_0} P_{p\circ\eta}^{\mathcal{B}}(B_\delta(p(a),r)) \;\middle|\; \forall r,r' < r_0,\ \iota'_{r,r'}(\sigma_r) = \sigma_{r'} \right\}.$$

where $\iota_{r,r'}$ (resp. $\iota'_{r,r'}$) stands for $\iota_{B_d(a,r),B_d(a,r')}$ (resp. for $\iota'_{B_\delta(p(a),r),B_\delta(p(a),r')}$).

Let now $(\sigma_r)_{r<r_0} \in \pi_1^{\mathrm{loc}}(X,p\circ\eta)_E$. For $r<r_0$, σ_r can be represented as a path $\gamma_r : [0,1] \to B_\delta(p(a),r))$ from $p(\eta(t_r))$ to itself. This path has a unique lift $\widetilde{\gamma}_r : [0,1] \to E$ which starts at $\eta(t_r)$.

We claim that the lift $\widetilde{\gamma}_r$ also ends at $\eta(t_r)$. Indeed, the path $(p\circ\eta)_{[0;t_r]} * \gamma_r * (p\circ\eta)_{[t_r;0]}$ represents the image in $\pi_1(X,p(*))$ of σ_r under the natural morphism $P_{p\circ\eta}^{p(\mathcal{B})}(B_\delta(p(a),r)) \to \pi_1(X,p(*))$. By assumption, this element lies in $\pi_1(E,*)$, i.e. the unique lift of $(p\circ\eta)_{[0;t_r]} * \gamma_r * (p\circ\eta)_{[t_r;0]}$ in E starting at $\eta(0) = *$ also ends at $\eta(0)$. However, if $g.\eta(t_r)$ is the endpoint of $\widetilde{\gamma}_r$ in E, then $\eta_{[0;t_r]} * \widetilde{\gamma}_r * g.\eta_{[t_r;0]}$ is a lift of $(p\circ\eta)_{[0;t_r]} * \gamma_r * (p\circ\eta)_{[t_r;0]}$ in E, which ends at $g.\eta(0) = g.*$. We then have $g.* = *$ and $g=1$ since $* \in E$.

By construction, γ_r is a path in $p^{-1}(B_\delta(p(a),r)) = \bigsqcup_{a'\in G.a} B_d(a',r)$. Since γ_r is continuous, and since $\gamma_r(0) = \eta(t_r) \in B_d(a,r)$, we obtain that γ_r is actually a path in $B_d(a,r)$. If s_r denotes the homotopy class of γ_r in $P_\eta^{\mathcal{B}}(B_d(a,r))$, then we obtain an element $(s_r)_{r<r_0} \in \prod_{r<r_0} P_\eta^{\mathcal{B}}(B_d(a,r))$. It remains to show that $(s_r)_{r<r_0}$ actually belongs to $\pi_1^{\mathrm{loc}}(E,\eta)$. For $r<r'<r_0$, the path γ_r is homotopic to the path $(p\circ\eta)_{[t_r;t_{r'}]} * \gamma_{r'} * (p\circ\eta)_{[t_{r'};t_r]}$ in $B_\delta(p(a),r)$ since $(\sigma_r)_{r<r_0} \in \pi_1^{\mathrm{loc}}(X,p\circ\eta)$. Since we have a covering map $E\to X$, we can lift a homotopy between those to paths to a homotopy between the lifts $\widetilde{\gamma}_r$ and $\eta_{[t_r;t_{r'}]} * \widetilde{\gamma}_{r'} * \eta_{[t_{r'};t_r]}$. Again, since $p^{-1}(B_\delta(a,r)) = \sqcup_{a'\in G.a} B_d(a',r)$, this lifted homotopy is actually a homotopy in $B_d(a,r)$, and we have $\iota_{r,r'}(s_r) = s_{r'}$, which is what we wanted to show.

By construction, $(s_r)_{r<r_0}$ is a preimage of (σ_r) under the morphism $\pi_1^{\mathrm{loc}}(E,\eta) \to \pi_1^{\mathrm{loc}}(X,p\circ\eta)$, which finishes the proof. ☐

Corollary 7.2.9 (Pseudoparabolic subgroups and group acting by isometries). *Let A be a metric space, and let G be a finite group acting by isometries on A. Let also E by the set of points of A with trivial stabilizer in G, and let $Y := A/G$, $X := E/G$.*
For $* \in E$, the pseudoparabolic subgroups of $\pi_1(E,*)$ (for the pair (E,A)) are exactly the intersections with $\pi_1(E,*)$ of the pseudoparabolic subgroups of $\pi_1(X,p(*))$ (for the pair (X,Y)).

Proof. First, let $H \subset \pi_1(E,*)$ be the pseudoparabolic subgroup induced by a capillary path η from $*$ to some $a\in A$. If H' is the pseudoparabolic subgroup of $\pi_1(X,p(*))$ induced by the capillary path $p\circ\eta$, then we have $H = H'\cap \pi_1(E,*)$ by Theorem 7.2.8.

Conversely, let $H' \subset \pi_1(X,p(*))$ be a pseudoparabolic subgroup induced by some capillary path γ from $p(*)$ to some $y\in Y$. Since $E\to X$ is a covering map, $\gamma_{[0,1[}$ can be lifted to a path $\widetilde{\gamma} : [0,1[\to X$ such that $\gamma(0) = *$. Now, $\gamma(1) = y = G.a$ for some $a\in A$, and we can consider $r>0$ small enough so that $p^{-1}(B_\delta(y,r)) = \bigsqcup_{a'\in G.a} B_d(a,r)$. As before, there is a unique $a' \in G.a$ such that $\widetilde{\eta}(t) \in B_d(a,r)$ for t big enough. We can then set $\widetilde{\eta}(1) = a'$ and we obtain a lift $\widetilde{\eta} : [0,1] \to A$, which is again a capillary path. Again by Theorem 7.2.8, the pseudoparabolic subgroup H of $\pi_1(E,*)$ induced by $\widetilde{\eta}$ is equal to $H' \cap \pi_1(E,*)$, which is then a pseudoparabolic subgroup of $\pi_1(E,*)$. □

Remark 7.2.10. Let A be a metric space, and let G be a finite group acting by isometries on A. Let also E by the set of points of A with trivial stabilizer in G, and let $Y := A/G$, $X := E/G$. Assume that the group $\pi_1(X, p(*))$ is endowed with a Garside group structure so that the parabolic subgroups of $\pi_1(X, p(*))$ in the Garside-theoretic sense coincide with the pseudoparabolic subgroups defined topologically. The pseudoparabolic subgroups of $\pi_1(E,*)$ as defined above coincide by Corollary 7.2.9 and Proposition 5.3.21 with the Garside-theoretic parabolic subgroups of $\pi_1(E,*)$, seen as a finite index subgroup of $\pi_1(X, p(*))$.

7.3 Parabolic subgroups of complex braid groups

Throughout this section we fix a finite dimensional complex vector space V of dimension n, along with a complex reflection group $W \subset \mathrm{GL}(V)$. We keep the notation of Chapter 6. We also fix a basepoint $* \in X$, so that $P(W) = \pi_1(X,*)$ and $B(W) = \pi_1(X/W, W.*)$.

There are two natural topological pairs (X, V) and $(X/W, V/W)$ attached to W. Using the tools of Section 7.1, we can define local fundamental groups for these topological pairs, and collections of pseudoparabolic subgroups of $P(W)$ and $B(W)$ (where $* \in X$ is some basepoint).

In [GM22], González-Meneses and Marin introduce a similar concept of parabolic subgroup of $B(W)$. The main difference between our definition and [GM22, Definition 2.3] is that, instead of capillary paths, the authors of *loc. cit.* use the more restrictive notion of *normal ray*. A normal ray η is a capillary path satisfying the additional assumption that there is some $\alpha_0 \in]0,1[$ such that $\eta([1-\alpha, 1[)$ is simply connected for every $\alpha \in]0, \alpha_0]$. Furthermore, they also require (in our notation) that the system P_η constructed using a normal ray η always stabilizes.

7.3.1 Pseudoparabolic subgroups and parabolic subgroups

Our first goal is to prove that the collection of pseudoparabolic subgroups of $B(W)$ defined in Definition 7.1.5 (local fundamental group) coincides with the collection of parabolic subgroups of $B(W)$ defined in [GM22, Definition 2.3]. In particular this will allow us to use the results of [GM22, Section 2] in our context.

Assume that the space V is endowed with a W-invariant hermitian scalar product. This scalar product endows V with a W-invariant norm $\|.\|$. We can then consider open balls for this W-invariant norm in the space V. For $v \in V$, we can consider a neighborhood basis $\mathcal{B}$ for $W.v$ in X/W given by is $\{(B(v,r) \cap X)/W \mid r > 0\}$. The following lemma, which is a consequence of the proof of [GM22, Proposition 2.1], shows that local fundamental groups for the topological pair $(X/W, V/W)$ can always be computed by systems which stabilize.

Lemma 7.3.1. *Let η be a capillary path based at $W.*$ and terminating at $W.v$. Let also $\varepsilon > 0$ be such that $B(v,\varepsilon)$ has a nonempty intersection with $H \in \mathcal{A}$ if and only if $v \in H$. Then the system P_η stabilizes at $(B(v,\varepsilon) \cap X)/W$.*

As a consequence of this, we obtain that the two possible definitions of parabolic subgroup of a complex braid groups, using normal rays or capillary paths, are actually equivalent.

Proposition 7.3.2. *The collection of pseudoparabolic subgroups of $B(W) = \pi_1(X/W, W.*)$ as defined in Definition 7.1.5 coincides with the collection of parabolic subgroups of $B(W)$ as defined* *in [GM22, Definition 2.3].*

Proof. Let $v \in V$, and let $\varepsilon > 0$ be as in Lemma 7.3.1. First, let $\eta : [0,1] \to V/W$ be a normal ray based at $W.*$ and terminating at $W.v$. By definition, the parabolic subgroup of $B(W)$ (in the sense of [GM22, Definition 2.3]) induced by η is the image of $\pi_1((B(v,\varepsilon) \cap X)/W, \eta(t_0))$ (for some $t_0 < 1$ big enough) in $B(W)$ under the natural morphism induced by η. Since the system P_η stabilizes at $(B(v,\varepsilon) \cap X)/W$, the pseudoparabolic subgroup of $B(W)$ induced by η is the same subgroup of $B(W)$.

Then, let $\eta_1, \eta_2 : [0,1] \to V/W$ be two capillary paths based at $W.*$ and terminating at $W.v$. Since X is the complement of a union of complex codimension 1 subspaces of V, we obtain that X is path connected, as well as $B(v,\varepsilon) \cap X$ and $(B(v,\varepsilon) \cap X)/W$. Since both P_{η_1} and P_{η_2} stabilize at $(B(v,\varepsilon) \cap X)/W$ by Lemma 7.3.1, we obtain that the pseudoparabolic subgroups induced by η_1 and η_2 are conjugate in $B(W)$ by Lemma 7.1.10. Since any point in V/W is the final point of some normal ray based at $W.*$, we can assume that η_2 is a normal ray. We obtain that the pseudoparabolic subgroup of $B(W)$ induced by η_1 is conjugate to the parabolic subgroup of $B(W)$ induced by η_2. Since parabolic subgroups of $B(W)$ are stable under conjugacy by [GM22, Proposition 2.5], we obtain that the pseudoparabolic subgroup of $B(W)$ induced by η_1 is actually a parabolic subgroup of $B(W)$. □

Remark 7.3.3. By construction of the norm on V, the finite group W acts on V by isometries, and the space X is the set of points of V with trivial stabilizer in W by Steinberg's Theorem. By Corollary 7.2.9 (pseudoparabolic subgroups and group acting by isometries), we obtain that the pseudoparabolic subgroups of $P(W)$ are exactly the intersections with $P(W)$ of the pseudoparabolic subgroups of $B(W)$.

In the case where W is a complexified real reflection group. We obtain by combining Proposition 7.3.2 and [GM22, Proposition 3.1] that the pseudoparabolic subgroups of $B(W)$ are the conjugates of the standard parabolic subgroups of $B(W)$ seen as an Artin group. In this case, the collection of pseudoparabolic subgroups of $P(W)$ is the same as the one giving the "algebraic curve complex" $\mathcal{C}_{\mathrm{alg}}$ of [DH21], which considers the more general case of simplicial hyperplane arrangements.

7.3.2 First results on parabolic subgroups

Using Proposition 7.3.2, we can now use the various results proven in [GM22, Section 2]. We recall these results here for the sake of self-containment.

First, we have that parabolic subgroups in $B(W)$ are braid groups of parabolic subgroups of the reflection group W, as defined in Definition 6.1.6.

Proposition 7.3.4. *[BMR98, Proposition 2.29] and [GM22, Proposition 2.5 and 2.8]*
Let $B_0 \subset B(W)$ be a parabolic subgroup, defined using a capillary path η terminating at some $W.v \in V/W$. Let also X_0 be the complement in V of the reflecting hyperplanes of W containing v.

(a) If $v' \in W.v$ is the terminating point of the unique lift of η in V starting from $*$, then the image W_0 of B_0 in W is the stabilizer of v' in W. In particular, it is a parabolic subgroup of W.

(b) The group B_0 is isomorphic to $B(W_0) \simeq \pi_1(X_0/W_0, W_0.*)$.

(c) *The embedding $B_0 \subset B(W)$ induces a bijection between the parabolic subgroups of $B_0 \simeq \pi_1(X_0/W_0, W_0.*)$ and the parabolic subgroups of $B(W)$ which are contained in B_0.*

In particular, assume that v is contained in a unique hyperplane $H \in \mathcal{A}$, then W_v is generated by a distinguished reflection of hyperplane H. One can then show that a parabolic subgroup B_0 induced by a capillary path terminating at $W.v$ is (almost by definition) of the form $\langle \sigma \rangle$, where σ is a distinguished braided reflection associated to s. In other words, parabolic subgroups of $B(W)$ of rank 1 are exactly the subgroups of $B(W)$ generated by distinguished braided reflections.

Since the image of a parabolic subgroup of $B(W)$ is a parabolic subgroup of W, we can define irreducible parabolic subgroups of $B(W)$ precisely as the parabolic subgroups of $B(W)$ whose image in W is irreducible in the sense of Remark 6.1.12.

Corollary 7.3.5. *[GM22, Corollary 2.9] Let $B_0, B_1 \subset B(W)$ be two parabolic subgroups with $B_0 \subset B_1$. Then B_0 and B_1 are equal if and only if their images in W are equal, if and only if they have the same rank.*

Another important result of [GM22] is that, since the definition of parabolic subgroups depends only on the topological pair $(X/W, V/W)$, two complex reflection groups W, W' for which the pairs $(X/W, V/W)$ and $(X'/W', V'/W')$ are isomorphic are such that $B(W)$ and $B(W')$ have the same parabolic subgroups. In particular, we have the following result:

Proposition 7.3.6. *[GM22, Proposition 2.7] Let W, W' be two isodiscriminantal complex reflection groups, and let f (resp. f') be a system of basic invariants for W (resp. for W') such that $\Delta(W, f) = \Delta(W', f')$. The isomorphism $B(W) \simeq B(W')$ induced by f and f' induces a bijection between the collections of parabolic subgroups of $B(W)$ and $B(W')$.*

Furthermore, it is possible to show that conjugacy of parabolic subgroups is completely determined by the conjugacy of parabolic subgroups in W. Note also that, as we mentioned earlier, parabolic subgroups of $B(W)$ are stable under conjugacy by [GM22, Proposition 2.5].

Proposition 7.3.7 (Conjugacy of parabolic subgroups). *[GM22, Proposition 2.6] Let B_1, B_2 be two parabolic subgroups of $B(W)$, and let W_1, W_2 be their images inside W.*

(a) *The groups W_1 and W_2 are equal if and only if B_1 and B_2 are conjugate by an element of $P(W)$.*

(b) *The groups W_1 and W_2 are conjugate in W if and only if B_1 and B_2 are conjugate in $B(W)$.*

Corollary 7.3.8. *Let $B_0 \subset B(W)$ be a parabolic subgroup induced by a capillary path η terminating at $W.v$. The $B(W)$-conjugacy class of B_0 depends only on the stratum of the discriminant stratification to which $W.v$ belongs. Likewise, the $P(W)$-conjugacy class of B_0 depends only on the stratum of the stratification $(F^0)_{F \in L(\mathcal{A})}$ of V.*

Note that it is possible to show (using in particular the tools of Section 5.3.3) that $P(W)$-conjugacy classes of parabolic subgroups of $B(W)$ are actually in bijection with $P(W)$ conjugacy classes of parabolic subgroups of $P(W)$.

A result which does not appear in [GM22] relates the strata of the discriminant stratification with generating systems of parabolic subgroups: As we announced in Section 6.2.1, we can relate the multiplicity of points in the discriminant stratification with the cardinalities of minimal generating sets of parabolic subgroups of $B(W)$, generalizing Proposition 6.2.19.

Proposition 7.3.9. *Let $B_0 \subset B(W)$ be a parabolic subgroup induced by a capillary path η terminating at $W.v$. Let also W_0 be the image of B_0 in W. The following integers are equal:*

- *The minimum number of reflections needed to generate W_0.*
- *The minimum number of braided reflections of $B(W)$ needed to generate B_0.*
- *The multiplicity of $W.v$ in the discriminant stratification.*

Proof. As usual, we can assume that W is irreducible, the result being easily deduced from this case. By Proposition 6.2.19 and Proposition 7.3.4, it is sufficient to show that the multiplicity of $W.v$ in the discriminant stratification is equal to the multiplicity of $W_0.0$ in the discriminant stratification of V/W_0 (attached to the discriminant of W_0).

First, let $\Sigma \subset V$ be the union of the reflecting hyperplanes of W. Let also $\mathcal{A}_0 \subset \mathcal{A}$ be the set of reflecting hyperplanes of W which contain v, and let Σ_0 be the union of the elements of $\mathcal{A}_0$. Let $F \in L(\mathcal{A})$ be the unique flat so that $v \in F^0$. By Lemma 6.2.7, we have a homeomorphism $F^0 \to F^0/W_0$, and a $|W/W_0|$-fold covering $F^0 \to F^0/W$. Thus, the natural map $\Sigma/W_0 \to \Sigma/W$ is unramified at $W_0.v$, and the multiplicity of $W.v$ in the discriminant stratification of V/W coincides with the multiplicity of $W_0.v$ in Σ/W_0.

Then, we show that the multiplicity of $W_0.v$ in Σ/W_0 is equal to the multiplicity of $W_0.v$ in the discriminant stratification of V/W_0. For $H \in \mathcal{A}$, let $\alpha_H \in V^*$ be a linear form with kernel H, and let e_H be the cardinality of the parabolic subgroup of W fixing H pointwise. An equation of Σ is given by $D(W) = \prod_{H\in\mathcal{A}}(\alpha_H)^{e_H}$, and an equation for Σ_0 is given by $D(W_0) = \prod_{H\in\mathcal{A}_0}(\alpha_H)^{e_H}$. We can then write $D(W) = D(W_0)P$ for $P = \prod_{H\in\mathcal{A}\setminus\mathcal{A}_0}(\alpha_H)^{e_H} \in \mathbb{C}[V]$.

Let $f = (f_1, \ldots, f_r)$ be a system of basic invariants for W_0 so that $\mathbb{C}[V]^{W_0} \simeq \mathbb{C}[X_1, \ldots, X_r]$. Since both $D(W)$ and $D(W_0)$ belong to $\mathbb{C}[V]^{W_0}$, so does P. Since $P(v) \neq 0$ by construction, we obtain that the multiplicity of $W_0.v$ in $V(D(W_0)P)/W_0 = \Sigma/W_0$ (i.e. the valuation of $D(W)$ seen as an element of $\mathbb{C}[X_1, \ldots, X_r]$) coincides with the multiplicity of $W_0.v$ in $V(D(W_0))/W_0 = \Sigma_0/W_0$, i.e. in the discriminant stratification of V/W_0.

Lastly, the multiplicity of $W_0.v$ in the discriminant stratification of V/W_0 is the same as that of $W_0.0$ since v and 0 both belong to the intersection of all the reflecting hyperplanes of W_0. □

7.3.3 The curve complex

Another important construction in [GM22] is the so-called curve complex attached to $B(W)$. It is a simplicial complex endowed with an action of W, built using a graph attached to the parabolic subgroups of $B(W)$.

Definition 7.3.10 (Curve graph). [GM22, Section 2.5] The curve graph of $B(W)$ is the graph Γ whose vertices are the irreducible parabolic subgroups of $B(W)$, and whose edges are the pairs $\{B_1, B_2\}$ such that $B_1 \neq B_2$ and either $B_1 \subset B_2, B_2 \subset B_1$ or $B_1 \cap B_2 = [B_1, B_2]$ is trivial.

Using this graph, the *curve complex* of $B(W)$ is defined as the clique complex of Γ. That is, the simplicial complex whose simplices are the sets of irreducible parabolic subgroups whose elements are all adjacent to one another in Γ.

The action of $B(W)$ on parabolic subgroups by conjugacy induces an action of $B(W)$ on Γ and on the curve complex. This action factorizes through $B(W)/Z(B(W))$. Since parabolic subgroups of rank 1 are given by distinguished braided reflections, and since distinguished braided reflections generate $B(W)$, we obtain the following proposition:

Proposition 7.3.11. *[GM22, Proposition 2.10] The action of $B(W)/Z(B(W))$ on Γ is faithful. It is actually already faithful on the set of vertices of rank 1.*

In order to study the curve complex (and its $B(W)$-action) further, González-Meneses and Marin give in [GM22, Theorem 1.3] a characterization of adjacency in the curve graph for every irreducible W but G_{31} (for which we will detail the proof in Section 10.2.2).

If $B_0 \subset B(W)$ is an irreducible parabolic subgroup, then $Z(B_0)$ is infinite cyclic by [BMR98, Theorem 2.24] and [Bes15, Theorem 12.3 and Corollary 12.7], and it is generated by a unique z_{B_0} such that $\ell(z_{B_0}) > 0$ (where $\ell : B(W) \to \mathbb{Z}$ is the length function of Lemma 6.2.17). Then [GM22, Theorem 1.3] states that two irreducible parabolic subgroups B_1, B_2 in $B(W)$ are adjacent in the curve graph Γ if and only if z_{B_1} and z_{B_2} are different and commute with one another.

Chapter 8

Dual braid monoids and related structures

In this chapter we study the dual braid monoid associated to a well-generated complex reflection group. This Garside monoid allows us in turn to study the braid groups of such a reflection group. The first two sections are not new. In Section 8.3, we study noncrossing partitions lattices, and how they are related to lattices of simple elements of dual braid monoids associated to irreducible well-generated complex reflection groups in the infinite series. In Section 5.2.1, we show that dual braid monoids are support-preserving in the sense of [GM22], allowing us to completely describe parabolic subgroups of the associated braid groups, and giving new proofs of results of González-Meneses and Marin regarding parabolic subgroups in this case.

Contents

In [BKL98], Birman Ko and Lee introduced a new Garside monoid with group of fractions isomorphic to the usual braid group (note that this was before the formal definition of Garside monoid in [DP99]). This monoid has more generators than the more classical Artin-Tits monoid, as a consequence it admits more symmetries in the sense of Section 4.1. This fact was used by Bessis, Digne and Michel [BDM02] to provide a “lift” of Springer theory of regular elements to the usual braid group.

In the subsequent years, the monoid of Birman-Ko-Lee was generalized to other Artin groups of spherical type by Bessis [Bes03], and then to well-generated complex braid groups [BC06], [Bes15], under the name “dual braid monoid”. These generalizations were the first Garside group structures on several exceptional complex braid groups (for most of these groups, no other Garside structures are known to this day). The many symmetries of these dual structures allowed Bessis to extend the result of [BDM02] to arbitrary well-generated complex braid groups, as we will see in Chapter 9.

As Garside structures naturally give rise to explicit description of $K(\pi, 1)$-spaces, the introduction of the dual braid monoid (and the associated dual structure on well-generated complex braid groups) was a key point in the proof of Bessis that finite complex reflection arrangements are $K(\pi, 1)$ [Bes15]. The general construction of dual braid monoids, along with their relationship with complex braid groups, relies on rather intricate arguments: On the one hand, geometrical constructions allow us to understand the regular orbit space of a well-generated complex reflection group, using the so-called Lyashko-Looijenga map. On the other hand, more combinatorial arguments allow us to understand the behavior of well-generated complex reflection group relative to the generating set made of all reflections. Both sides are crucial in Bessis’ works, and several arguments on these matters were clarified in [Dou17] and [Rip10].

In the first section, we consider the (extended) Lyashko-Looijenga morphism associated to a well-generated irreducible complex reflection group W (under a convenient choice of basic invariants). We detail how this morphism can be used to give a description of the regular orbit space attached to W, mostly following [Dou17] and [Bes15]. Note however that we extend the description given in [Bes15, Section 11] to the whole orbit space, and not just the regular orbit space as in [Bes15].

In the second section, we introduce dual braid monoids as Garside interval monoids attached to well-generated complex reflection groups, and we consider the first general properties of their lattices of simples. In the case of irreducible groups belonging to the infinite series, the lattice of simples of the dual braid monoid can be understood as a noncrossing partition lattice of some subset of the complex plane [BKL98], [BW02], [BC06]. As this interpretation will be useful to us later, we detail it in the third section.

The fourth section is devoted to proving support-preservingness of dual structures on well-generated complex braid groups. In fact we obtain a description of minimal positive conjugators, which implies in particular support-preservingness. This result allows us to give new proofs of [GM22, Theorem 1.1 and Theorem 1.2] for well-generated complex braid groups. Our description of minimal positive conjugators is obtained in a case-by-case fashion, using in particular the results of the third section when considering members of the infinite series.

Lastly, we detail in the fifth section how to generalize the dual structure in order to study irreducible badly-generated complex braid groups belonging to the infinite series. In particular, we obtain new proofs of [GM22, Theorem 1.1 and Theorem 1.2] in this case.

8.1 The Lyashko–Looijenga morphism

Let W be an irreducible well-generated complex reflection group. The starting point of Bessis' construction of the dual braid monoid attached to W is a convenient description of its associated orbit space, obtained using the Lyashko-Looijenga morphism. This morphism is a ramified covering map [Bes15, Theorem 5.3], and it is possible to completely describe its fibers [Bes15, Theorem 7.20]. However, Bessis later suggested that it was far more efficient to consider the *extended* Lyashko-Looijenga map for all purposes (this already appears in [Bes15, Section 11]).

Here we follow this point of view and we study almost exclusively the extended Lyashko-Looijenga morphism. We detail how it enjoys similar properties as the classical Lyashko-Looijenga map, following mostly [Bes15] and [Dou17]. We also detail how to use the extended Lyashko-Looijenga morphism to obtain a complete description of the orbit space of W (and not only of its regular orbit space as in [Bes15]).

8.1.1 Reminders on multisets and configurations

Let V be a finite dimensional complex vector space of dimension n, and let $W \subset \mathrm{GL}(V)$ be an irreducible well-generated complex reflection group. By construction, the image under the Lyashko-Looijenga morphism of a point in V/W will be a multiset of points in $\mathbb{C}^n$ (defined as the roots of a certain polynomial). We introduce in this short section the various notation we will use when considering multisets.

Recall that $\mathfrak{S}_n$ acting on $\mathbb{C}^n$ by permuting the coordinates is the complex reflection group $G(1,1,n)$ (Example 6.1.14). A system of basic invariants for this group is given by the elementary symmetric polynomials $\omega_1, \dots, \omega_n$. If we restrict the action of $\mathfrak{S}_n$ to the hyperplane $H = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \omega_1(x_1, \dots, x_n) = 0\}$, we obtain the irreducible complex reflection group $\widetilde{G}(1,1,n)$, and a system of basic invariants for this group is given by $\omega_2, \dots, \omega_n$.

The orbit space $E_n := \mathbb{C}^n/\mathfrak{S}_n$ is by definition the set of *multisets* in $\mathbb{C}$ of (multiset) cardinality n. The multiset attached to a vector $(x_1, \dots, x_n)$ will be denoted by $\{\!\{x_1, \dots, x_n\}\!\}$. We know by the beginning of Section 6.2.1 that the map $E_n \to \mathbb{C}^n$ defined by

$$\{\!\{x_1, \dots, x_n\}\!\} \mapsto (\omega_1(x_1, \dots, x_n), \dots, \omega_n(x_1, \dots, x_n))$$

is a homeomorphism. Actually, up to a sign, the $\omega_i(x_1, \dots, x_n)$ are the coefficients of the unique monic polynomial $P(T) \in \mathbb{C}[T]$ whose multiset of roots is $\{\!\{x_1, \dots, x_n\}\!\}$.

Likewise, sending $\{\!\{x_1, \dots, x_n\}\!\}$ to the vector $(\omega_2(x_1, \dots, x_n), \dots, \omega_n(x_1, \dots, x_n))$ induces a homeomorphism between the set $E_n^{\mathrm{c}} := H/\mathfrak{S}_n$ of *centered multisets* and $\mathbb{C}^{n-1}$. Actually, we have an isomorphism of topological pairs $(E_n^{\mathrm{c}}, E_n) \simeq (\{0\} \times \mathbb{C}^{n-1}, \mathbb{C}^n)$.

The complement X in $\mathbb{C}^n$ of the reflecting hyperplanes of $\mathfrak{S}_n \simeq G(1,1,n)$ is the set of vectors containing no equal entries. The quotient $E_n^{\mathrm{reg}} := X/\mathfrak{S}_n$ is then the set of multisets of n points containing no multiple points. In other words, E_n^{reg} is the set of configurations of n points in $\mathbb{C}$. Likewise, $(E_n^{\mathrm{c}})^{\mathrm{reg}} := (X \cap H)/\mathfrak{S}_n$ is the set of *centered configurations* of n points in $\mathbb{C}$.

Lastly, we define E_n° (resp. $E_n^{\mathrm{c}\circ}$) as the set of multisets (resp. centered multisets) not containing 0.

8.1.2 Main properties of the Lyashko-Looijenga morphism

In this section, we fix a finite dimensional complex vector space V with $n := \dim(V)$, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We denote by h the highest degree of W, and we otherwise keep the notation of Chapter 6.

By Theorem 6.2.21, we know that, for any system of basic invariants f for W, the discriminant $\Delta(W, f) \in \mathbb{C}[f_1, \ldots, f_n] \simeq \mathbb{C}[X_1, \ldots, X_n]$, viewed as a polynomial in X_n, with coefficients in $\mathbb{C}[X_1, \ldots, X_n]$, is monic of degree n. We also know that, up to changing the last term of f, we can actually write

$$\Delta(W, f) = X_n^n + \alpha_2(X_1, \ldots, X_{n-1})X_n^{n-2} + \cdots + \alpha_n(X_1, \ldots, X_{n-1}), \tag{8.1.1}$$

where $\alpha_i \in \mathbb{C}[X_1, \ldots, X_{n-1}]$ is weighted homogeneous of weighted degree ih.

We also fix in this section a system of basic invariants f for W such that $\Delta(W, f)$ has the above form, and we consider the $\alpha_i \in \mathbb{C}[X_1, \ldots, X_{n-1}]$ for $i \in [\![2, n]\!]$ as above. We also define $\alpha_1 = 0$ and $\alpha_0 := 1$, so that $\Delta(W, f) = \sum_{i=0}^{n} \alpha_{n-i} X_n^i$.

The homeomorphism $V/W \simeq \mathbb{C}^n$ induced by f allows us to write any element $W.v$ of V/W as a pair (y, z), where $z = f_n(v) \in \mathbb{C}$, and $y = (f_1(v), \ldots, f_{n-1}(v)) \in Y := \mathbb{C}^{n-1}$.

Definition 8.1.1 (Lyashko-Looijenga morphism). [Bes15, Definition 5.1 and 7.23]
Let $x := W.v \in V/W$ be identified with $(y, z) \in Y \times \mathbb{C}$. The multiset $\mathrm{LL}(y) \in E_n^c$ is defined as the solutions of the univariate polynomial

$$P(T) := \Delta(W, f)(y_1, \ldots, y_{n-1}, T) \in \mathbb{C}[T].$$

The multiset $\overline{\mathrm{LL}}(x) \in E_n$ is defined as the solutions of the univariate polynomial

$$\overline{P}(T) := \Delta(W, f)(y_1, \ldots, y_{n-1}, T + z) \in \mathbb{C}[T].$$

The map $\mathrm{LL} : Y \to E_n^c$ (resp. $\overline{\mathrm{LL}} : V/W \to E_n$) is called the *Lyashko-Looijenga morphism* (resp. the *extended Lyashko-Looijenga morphism*).

On an algebraic point of view, we know that $Y = \operatorname{Specmax} \mathbb{C}[X_1, \ldots, X_{n-1}]$. Under the identification $E_n^c = \operatorname{Specmax} \mathbb{C}[\omega_2, \ldots, \omega_n]$, the roots of the polynomial $P(T)$ are sent (up to a sign) to its coefficients. The Lyashko-Looijenga morphism can then be described algebraically as the graded morphism $\mathbb{C}[\omega_2, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_{n-1}]$ sending ω_i to $(-1)^i \alpha_i(X_1, \ldots, X_{n-1})$. This is the original version of [Bes15, Definition 5.1]. Similarly, one can describe the extended Lyashko-Looijenga morphism algebraically:

Lemma 8.1.2 (Algebraic description of $\overline{\mathrm{LL}}$). *Under the identifications of V/W with $\operatorname{Specmax} \mathbb{C}[X_1, \ldots, X_n]$ and of E_n with $\operatorname{Specmax} \mathbb{C}[\omega_1, \ldots, \omega_n]$, the extended Lyashko-Looijenga morphism becomes the graded algebra morphism $\mathbb{C}[\omega_1, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_n]$ defined by*

$$\forall i \in [\![1, n]\!],\ \omega_i \mapsto (-1)^i \sum_{k=0}^{i} \binom{n-k}{i-k} X_n^{i-k} \alpha_k(X_1, \ldots, X_{n-1}).$$

In particular, this morphism sends ω_1 to $-nX_n$.

Proof. By the discussions before the lemma, $\overline{\mathrm{LL}}$ is identified with the algebra morphism $\mathbb{C}[\omega_1, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_n]$ sending ω_i to $(-1)^i$ times the $(n - i)$-th coefficient in T of the polynomial $\Delta(W,f)(X_1,\ldots,X_{n-1},X_n+T)$. Developing this polynomial yields

$$\begin{aligned}\overline{P}(T) &= (X_n+T)^n+\alpha_2(X_n+T)^{n-2}+\cdots+\alpha_{n-1}(X_n+T)+\alpha_n\\ &= \sum_{j=0}^{n}\binom{n}{j}X_n^{n-j}T^j+\sum_{p=0}^{n-2}\alpha_{n-p}\sum_{j=0}^{p}\binom{p}{j}X_n^{p-j}T^j\\ &= \sum_{j=0}^{n}\binom{n}{j}X_n^{n-j}T^j+\sum_{j=0}^{n-2}\sum_{p=j}^{n-2}\alpha_{n-p}\binom{p}{j}X_n^{p-j}T^j\\ &= T^n+nX_nT^{n-1}+\sum_{j=0}^{n-2}T^j\left(\binom{n}{j}X_n^{n-j}+\sum_{p=j}^{n-2}\binom{p}{j}X_n^{p-j}\alpha_{n-p}\right)\\ &= T^n+nX_nT^{n-1}+\sum_{j=0}^{n-2}T^j\left(\sum_{p=j}^{n}\binom{p}{j}X_n^{p-j}\alpha_{n-p}\right)\\ &= T^n+\sum_{j=0}^{n-1}T^j\left(\sum_{p=j}^{n}\binom{p}{j}X_n^{p-j}\alpha_{n-p}\right),\end{aligned}$$

since we defined α_1 as 0 and α_0 as 1. For $i\in[\![1,n]\!]$, the $n-i$-th coefficient of this polynomial is given by

$$\sum_{p=n-i}^{n}\binom{p}{n-i}X_n^{p-n+i}\alpha_{n-p}=\sum_{k(=n-p)=i}^{0}\binom{n-k}{n-i}X_n^{i-k}\alpha_k=\sum_{k=0}^{i}\binom{n-k}{i-k}X_n^{i-k}\alpha_k.$$

We then obtain the description we wanted of the extended Lyashko-Looijenga morphism. Now, it remains to show that $\overline{\mathrm{LL}}$ is quasi-homogeneous. Let $i\in[\![1,n]\!]$ and let $k\in[\![0,i]\!]$. We know that $X_n^{i-k}\alpha_{n-k}(X_1,\ldots,X_{n-1})$ is weighted homogeneous of weighted degree $(i-k)h+kh=ih$, which does not depend on k. In particular, the image of ω_i under $\overline{\mathrm{LL}}$ is weighted homogeneous of weighted degree ih. □

More explicitly, if $x\in V/W$ is identified with the couple (y,z), then we have

$$\overline{\mathrm{LL}}(x)=\mathrm{LL}(y)-z$$

as multisets in $\mathbb{C}$.

Remark 8.1.3. The Lyashko-Looijenga morphism is studied before the extended Lyashko-Looijenga morphism in [Bes15], maybe because the algebraic definition of LL is easier to follow than that of $\overline{\mathrm{LL}}$. However, Bessis suggested afterwards that using LL instead of $\overline{\mathrm{LL}}$ was "plainly dumb, as working with $\overline{\mathrm{LL}}$ is much easier for all purposes" [Bes16]. Furthermore, since $\overline{\mathrm{LL}}$ is easily deduced from LL, we will see that these two maps share the same properties.

An immediate consequence of the definitions is that the map $\overline{\mathrm{LL}}$ is homogeneous (a property which is not shared by LL):

Lemma 8.1.4. *[Bes15, Lemma 11.1] Let $x\in V/W$, and let $\lambda\in\mathbb{C}^*$. We have $\overline{\mathrm{LL}}(\lambda.x)=\lambda^h\overline{\mathrm{LL}}(x)$, where the action of $\mathbb{C}^*$ on V/W is the quotient of the action of $\mathbb{C}^*$ on V by scalar multiplication.*

Another immediate consequence is that, if $x\in W/V$ is associated to (y,z) in $Y\times\mathbb{C}$, then x belongs to the discriminant hypersurface if and only if $0\in\overline{\mathrm{LL}}(x)$, which is equivalent to $z\in\mathrm{LL}(y)$. In other words, we have $X/W=\mathrm{LL}^{-1}(E_n^\circ)$. The following lemma shows that we can me more precise, and relate the multiplicity of x in the discriminant stratification with the multiplicity of 0 in $\overline{\mathrm{LL}}(x)$ (or, equivalently, of z in $\mathrm{LL}(y)$).

Lemma 8.1.5. *[Bes15, Corollary 5.9] Let $x := W.v \in V/W$. The following integers are equal:*

- *The minimum number of reflections needed to generate the stabilizer W_0 of v in W.*
- *The multiplicity of x in the discriminant stratification.*
- *The multiplicity of 0 in the multiset $\overline{\mathrm{LL}}(x)$.*
- *The codimension in V of the unique flat $F \in L(\mathcal{A})$ such that $v \in F^0$.*

Proof. The equality between the first two integers is Proposition 7.3.9. The equality between the second and the third is by definition of $\overline{\mathrm{LL}}(x)$. The equality between the first and last integer comes from the fact that, since W is well-generated, all its parabolic subgroups are also well-generated by Proposition 6.1.23 (parabolic subgroup of a well-generated group), thus, the number of reflections needed to generate the stabilizer of v is equal to its rank, which is precisely the fourth integer. □

Theorem 8.1.6 (Finiteness of $\overline{\mathrm{LL}}$). *[Bes15, Theorem 5.3]*
Let V be a finite dimensional complex vector space of dimension n, and let $W \subset \mathrm{GL}(V)$ be an irreducible well-generated complex reflection group. Let also f be a system of basic invariants for W such that $\Delta(W, f)$ has the form 8.1.1.
The extended Lyashko-Looijenga morphism $\overline{\mathrm{LL}} : V/W \to E_n$ is a finite map. In other words, the algebra morphism $\mathbb{C}[\omega_1, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_n]$ induced by $\overline{\mathrm{LL}}$ makes the latter into a finitely generated module on the former.

Proof. Let us write by f the map $\mathbb{C}[\omega_2, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_{n-1}]$ induced by LL, and by $\overline{f} : \mathbb{C}[\omega_1, \ldots, \omega_n] \to \mathbb{C}[X_1, \ldots, X_n]$ the map induced $\overline{\mathrm{LL}}$. Let also A (resp. $\overline{A}$) be the image of f (resp. of $\overline{f}$). By construction, the algebra A is generated by $\alpha_2, \ldots, \alpha_n$. By Lemma 8.1.2, we have

$$\overline{f}(\omega_i) = (-1)^i \sum_{k=0}^{i} \binom{n-k}{i-k} X_n^{i-k} \alpha_k(X_1, \ldots, X_{n-1}) \in A \otimes_{\mathbb{C}} \mathbb{C}[X_n].$$

Thus, we have $\overline{A} \subset A \otimes_{\mathbb{C}} \mathbb{C}[X_n]$ since $\overline{A}$ is by definition the subalgebra of $\mathbb{C}[X_1, \ldots, X_n]$ generated by the $\overline{f}(\omega_i)$ for $i \in [\![1, n]\!]$. Conversely, we show by induction that $\alpha_i \in \overline{A}$ for all $i \in [\![0, n]\!]$. First, we have that $\alpha_0 = 1$ and $\alpha_1 = 0$ lie in $\overline{A}$ as it is the image of the algebra $\mathbb{C}[\omega_1, \ldots, \omega_n]$. Then if $\alpha_0, \ldots, \alpha_i - 1 \in \overline{A}$ for some $0 < i$, then we have

$$\overline{f}(\omega_i) = (-1)^i \left(\sum_{k=0}^{i-1} \binom{n-k}{i-k} X_n^{i-k} \alpha_k + \alpha_i \right).$$

Since the right-hand part of the sum on the left belongs to $\overline{A}$ by induction hypothesis, we obtain that $\alpha_i \in \overline{A}$. Thus we have $\overline{A} = A \otimes_{\mathbb{C}} \mathbb{C}[X_n]$.

By [Bes15, Theorem 5.3], there is a finite family $R_1, \ldots, R_m \in \mathbb{C}[X_1, \ldots, X_{n-1}]$ which generates $\mathbb{C}[X_1, \ldots, X_{n-1}]$ as a A-module (a detailed proof is given in [Dou17, Theorem 51]). Let now $P = P(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]$. Developing P as a polynomial in X_n with coefficients in $\mathbb{C}[X_1, \ldots, X_{n-1}]$, we get

$$P = \sum_{i=1}^{d} P_i(X_1, \ldots, X_{n-1}) X_n^i.$$

Since all the P_i lie in the sub-A-module of $\mathbb{C}[X_1, \ldots, X_{n-1}]$ generated by $R_1, \ldots, R_m$, and since $\overline{A} = A \otimes_{\mathbb{C}} \mathbb{C}[X_n]$, we obtain that P lies in the sub-$\overline{A}$-module of $\mathbb{C}[X_1, \ldots, X_n]$ generated by

$R_1, \dots, R_m$. Thus $\mathbb{C}[X_1, \dots, X_n]$ is generated by the family $R_1, \dots, R_m$, which finishes the proof. $\square$

Since $\overline{\mathrm{LL}}$ is a quasi-homogeneous finite morphism, we can apply the arguments [Dou17, Proposition 46, Corollary 52, Proposition 55] to $\overline{\mathrm{LL}}$ to obtain the following result:

Corollary 8.1.7. *The extended Lyashko-Looijenga morphism $\overline{\mathrm{LL}}$ is a flat morphism, which is a proper map for the usual topology. Moreover, for $x \in V/W$, we have $\overline{\mathrm{LL}}(x) = \{\!\{0, \dots, 0\}\!\}$ if and only if $x = W.0$.*

Note that, since $\overline{\mathrm{LL}}$ is a quasi-homogeneous map from $\mathbb{C}^n$ to itself, the last statement of the above Corollary is actually equivalent to $\overline{\mathrm{LL}}$ being a finite morphism by [Dou17, Proposition 46]. In Theorem 8.1.6, we proved that $\overline{\mathrm{LL}}$ is finite using that LL is finite. This is proven in [Dou17] using precisely the characterization of [Dou17, Proposition 46].

Since LL is finite, and proper for the usual topology, we can apply the arguments of [Dou17, Corollary 53] to obtain another result on $\overline{\mathrm{LL}}$.

Corollary 8.1.8. *The extended Lyashko-Looijenga morphism is surjective. Moreover, any continuous path in E_n admits a (not necessarily unique) lift under $\overline{\mathrm{LL}}$.*

Lastly, the finiteness of $\overline{\mathrm{LL}}$ can be used to recover covering map properties on $\overline{\mathrm{LL}}$, as in the following corollary:

Corollary 8.1.9 (Covering map). *[Bes15, Lemma 5.7] and [Dou17, Corollary 59] Consider the set*

$$\mathcal{L} := \{x \in V/W \mid \overline{\mathrm{LL}}(x) \notin E_n^{\mathrm{reg}}\}$$

(in other words $\mathcal{L} \subset V/W$ is the set of points x such that $\overline{\mathrm{LL}}(x)$ contains multiple points). The restriction of $\overline{\mathrm{LL}}$ to $(V/W) \setminus \mathcal{L}$ induces a covering map $(V/W) \setminus \mathcal{L} \to E_n^{\mathrm{reg}}$ of degree $n!h^n/|W|$.

Proof. Let $\mathcal{K} \subset Y \simeq \mathbb{C}^{n-1}$ be the set of points y such that $\mathrm{LL}(y) \notin E_n^{\mathrm{creg}}$. By [Dou17, Corollary 59], the restriction of LL to $Y \setminus \mathcal{K}$ induces a covering map $Y \setminus \mathcal{K} \to E_n^{\mathrm{creg}}$ of degree $\frac{n!h^n}{|W|}$.

By construction, the function defined on E_n by $b(\kappa) := \omega_1(\kappa)/n$ is a continuous function $E_n \mapsto \mathbb{C}$ giving the (weighted) barycenter of a multiset. We then have a homeomorphism $\varphi : E_n \simeq E_n^c \times \mathbb{C}$ sending $\kappa \in E_n$ to $(\kappa - b(\kappa), b(\kappa))$. The inverse homeomorphism ψ sends (κ, z) to $\kappa + z$. Let $x = (y, z) \in V/W \simeq Y \times \mathbb{C}$. By definition, we have $\overline{\mathrm{LL}}(x) = \mathrm{LL}(y) - z = \psi(\mathrm{LL}(y), -z)$, and thus the following diagram is commutative:

$$\begin{tikzcd}
 & E_n \arrow[dd, bend right, "\varphi"'] \\
V/W \arrow[ur, "\overline{\mathrm{LL}}"] \arrow[dr, "\mathrm{LL} \times -\mathrm{Id}_{\mathbb{C}}"'] & \\
 & E_n^c \times \mathbb{C} \arrow[uu, bend right, "\psi"']
\end{tikzcd}$$

The homeomorphism $V/W \simeq Y \times \mathbb{C}$ identifies $\mathcal{L}$ with $\mathcal{K} \times \mathbb{C}$. The restriction of $\overline{\mathrm{LL}}$ to the complement of $\mathcal{L}$ in V/W is then the composition of ψ with the restriction of $\mathrm{LL} \times -\mathrm{Id}_{\mathbb{C}}$ to $\mathcal{K} \times \mathbb{C}$, which is a covering map of degree $\frac{n!h^n}{|W|}$. $\square$

Of course, having covering maps allows us to lift paths in E_n which remain in E_n^{reg}. However, we have the following much stronger property, which we will use often for constructing paths in V/W in the sequel.

Proposition 8.1.10 (Path lifting). *[Bes15, Remark 7.21] Let $\gamma : [0,1] \to E_n$ be a path such that points are not unmerged as t increases (they can be merged). If $x_0 \in V/W$ is such that $\overline{\mathrm{LL}}(x_0) = \gamma(0)$, then there exists a unique lift $\widetilde{\gamma} : [0,1] \to V/W$ of γ such that $\widetilde{\gamma}(0) = x_0$.*

Proof. We consider the homeomorphisms φ, ψ introduced in the proof of Corollary 8.1.9. Let $\gamma : [0,1] \to E_n$ be a continuous path. Since φ is a homeomorphism, a lift of γ under $\overline{\mathrm{LL}}$ is the same thing as a lift of $\varphi \circ \gamma$ under $\mathrm{LL} \times -\mathrm{Id}_{\mathbb{C}}$. Since $-\mathrm{Id}_{\mathbb{C}}$ is a homeomorphism, the data of such a lift is equivalent to the data of a lift of the first coordinate of $\varphi \circ \gamma$ under LL. Moreover, since the first coordinate of $\varphi \circ \gamma$ is obtained by simply translating γ, we obtained that points cannot be unmerged in the first coordinate of $\varphi \circ \gamma$ as t increases. The result is then an immediate application of [Bes15, Remark 7.21]. □

Remark 8.1.11. This proposition could be seen as a cheat, as it relies on the very strong trivialization theorem [Bes15, Theorem 7.25], which itself relies on the notion of reduced label, which we have not introduced. We chose to use this result as a black box in order to facilitate the exposition and the definition of the (extended) cyclic label.

Remark 8.1.12. Assume that W has rank 2. Let $d \leqslant h$ be de degrees of W, and let d^* be its nonzero codegree. We saw in Section 6.2.3 that the discriminant of W can be written $B^2 - A^{2h/d}$ for some convenient system of basic invariants. The Lyashko-Looijenga morphism $\mathrm{LL} : Y \simeq \mathbb{C} \to E_2^c$ then sends $a \in \mathbb{C}$ to the set of square roots of $a^{2h/d}$. The second elementary polynomial $\omega_2(x,y) = xy$ allows us to identify E_2^c with $\mathbb{C}$. Under this identification, LL becomes a map $\mathbb{C} \to \mathbb{C}$ sending a to $a^{2h/d}$, and we know that this map is a ramified covering of degree $\frac{2h}{d} = \frac{2h^2}{hd} = \frac{2!h^2}{|W|}$.

On the other hand, the extended Lyashko-Looijenga morphism sends $(a,b) \in \mathbb{C}^2$ to the set of roots of the polynomial $(T+b)^2 - a^{2h/d} = T^2 + 2bT + b^2 - a^{2h/d}$. Again using the elementary symmetric polynomials, we obtain that $\overline{\mathrm{LL}} : \mathbb{C}^2 \to \mathbb{C}^2$ sends (a,b) to $(2b, a^{2h/d})$. Although it is a bit less obvious, this map is also a ramified covering map of degree $\frac{2h}{d}$.

8.1.3 Circular tunnels, Hurwitz rule and labels

In this section, we fix a finite dimensional complex vector space V with $n := \dim(V)$, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We denote by h the highest degree of W, and we otherwise keep the notation of Chapter 6. We also fix a system f of basic invariants for W such that $\Delta(W,f)$ has the form 8.1.1, and we consider again the (extended) Lyashko-Looijenga morphism $\overline{\mathrm{LL}} : V/W \to E_n$.

On top of its remarkable topological and geometrical properties, the map $\overline{\mathrm{LL}}$ allows for good descriptions of the space X/W. In [Bes15], the map LL is used to completely describe the topological pair $(X/W, V/W)$, and the description of X/W is then converted into a description using $\overline{\mathrm{LL}}$.

In order to study parabolic subgroups of centralizers of regular braids in Chapter 9, we need a complete description of the topological pair $(X/W, V/W)$ using only the map $\overline{\mathrm{LL}}$, which we will solely use from now on.

The open subset $\mathcal{U} := \{x \in V/W \mid \overline{\mathrm{LL}}(x) \cap i\mathbb{R}_{\geqslant 0} = \varnothing\}$ of X/W is contractible [Bes15, Lemma 6.3], thus $\pi_1(X/W,\mathcal{U})$ is well-defined as the set of classes of paths from some point of $\mathcal{U}$ to some other one, up to a homotopy leaving the endpoints in $\mathcal{U}$. Following [Bes15, Definition 6.4], we set $B(W) = \pi_1(X/W,\mathcal{U})$ for the remainder of this section.

Definition 8.1.13 (Circular tunnel). [Bes15, Definition 11.24] A *circular tunnel* is a couple $T = (x, L) \in \mathcal{U} \times [0, 2\pi/h]$ such that $e^{iL}x \in \mathcal{U}$. It is identified with the path $\gamma_T : [0, 1] \to X/W$ sending t to $e^{itL}x$.

By definition of $\pi_1(X/W,\mathcal{U})$, the path γ_T associated to a circular tunnel T induces a well-defined element of $B(W) = \pi_1(X/W,\mathcal{U})$, which we call a *simple element*. Of course, several distinct circular tunnels may represent the same element of $B(W)$. In particular, we have the following useful lemma, which is a variation of [Bes15, Lemma 6.15].

Lemma 8.1.14 (Hurwitz rule). *Let $\lambda : [0, 1] \to [0, \frac{2\pi}{h}]$ and let $\gamma : [0, 1] \to \mathcal{U}$ be continuous paths. If $T_t := (\gamma(t), \lambda(t))$ is a circular tunnel for all $t \in [0, 1]$, then all the T_t for $t \in [0, 1]$ represent the same element of $B(W)$.*

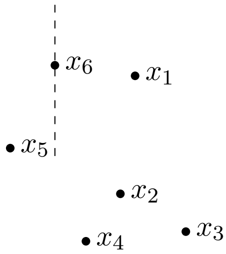
Proof. Let $t \in [0, 1]$. Defining $H(r, s) := e^{is\lambda(rt)}\gamma(rt)$ for $r, s \in [0, 1]$ yields a homotopy between the paths $H(0, s) : s \mapsto e^{i\lambda(0)}\gamma(0)$ and $H(1, s) : s \mapsto e^{is\lambda(t)}\gamma(t)$ in X/W. By assumptions, the endpoints of this homotopy, namely $H(r, 0) = \gamma(rt)$ and $H(r, 1) = e^{i\lambda(rt)}\gamma(rt)$, lie in $\mathcal{U}$. Thus T_0 and T_t represent the same element of $B(W)$ by definition. $\square$

Corollary 8.1.15. *The homotopy class in $B(W) = \pi_1(X/W,\mathcal{U})$ of the circular tunnel $(x, \frac{2\pi}{h})$ does not depend on $x \in \mathcal{U}$. It is denoted by Δ.*

Proof. First, notice that $(x, \frac{2\pi}{h})$ is always a circular tunnel for $x \in \mathcal{U}$, since $\overline{\mathrm{LL}}(e^{\frac{2i\pi}{h}}x) = \overline{\mathrm{LL}}(x)$ for $x \in \mathcal{U}$ by Lemma 8.1.4. Then, for $x, y \in \mathcal{U}$, we can consider a path $\gamma : [0, 1] \to \mathcal{U}$ from x to y since $\mathcal{U}$ is path connected. We then have that $(x, \frac{2\pi}{h})$ and $(y, \frac{2\pi}{h})$ represent the same element of $B(W)$ by Lemma 8.1.14. $\square$

We aim to describe the fibers of the extended Lyashko-Looijenga morphism on V/W, in the spirit of [Bes15, Proposition 11.13] (which only covers the case of X/W). In [Bes15, Proposition 11.13] this is done by defining a notion of cyclic label of an element x of X/W which, coupled with $\overline{\mathrm{LL}}(x)$, completely characterizes x. We will extend the definition of the cyclic label to points belonging to the discriminant hypersurface $\mathcal{H}$. In [Bes15], the notion of cyclic label of an element of X/W is defined using an earlier notion of label [Bes15, Definition 6.9]. We provide a direct definition here.

Let $\kappa \in E_n^\circ$ be a multiset not containing 0. The arguments of the points composing κ are seen in $[\frac{-3\pi}{2}, \frac{\pi}{2}[$. Following [Bes15, Definition 11.8], the points composing κ are ordered by decreasing argument. Points with identical arguments are ordered by increasing modulus. Since we consider arguments in $[\frac{-3\pi}{2}, \frac{\pi}{2}[$, points on the vertical half-line $i\mathbb{R}_+$ always come last, as in the following example:



The *cyclic support* of κ is then defined as the sequence $(\kappa_1,\dots,\kappa_m)$ of distinct points in κ, ordered as above [Bes15, Definition 11.8]. If κ contains 0, then we extend this definition by defining $(\kappa_1,\dots,\kappa_m,0)$ as the cyclic support of κ, where $(\kappa_1,\dots,\kappa_m)$ is the cyclic support of $\kappa\setminus\{0\}$. These definitions are extended to V/W by defining the cyclic support of $x\in V/W$ as that of $\overline{\mathrm{LL}}(x)$.

Let $x\in X/W$, with cyclic support $(\kappa_1,\dots,\kappa_m)$. For $j\in[\![1,m]\!]$, we set $\theta_j:=\frac{\pi}{2}-\arg(\kappa_j)\in\,]0,2\pi]$ (again, we see the argument of κ_j in $[\frac{-3\pi}{2},\frac{\pi}{2}[$), so that $e^{i\theta_j}\kappa_j\in i\mathbb{R}_+$. The sequence $(\theta_1,\dots,\theta_m)$ is the *cyclic argument* of x [Bes15, Definition 11.8].

Assume at first that $0<\theta_1<\theta_2<\ldots<\theta_m<2\pi$, and let $\theta\in\,]0,\frac{2\pi}{h}]$. By definition, the couple (x,θ) is a circular tunnel if and only if θ is different from all the $\frac{\theta_j}{h}$ for $j\in[\![1,m]\!]$. For $j\in[\![0,m]\!]$, Lemma 8.1.14 (Hurwitz rule) gives that the class of the circular tunnel (x,θ) does not depend on θ such that $\frac{\theta_j}{h}<\theta<\frac{\theta_{j+1}}{h}$ (with the convention that $\theta_0=0$ and $\theta_{m+1}=2\pi$). We denote this circular tunnel by p_j. The Hurwitz rule again gives that $p_0=1$ and that p_m is the element Δ of Corollary 8.1.15. The *cyclic label* $\mathrm{clbl}(x)$ of x is then defined as the sequence $(s_1,\dots,s_m)$ where $s_1=p_1$ and $s_i=(s_1\cdots s_{i-1})^{-1}p_i$ for $i\in[\![2,m]\!]$ [Bes15, Definition 11.9]. Note that, by construction, the cyclic label $\mathrm{clbl}(x)$ is composed of simple elements in $B(W)$.

Now, if the above condition on the cyclic argument is not satisfied, or if $x\in\mathcal{H}$, we will define the cyclic label of x as that of a desingularization of x. In order to show that such a definition is valid, we need to show that it doesn't depend on the desingularization we choose.

Let $x\in V/W$, with cyclic support $(\kappa_1,\dots,\kappa_m)$. For a family of paths $\gamma_1,\dots,\gamma_m:[0,1]\to\mathbb{C}$, we define a path $\gamma:[0,1]\to E_n$ by setting $\gamma(t)$ as the multiset $\{\!\{\gamma_1(t),\dots,\gamma_m(t)\}\!\}$, where the multiplicity of $\gamma_j(t)$ is the same as that of κ_j in $\overline{\mathrm{LL}}(x)$. We consider the following conditions on the family $(\gamma_1,\dots,\gamma_m)$:

- For all $j\in[\![1,m]\!]$, we have $\gamma_j(t)=\kappa_j$ (in particular, $\gamma(1)=\overline{\mathrm{LL}}(x)$).
- For all $t\in[0,1]$, we have $\gamma_i(t)\neq\gamma_j(t)$ for $i\neq j$.
- For $t<1$, we have $\gamma(t)\in E_n^\circ$.
- For $t\in[0,1]$, the cyclic support of $\gamma(t)$ is $(\gamma_1(t),\dots,\gamma_m(t))$ (that is, the ordering of the cyclic support is preserved along γ), and the cyclic argument $(\theta_1(t),\dots,\theta_m(t))$ of $\gamma(t)$ respects $0<\theta_1(t)<\ldots<\theta_m(t)<2\pi$.

The first two conditions ensure in particular that the path γ has a unique lift in V/W terminating at x (by Proposition 8.1.10). If all these conditions are met, the lift of γ in V/W which terminates at x is called a *desingularization path* for x. One easily sees that every point of V/W admits a desingularization path. For instance, if $\theta_m=2\pi$, then a path of the form $t\mapsto e^{i\varepsilon(1-t)}x$ for $\varepsilon>0$ small enough can give a desingularization path. If $\theta_j=\theta_{j+1}$, for some $j\in[\![1,m]\!]$ then we can replace κ_i by $\gamma_j(t)=e^{i\varepsilon(1-t)}\kappa_j$ for $\varepsilon>0$ small enough. And, if $x\in\mathcal{H}$ (i.e. if $\kappa_m=0$), then we can replace κ_m by a path of the form $\gamma_m(t)=e^{i\varepsilon(1-t)}i\varepsilon(1-t)$ for $\varepsilon>0$ small enough. Note that, if $x \in X/W$ respects the previous condition on the cyclic argument, then the constant path equal to x is a desingularization path for x.

Let γ_1 and γ_2 be two desingularization paths for x. We need to show that $\mathrm{clbl}(\gamma_1(t))$ and $\mathrm{clbl}(\gamma_2(t))$ are equal for t big enough. This is ensured by the following lemma, which actually establishes the stronger result that $\mathrm{clbl}(\gamma_1(t))$ is constant.

Lemma 8.1.16. *Let $x \in V/W$, and let $\gamma : [0,1] \to V/W$ be a desingularization path for x. For all $t, t' < 1$, we have $\mathrm{clbl}(\gamma(t)) = \mathrm{clbl}(\gamma(t'))$. Furthermore, if γ_1, γ_2 are two desingularization paths for x, then we have $\mathrm{clbl}(\gamma_1(t)) = \mathrm{clbl}(\gamma_2(t))$ for all $t < 1$.*

Proof. Let $\gamma : [0,1] \to V/W$ be a desingularization path for x, and let $(\theta_1(t), \ldots, \theta_m(t))$ be the cyclic argument of $\gamma(t)$. By definition of a desingularization path, $\theta_j : [0,1] \to]0, 2\pi]$ is a continuous path for $j \in [\![1, m]\!]$, and we have $0 < \theta_1(t) < \ldots < \theta_m(t) < 2\pi$. For $j \in [\![1, m]\!]$ and $t < 1$, the circular tunnel $(\gamma(t), \frac{\theta_j(t)+\theta_{j+1}(t)}{2})$ always represent the same element of $B(W)$ by the Hurwitz rule (with the convention that $\theta_0 = 0$ and $\theta_{m+1} = 2\pi$). Since this element is the product of the first j terms of $\mathrm{clbl}(\gamma(t))$, we have the first result.

Now, let γ_1, γ_2 be two desingularization paths for x. For $t < 1$, let us write the cyclic argument $(\theta_1^i(t), \ldots, \theta_m^i(t))$ of $\gamma_i(t)$ for $i = 1, 2$. By assumption, we have $0 < \theta_1^i(t) < \ldots < \theta_m^i(t) < 2\pi$ for $i = 1, 2$. For $j \in [\![1, m]\!]$, and $s \in [0,1]$, we set $\sigma_j(s) := s(\theta_j^1(t)) + (1-s)(\theta_j^2(t))$. By construction, we have $0 < \sigma_1(s) < \ldots < \sigma_m(s) < 2\pi$ for all $s \in [0,1]$. We obtain a path from $\gamma_1(t)$ to $\gamma_2(t)$. Applying the Hurwitz rule to this path gives that $\gamma_1(t)$ and $\gamma_2(t)$ share the same cyclic label. □

Thanks to this lemma, the following definition is valid and doesn't depend on the choice of a desingularization path.

Definition 8.1.17 (Cyclic and outer label). Let $x \in V/W$, and let $\gamma : [0,1] \to V/W$ be a desingularization path for x. The *cyclic label* $\mathrm{clbl}(x)$ of x is defined as $\mathrm{clbl}(\gamma(t))$ for $t < 1$ big enough. If $x \in \mathcal{H}$, then the *outer label* $\mathrm{olbl}(x)$ is defined as $\mathrm{clbl}(x)$ minus its last entry. If $x \in X/W$, then $\mathrm{olbl}(x)$ is defined to be equal to $\mathrm{clbl}(x)$.

Since the product of the terms of the cyclic label in $B(W)$ is always equal to the element Δ of Corollary 8.1.15, it is easy to recover olbl from clbl and vice versa. Note that $\mathrm{olbl}(x)$ is empty if and only if $\mathrm{LL}(x) = \{\!\{0\}\!\}$, that is if and only if $x = 0$ by [Bes15, Lemma 5.6]. We now show that, as we expected, the cyclic label completely describes the fiber of the Lyashko-Looijenga morphism (recall that $\ell : B(W) \to \mathbb{Z}$ is the length function of Lemma 6.2.17).

Proposition 8.1.18 (Trivialization of $\overline{\mathrm{LL}}$). *Let $x, x' \in V/W$. If $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$ and $\mathrm{clbl}(x) = \mathrm{clbl}(x')$, then $x = x'$. Conversely, let $\kappa \in E_n$ with cyclic support $(\kappa_1, \ldots, \kappa_m)$ and let $s := (s_1, \ldots, s_m)$ be a tuple of simple elements of $B(W)$ such that $s_1 \cdots s_m = \Delta$. If the multiplicity of κ_i in κ is equal to $\ell(s_i)$ for all $i \in [\![1, m]\!]$, then there is a unique $x \in V/W$ such that $\overline{\mathrm{LL}}(x) = \kappa$ and $\mathrm{clbl}(x) = s$.*

Proof. Assume that $x, x' \in V/W$ are such that $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$ and $\mathrm{clbl}(x) = \mathrm{clbl}(x')$. Since $x \in \mathcal{H}$ if and only if $0 \in \overline{\mathrm{LL}}(x)$, we have that if either x or x' lie in X/W, then so does the other. In this case, we have $x = x'$ by [Bes15, Proposition 11.13]. We can thus restrict our attention to the case where $x, x' \in \mathcal{H}$. Let $\gamma : [0,1] \to V/W$ be a desingularization path for x. Since $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$, and since the condition of being a desingularization path only depends on $\overline{\mathrm{LL}} \circ \gamma$, the unique lift γ' of $\overline{\mathrm{LL}} \circ \gamma$ in V/W which terminates at x' is a desingularization path for x'. By definition, we have $\mathrm{clbl}(\gamma(t)) = \mathrm{clbl}(x) = \mathrm{clbl}(x') = \mathrm{clbl}(\gamma'(t))$ for t big enough. Since

$\overline{\mathrm{LL}}(\gamma(t)) = \overline{\mathrm{LL}}(\gamma'(t))$ by assumption, the first part of the proof gives that $\gamma(t) = \gamma'(t)$ for $t < 1$ big enough. The respective endpoints of γ and γ', that is, x and x', are then equal.

Let now (κ, s) be a couple, with $\kappa \in E_n$ and s a tuple of simple elements of $B(W)$, such that the cyclic support of κ has the same length as s, and that the multiplicity of κ_i in κ is equal to $\ell(s_i)$ for $i \in [\![1, m]\!]$. If the cyclic argument of κ is strictly increasing and in $]0, 2\pi[$, then in particular, $\kappa \in E_n^\circ$ and [Bes15, Proposition 11.13] gives that there is a unique $x \in X/W$ such that $\overline{\mathrm{LL}}(x) = \kappa$ and $\mathrm{clbl}(x) = s$. Otherwise, let $\gamma_1, \dots, \gamma_m : [0,1] \to \mathbb{C}$ be a family of paths which respects the conditions of a desingularization path for $(\kappa_1, \dots, \kappa_m)$. By [Bes15, Proposition 11.13], there is a unique $x \in X/W$ such that $\overline{\mathrm{LL}}(x) = \gamma(0)$ and $\mathrm{clbl}(x) = s$. By construction, the unique lift $\tilde{\gamma}$ of γ inside V/W starting at x is a desingularization path for $\gamma(1)$. By Lemma 8.1.16, we have $\mathrm{clbl}(\gamma(t)) = s$ for all $t < 1$. By definition, $\gamma(1)$ is then a point of V/W such that $\overline{\mathrm{LL}}(\gamma(1)) = \kappa$ and $\mathrm{clbl}(\gamma(1)) = s$. $\square$

We can reformulate the first statement in the above proposition using outer labels instead of cyclic labels as follows.

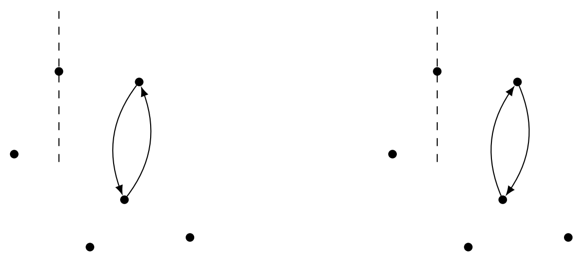
Corollary 8.1.19. *Let $x, x' \in V/W$. If $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$ and $\mathrm{olbl}(x) = \mathrm{olbl}(x')$, then $x = x'$.*

Proof. If $x \in X/W$, then $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$ implies that $x' \in X/W$. In this case, we have $\mathrm{clbl}(x) = \mathrm{olbl}(x) = \mathrm{olbl}(x') = \mathrm{clbl}(x')$ and $x = x'$ by Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$). If $x' \in \mathcal{H}$, then $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x')$ implies that $x' \in \mathcal{H}$. Let $\mathrm{olbl}(x) = (s_1, \dots, s_m)$. Since the cyclic label must be a length-additive decomposition of c, we have

$$\mathrm{clbl}(x) = (s_1, \dots, s_m, (s_1 \cdots s_m)^{-1}\Delta) = \mathrm{clbl}(x'),$$

thus $x = x'$ by Proposition 8.1.18. $\square$

We now investigate how some (lifts of) paths in E_n impact the cyclic label of an element of V/W. Let $x \in X/W$, with cyclic support $(\kappa_1, \dots, \kappa_m)$. Suppose that we swap two consecutive points κ_i and κ_{i+1} of $\overline{\mathrm{LL}}(x)$, there are two natural ways to do so, one going "farther" than the other:



By Proposition 8.1.10 (path lifting), both of these paths in E_n have a unique lift in V/W, that we denote by γ_i^+ and γ_i^-, respectively.

Lemma 8.1.20 (Hurwitz moves). *Let $x \in X/W$, with cyclic label $(s_1, \dots, s_m)$. The cyclic labels of $\gamma_i^+(1)$ and $\gamma_i^-(1)$ are given by*

$$\mathrm{clbl}(\gamma_i^+(1)) = (s_1, \dots, s_{i-1}, s_{i+1}, s_i^{s_{i+1}}, s_{i+2}, \dots, s_m),$$
$$\mathrm{clbl}(\gamma_i^-(1)) = (s_1, \dots, s_{i-1}, {}^{s_i}s_{i+1}, s_i, s_{i+2}, \dots, s_m).$$

Proof. The concatenation of γ_i^- and γ_i^+ gives a homotopically trivial path from x to itself. In particular the assertion about $\mathrm{clbl}(\gamma_i^-(1))$ follows from that on $\mathrm{clbl}(\gamma_i^+(1))$. By construction of the cyclic label, we can assume that the cyclic argument $(\theta_1, \dots, \theta_m)$ of x satisfies $0 < \theta_1 < \dots < \theta_m < 2\pi$. Let then θ be such that $\frac{\theta_{i-1}}{h} < \theta < \frac{\theta_i}{h}$. By [Bes15, Lemma 11.1], one can replace x by $e^{i\theta}x$ and consider $x' := \gamma_1^+(x)$.

Now if θ' is such that $\frac{\theta_2}{h} < \theta' < \frac{\theta_3}{h}$, then the circular tunnels (x, θ') and (x', θ') represent the same element in $B(W)$ by the Hurwitz rule. That is the product of the first two terms of $\mathrm{clbl}(x)$ and $\mathrm{clbl}(x')$ are equal. The Hurwitz rule also shows that the terms of $\mathrm{clbl}(x)$ and $\mathrm{clbl}(x')$ are equal for $i > 2$.

Lastly, the assertion that the first two terms of $\mathrm{clbl}(x')$ are $(s_2, s_1^{s_2})$ follows from [Bes15, Lemma 11.11] and [Bes15, Corollary 6.20]. □

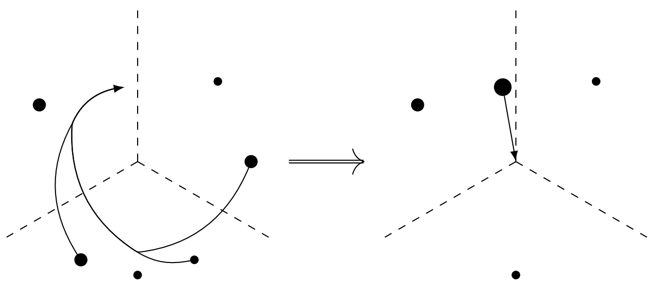
By induction, we see that in general, moving a point of the cyclic support of some $x \in X/W$ may only change the terms of the cyclic label corresponding to points of the cyclic support with lower modulus.

Another result that will be useful to us is the behavior of the cyclic label when one brings points of the cyclic support affinely towards the center.

Lemma 8.1.21. *Let $x \in X/W$ with cyclic support $(\kappa_1, \dots, \kappa_m)$ and cyclic label $(s_1, \dots, s_m)$. We assume that all the κ_i have distinct arguments. Let $I \sqcup J = [\![1, m]\!]$ be a partition of $[\![1, m]\!]$, with $J = \{j_1 < \dots < j_p\}$. Let $\gamma_I : [0, 1] \to E_n$ be the path starting at $\overline{\mathrm{LL}}(x)$ and such that the cyclic support of $\gamma(t)$ is made of the points κ_j for $j \in J$ and the points $(1-t)\kappa_i$ for $i \in I$. There is a unique lift $\widetilde{\gamma}_I$ of γ_I in V/W, and we have*

$$\mathrm{clbl}(\widetilde{\gamma}_I(1)) = (s_{j_1}, \dots, s_{j_p}, (s_{j_1} \cdots s_{j_p})^{-1}\Delta).$$

Proof. First, up to a rotation, we can assume that the last term of the cyclic argument of x is strictly smaller than 2π. The existence and uniqueness of $\widetilde{\gamma}_I$ comes from Proposition 8.1.10 (path lifting). Let γ_1 be the path starting from $\overline{\mathrm{LL}}(x)$ consisting in sliding the points x_i with $i \in I$ together clockwise (and going closer than 0 than all the x_j with $j \in J$) and next to the vertical half-line $i\mathbb{R}_{\geqslant 0}$ (closer than $\kappa_{j_{\max}}$, with $j_{\max} := \max J$). Let then γ_2 be the path starting from $\gamma_1(1)$ consisting in sliding this last point down towards 0.



The path $\widetilde{\gamma}_I$ is homotopic to the unique lift in V/W starting at x of the concatenation $\gamma_1 * \gamma_2$. Since moving points in the cyclic support of some $x \in V/W$ may only affect the terms of the cyclic label corresponding to points of the cyclic support with lower modulus by Lemma 8.1.20, and since the product of the terms of the cyclic label is equal to Δ, we have

$$\mathrm{clbl}(\gamma_1(1)) = (s_{j_1}, \dots, s_{j_p}, (s_{j_1} \cdots s_{j_p})^{-1}\Delta).$$

The path γ_2 is a desingularization path for $\gamma_2(1)$. For all $t < 1$, we have by the Hurwitz rule that $\mathrm{clbl}(\gamma_2(t)) = \mathrm{clbl}(\gamma_2(0)) = \mathrm{clbl}(\gamma_1(1)) = (s_{j_1}, \ldots, s_{j_p}, (s_{j_1} \cdots s_{j_p})^{-1}\Delta)$. By definition of $\mathrm{clbl}(\gamma_2(1))$, we deduce that $\mathrm{clbl}(\gamma_2(1)) = (s_{j_1}, \ldots, s_{j_p}, (s_{j_1} \cdots s_{j_p})^{-1}\Delta)$ as claimed. □

Now, Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$) gives a complete description of the topological pair $(X/W, V/W)$ via the map $(\overline{\mathrm{LL}}, \mathrm{clbl})$.

8.2 The dual braid monoid

Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be a well-generated complex reflection group.

In the last section, we were able to completely describe the orbit space V/W in terms of the Lyashko-Looijenga morphism, along with families of particular elements of $B(W)$, called simple elements. Of course, this terminology is not innocent: the simple elements of $B(W)$ as defined above should generate a Garside submonoid of $B(W)$, making it into a Garside group.

While we could start from these topologically defined simple elements, and prove that they generate a Garside monoid, it is also possible (and possibly more clear) to define a Garside monoid in a purely algebraic way as an interval monoid attached to the group W, and then show that this monoid is the same as the submonoid of $B(W)$ generated by simple elements. We prefer this last approach.

In the first section, we describe elementary properties of Coxeter elements in well-generated complex reflection groups, as these elements play a key role in studying dual braid monoids (we follow mostly [Rip10]). In the second section, we introduce the dual braid monoid as an interval monoid, we show following [Bes15, Section 8] that it is a Garside monoid, and we study elementary properties of its germ of simples. Lastly, we provide in the third section the connection between this algebraic approach and the topological construction of Section 8.1, showing that the dual braid monoid provides a Garside structure on $B(W)$.

8.2.1 Coxeter elements and parabolic Coxeter elements

In this section, we fix a finite dimensional complex vector space V with $n := \dim(V)$, along with a well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We denote by T the set of reflections of W. We otherwise keep the notation of Chapter 6. Like in Section 2.2.2, the generating set T is attached to a function ℓ_T on W, giving the smallest length of a word in T expressing a given element. We call ℓ_T the *reflection length* on W. This function is invariant under conjugacy as the set T is itself invariant under conjugacy (Lemma 2.2.8).

If W is irreducible, then the highest degree h of W is unique by Theorem 6.1.22 (characterization of well-generated groups), and it is a regular number for W by Theorem 6.1.27 (characterization of regular numbers). A ζ_h-regular element of W is called a *Coxeter element*.

Assume now that W is reducible, say with $W = W_1 \times \cdots \times W_r$ with W_i irreducible (and well-generated as W is well-generated). A Coxeter element for W is simply defined as a product $c_1 \cdots c_r$, where c_i is a Coxeter element of W_i. Since any two Coxeter elements in an irreducible well-generated complex reflection group are conjugate by Corollary 6.1.28, we deduce that any two Coxeter elements in an arbitrary well-generated complex reflection group are also conjugate.

Let now $W_0 \subset W$ be a parabolic subgroup. Recall that W_0 is well-generated by Proposition 6.1.23 (parabolic subgroup of a well-generated group). In particular it is eligible to have Coxeter elements of its own, giving rise to the following definition:

Definition 8.2.1 (Parabolic Coxeter element). [Rip10, Proposition 6.3]
An element $w \in W$ is a *parabolic Coxeter element* if there is some parabolic subgroup $W_0 \subset W$ such that $w \in W_0$ is a Coxeter element in W_0.

Let $W_H \subset W$ be a parabolic subgroup of rank 1. If we write $k = |W_H|$, then $W_H \simeq \mathbb{Z}/k\mathbb{Z}$ is generated by a distinguished reflection s of determinant ζ_k. By definition, a Coxeter element in W_H is a ζ_k-regular element, and the only such element is s. In particular, a reflection of W is a parabolic Coxeter element if and only if it is a distinguished reflection.

Parabolic Coxeter elements will provide the basis for simple elements in the dual braid monoid. We give a few general results on parabolic Coxeter elements, proven in [Rip10]. First we have a convenient characterization of Coxeter elements.

Proposition 8.2.2. *[Rip10, Proposition 6.3] An element $w \in W$ is a parabolic Coxeter element if and only if there is some Coxeter element $c \in W$ such that $\ell_T(w) + \ell_T(w^{-1}c) = \ell_T(c)$.*

In other words, we have $w \preccurlyeq_T c$ with the notation of Section 2.2.2. This is also equivalent to saying that there is a reduced T-word expressing c whose first terms make up a reduced T-word expressing w.

Another property of parabolic Coxeter elements is that their length relative to the generating set T admits an easy geometrical description.

Proposition 8.2.3. *[Rip10, Proposition 1.3] The reflection length of a parabolic Coxeter element $w \in W$ is given by $\ell_T(w) = \operatorname{codim}(\operatorname{Ker}(w-1))$.*

Of course, parabolic Coxeter elements are not the only elements which satisfy this relations (for instance, all reflections do). However, it is possible to find elements of W which do not satisfy this relation when W is not a complexified real reflection group.

If $W_0 \subset W$ is a parabolic subgroup, with $T_0 := W_0 \cap T$ as its set of reflections, then we can consider the reflection length ℓ_T in W and ℓ_{T_0} in W_0. These two functions are defined in particular on W_0, and we may ask whether or not they coincide. By the following Lemma, the answer is yes if we consider parabolic Coxeter elements.

Lemma 8.2.4. *[Rip10, Proposition 6.5] Let $W_0 \subset W$ be a parabolic subgroup, and let $T_0 := T \cap W_0$ be the set of reflections of W_0. If $w \in W$ is a parabolic Coxeter element of W lying in W_0, then $\ell_{T_0}(w) = \ell_T(w)$.*

Lastly, we obtain that the correspondence with parabolic Coxeter elements and parabolic subgroups is one-to-one.

Proposition 8.2.5. *[Rip10, Proposition 6.5] Let $w \in W$ be a parabolic Coxeter element, and let $W_0 \subset W$ be the parabolic subgroup of W associated to the flat $\operatorname{Ker}(w-1) \in L(\mathcal{A})$.*

(a) The group W_0 is the unique parabolic subgroup of W of which w is a Coxeter element.

(b) If $r_1 \cdots r_{\ell_T(w)}$ is a T-word of minimal length expressing w, then $\langle r_1, \ldots, r_{\ell_T(w)} \rangle = W_0$.

8.2.2 Algebraic definition

In this section, we fix a finite dimensional complex vector space V with $n := \dim(V)$, along with a well-generated complex reflection group $W \subset \mathrm{GL}(V)$, and a Coxeter element $c \in W$. We denote by T the set of reflections of W. We otherwise keep the notation of Chapter 6.

Definition 8.2.6 (Dual braid monoid). The *dual braid monoid* attached to W, c is the interval monoid $M(c) = M(W, c)$ attached to the data (W, T, c). The enveloping group $G(c) = G(W, c)$ of $M(W, c)$ will be called the *dual group* (attached to W, c).

Theoretically, it would be reasonable to call $G(W, c)$ the "dual braid group", however we prefer the designation of "dual group" to avoid any confusions.

The choice of a Coxeter element in the definition of dual braid monoid is not very consequential by the following lemma:

Lemma 8.2.7. *Let $c, c' \in W$ be two Coxeter elements. The interval germs $(I(c)_T, \cdot)$ and $(I(c')_T, \cdot)$ respectively attached to the data (W, T, c) and (W, T, c'), are isomorphic. In particular, we have $M(W, c) \simeq M(W, c')$ and $G(W, c) \simeq G(W, c')$.*

Proof. The discussion at the beginning of the section proves that there is an element $g \in W$ such that $gcg^{-1} = c'$. Let $w, w' \in W$, since the reflection length is invariant under conjugacy, we have

$$w \preccurlyeq_T w' \Leftrightarrow \ell_R(w) + \ell_R(w^{-1}w') = \ell_R(w') \quad \Leftrightarrow \ell_R(gwg^{-1}) + \ell_T(gw^{-1}g^{-1}gw'g^{-1}) = \ell_T(gw'g^{-1}).$$

In particular, for $w' = c$, we obtain that conjugation by g identifies the sets $I(c)_T$ and $I(c')_T$. Moreover, it induces an isomorphism of posets $(I(c)_T, \preccurlyeq_T) \simeq (I(c')_T, \preccurlyeq_T)$. Since the germ structure on an interval germ depends only on its poset structure, we obtain the result. □

Notation 8.2.8. Using Lemma 8.2.7, we will allow ourselves to abuse the notation and terminology, and to talk about "the" dual braid monoid (resp. dual group) of type W (without referencing a chosen Coxeter element). In particular we may denote $M(W, c)$ by $M(W)$ (resp. $G(W, c)$ by $G(W)$).

In order to show that the dual braid monoid $M(W, c)$ is a Garside monoid, we need to show that the interval $(I(c)_T, \preccurlyeq_T)$ (which is isomorphic to $(I(c)_T, \succcurlyeq_T)$) is a lattice. In the case where W is irreducible and contains only involutive reflections, this is done in [Bes15, Section 8]. If W is irreducible and contains noninvolutive reflections, then we know by [Bes15, Theorem 2.2] that W is isodiscriminantal to a group W' containing only involutive reflections. It is then sufficient to prove that the intervals associated to Coxeter elements of W and W' are isomorphic.

Proposition 8.2.9. *Let W, W' be irreducible well-generated complex reflection groups. Let also $T := \mathrm{Ref}(W)$ (resp. $T' := \mathrm{Ref}(W')$) and let c (resp. c') be a Coxeter element of W (resp. of W'). If W and W' are isodiscriminantal, then the interval germs $(I(c)_T, \cdot)$ and $(I(c')_{T'}, \cdot)$ are isomorphic.*

Proof. Let us consider systems of basic invariants f, f' for W, W' respectively, such that $\Delta(W, f) = \Delta(W', f')$. We saw at the end of Section 6.2.2 that we could change both f, f' in order to have $\Delta(W, f) = \Delta(W', f')$ written as monic polynomials of degree n with no term of degree $n - 1$. The construction of the Lyashko-Looijenga morphism is then the same for both W and W', and the isomorphism $B(W) \simeq B(W')$ induced by isodiscriminantality preserves the simple elements (in the sense of [Bes15, Definition 6.7]).

We denote by M the submonoid of $B(W) \simeq B(W')$ generated by simple elements (in the sense of [Bes15, Definition 6.7]). The set of simple elements in M is endowed with a germ structure, simply given by the product on M restricted to the set of simple elements.

By [Bes15, Proposition 8.5], the projection map $p : B(W) \to W$ (resp. $p' : B(W') \to W'$) induces an isomorphism of germs between $(I(\widetilde{c})_T, \preccurlyeq_T)$ (resp. $(I(\widetilde{c}')_{T'}, \preccurlyeq_{T'})$), for some Coxeter element $\widetilde{c}$ (resp. $\widetilde{c}'$) and this germ of simple elements. Note that even though Bessis only considers groups containing involutive reflections in [Bes15, Section 8], this assumption is not needed in the proof of [Bes15, Proposition 8.5]. Composing these two germ isomorphisms gives the result. □

This proves in particular that $(I(c)_T, \preccurlyeq_T)$ is always a lattice in the case where W is irreducible. Of course this is a bit convoluted as, in proving that we did not need any assumption regarding the order of the reflections of W, we actually showed that groups containing reflections of order > 2 did not give rise to any new monoid. Nonetheless, we are now able to skip any assumption regarding the order of the reflections of W.

The case where W is reducible is easily deduced from the irreducible case thanks to the following lemma:

Lemma 8.2.10. *Let W be written as $W = W_1 \times \cdots \times W_r$, where W_i is an irreducible complex reflection group. For $i \in [\![1, r]\!]$, let $c_i \in W_i$ be a Coxeter element, and let $c := (c_1, \ldots, c_r) \in W$ be a Coxeter element. The interval germ $(I(c)_T, \cdot)$ is isomorphic to the direct product $(I(c_1)_{T_1}, \cdot) \times \cdots \times (I(c_r)_{T_r}, \cdot)$, where $T_i = T \cap W_i$ is the set of reflections of W_i.*

Proof. Let $x = (x_1, \ldots, x_r) \in W = W_1 \times \cdots \times W_r$. By construction of the W_i (see Proposition 6.1.13), the reflections of W are given by the disjoint union of the T_i. In particular, a T-word of minimal length expressing x is just a r-tuples $(m_1, \ldots, m_r)$, where m_i is a T_i-word of minimal length expressing x_i, and we have

$$\ell_T(x) = \sum_{i=1}^{r} \ell_{T_i}(x_i).$$

Let now $y = (y_1, \ldots, y_r) \in W$. By definition, we have

$$x \preccurlyeq_T y \Leftrightarrow \ell_T(x) + \ell_T(x^{-1}y) = \ell_T(y) \Leftrightarrow \sum_{i=1}^{r} \ell_{T_i}(x_i) + \ell_{T_i}(x_i^{-1}y) = \sum_{i=1}^{r} \ell_{T_i}(y_i)$$

In particular, if $x_i \preccurlyeq y_i$ for all $i \in [\![1, i]\!]$, then we have $x \preccurlyeq_T y$ in W. Conversely, assume that $x \preccurlyeq_T y$ in W, since we always have $\ell_{T_i}(x_i) + \ell_{T_i}(x_i^{-1}y) \geqslant \ell_{T_i}(y_i)$, the equality $\ell_T(x) + \ell_T(x^{-1}y) = \ell_T(y)$ forces the equality $\ell_{T_i}(x_i) + \ell_{T_i}(x_i^{-1}y_i) = \ell_{T_i}(y_i)$, and thus $x_i \preccurlyeq_{T_i} y_i$, for all $i \in [\![1, r]\!]$.

In particular, the isomorphism $W \simeq W_1 \times \cdots \times W_k$ induces an isomorphism of posets between $I(c)_T$ and $I(c_1)_{T_1} \times \cdots I(c_k)_{T_k}$. Moreover, since we have a group morphism, this isomorphism actually preserves the germ structure, as it depends only on the group structures of W and of $W_1 \times \cdots \times W_k$. □

Combining this Lemma with Proposition 8.2.9 and [Bes15, Section 8], we obtain the following general result:

Theorem 8.2.11 (Dual group is Garside). *[Bes15, Theorem 8.2]*
Let W be a well-generated complex reflection group with $T := \mathrm{Ref}(W)$, and let $c \in W$ be a Coxeter

element. The interval germ $(I(c)_T, \cdot)$ is a Garside germ, and the dual group $(G(W,c), M(W,c), c)$ is a Garside group.

In the next section, we will show that $G(W,c)$ is isomorphic to the complex braid group $B(W)$, making the latter into a Garside group. For now, $G(W,c)$ is only a Garside group whose germ of simples has interesting properties. By definition, the simple elements of $M(W,c)$ are the elements of the interval $I(c)_T$. In order to avoid confusion, we will use the notation $\mathcal{S}$ to differentiate whether we see $I(c)_T$ as a subset of $M(W,c)$ or as a subset of W.

First, we notice that, just like in Artin-Tits monoids, the powers of a nontrivial simple element in $G(W,c)$ are never simple elements.

Lemma 8.2.12. *Simple elements of $M(W,c)$ are squarefree in the sense that, if $s \in \mathcal{S}$, then $s^2 \in \mathcal{S}$ is also simple if and only if $s = 1 \in M(W,c)$.*

Proof. Let $s \in \mathcal{S}$ be associated with some $\overline{s} \in I(c)_T \subset W$. If $s^2 \in \mathcal{S}$, then $ss = s^2$ is a defining relation in the germ of simples of $M(c)$, which is the germ $(I(c)_T, \cdot)$ by definition. We then have that $\overline{s}^2 \in I(c)_T$, and we must have $2\ell_T(\overline{s}) = \ell_T(\overline{s}^2)$. If $s \neq 1$, we can consider $r \in T_c$ such that $r \preccurlyeq_T \overline{s}$, with $r\overline{s}' = \overline{s}$ and $\overline{s}''r = \overline{s}$. We have $\overline{s}^2 = \overline{s}''rr\overline{s} = \overline{s}''r^2\overline{s}$, so $\ell_T(\overline{s}^2) \leqslant \ell_T(\overline{s}'') + \ell_T(\overline{s}') + 1 = 2\ell_T(\overline{s}) - 1$, which contradicts $2\ell_T(\overline{s}) = \ell_T(\overline{s}^2)$. □

Note the distinction between $\mathcal{S}$ and $I(c)_T$: If $r \in I(c)_T$ is an involution (for instance, a reflection with order 2), then we have $r^2 = 1 \in I(c)_T$ without having $r = 1$. This does not contradict the lemma because the relation $r^2 = 1$ does not hold in $M(W,c)$.

The following proposition gives an easy description of the germ structure on $\mathcal{S}$.

Proposition 8.2.13. *If $s, t \in \mathcal{S}$ are such that $st \in \mathcal{S}$, then $st = s \vee t$ in $\mathcal{S}$ and $s \wedge t$ is trivial.*

Proof. First, note that in $I(c)_T$, we have $st = ts^t = {}^s ts$. Now, because ℓ_T is constant on conjugacy classes, we have

$$\ell_T(st) = \ell_T(s) + \ell_T(t) = \ell_T(t) + \ell_T\left(s^t\right) = \ell_T\left({}^s t\right) + \ell_T(s).$$

This means that $st \succcurlyeq_T s^t$ and ${}^s t \preccurlyeq_T st$. Since $s \cdot t \in Ss$, we have $st \in I(c)_t$ and thus $s^t, {}^s t \in I(c)_T$, and st is a common left and right multiple of s and t. Now let $s \cdot u = t \cdot u'$ be the right-lcm of s and t in $\mathcal{S}$. By definition, there is some $x \in \mathcal{S}$ such that $u \cdot x = t$ and $u' \cdot x = s^t$. In particular, we have $t \succcurlyeq_T x$, $s^t \succcurlyeq_T x$ and $x \preccurlyeq_T s^t$ since every element is balanced. So st admits x^2 as a factor and x^2 is simple. By Lemma 8.2.12, we have $x = 1$, and thus $t = u$, $u' = s^t$ and $st = s \vee t$.

If x is a common divisor of s and t in $\mathcal{S}$, then we have $s \succcurlyeq_T x$ and $x \preccurlyeq_T t$. We get that x^2 is a simple element in $M(c)$, it must then be trivial. □

As a corollary, for $s, t \in I(c)_T$, the product $s \cdot t$ is defined in $I(c)_T$ if and only if $t \preccurlyeq_T s^{-1}c$, in which case it is equal to $s \vee t$. In particular, this implies that the germ structure on $\mathcal{S}$ (i.e. on $I(c)_T$) depends only on its poset structure, and on the complement to c.

Note that, since simple elements of $M(W,c)$ are all balanced since T is stable under conjugacy, we have $s \vee t = s \vee_L t$ and $s \wedge t = s \wedge_R t$ for all $s, t \in \mathcal{S}$. Using these result, we obtain a characterization of parabolic Garside elements in dual braid monoids:

Lemma 8.2.14. *The parabolic Garside elements in $M(W,c)$ are exactly the simple elements.*

Proof. By definition, parabolic Garside elements are simple elements, thus we only have to prove that all simple elements are parabolic Garside elements. Furthermore, since T is stable under conjugacy, we have that every simple element of $M(W,c)$ is balanced. Thus, we only have to show that, for $s,t,u \in \mathcal{S}$, if $s,t \preccurlyeq u$ and if $st \in \mathcal{S}$, then $st \preccurlyeq u$. However, if st is simple, then it is equal to $s \vee t$ by Proposition 8.2.13, and thus $s \vee t = st \preccurlyeq u$ is obvious. □

Corollary 8.2.15. *Let $x \in M(W,c)$, be written as a (non necessarily greedy) product of simples $x = s_1 \cdots s_r$. The standard parabolic closure of x in $G(W,c)$ is $G(W,c)_s$, where s is the left-lcm of all the s_i.*

Proof. Let $t \in \mathcal{S}$ be a parabolic Garside element such that x belongs to the standard parabolic subgroup $G(W,c)_t$. By Proposition 5.1.10, all the left- or right-divisors of x lie in $G(W,c)_t$. In particular, we have $s_i \in M(W,c)_t$ for all $i \in [\![1,r]\!]$, and thus $s \in M(W,c)_T$, whence $s \preccurlyeq t$. Conversely, since s is a parabolic Garside element by Lemma 8.2.14, we have that $G(W,c)_s$ is a standard parabolic subgroup of $G(W,c)$ containing x, and thus $s \preccurlyeq t$. We then have $s = t$ and $G(W,c)_s = G(W,c)_t$ is the standard parabolic closure of x. □

We finish this section by relating the standard parabolic submonoids of $M(W,c)$ with the dual braid monoids attached to parabolic subgroups of W. This will be obtained by considering parabolic Coxeter elements:

If $c_0 \in W$ is a parabolic Coxeter element, say for a parabolic subgroup $W_0 \subset W$, then we can consider the dual braid monoid $M(W_0,c_0)$. If $c_0 \preccurlyeq_T c$, then c_0 seen as a simple element of $M(W,c)$ is a parabolic Garside element by Lemma 8.2.14. We then want to show that the dual braid monoid $M(W_0,c_0)$ is isomorphic to the standard parabolic submonoid of $M(W,c)$ induced by c_0.

By Proposition 5.1.5 (standard parabolic subcategories are Garside categories), the lattice of simple elements of the standard parabolic submonoid $M(W,c)_{c_0}$ associated to c_0 is given by $(\{s \in I(c)_T \mid s \preccurlyeq_T c_0\}, \preccurlyeq_T)$, and its germ structure is simply restricted from that of $I(c)_T$. The following proposition gives that this germ is the same as the interval germ defining the dual braid monoid of W_0.

Proposition 8.2.16. *Let $c_0 \in I(c)_T$. Let also W_0 be the parabolic subgroup of W of which c_0 is a Coxeter element.*

(a) The embedding $W_0 \to W$ induces a bijection between the set $(I(c_0)_{T_0}, \preccurlyeq_{T_0})$ and $(\{s \in I(c)_T \mid s \preccurlyeq c_0\}, \preccurlyeq_T)$.

(b) This bijection is an isomorphism between the associated germs, which induces an isomorphism between the dual braid monoid $M(c_0)$ associated to W_0 is isomorphic to the standard parabolic submonoid $M(W,c)_{c_0}$ of $M(W,c)$ associated to c_0.

Proof. Note that the group W_0 is unique by Proposition 8.2.5.

Statement (a) is [GM22, Proposition 3.12]. For (b), first, the bijection is an isomorphism of posets since, on $I(c_0)_{T_0}$, ℓ_T and ℓ_{T_0} coincide by Lemma 8.2.4. As we already said, since T and T_0 are invariant under conjugacy (in their respective groups), the germs structures on $I(c_0)_{T_0}$ and $\{s \in I(c)_T \mid s \preccurlyeq c_0\}$ are entirely determined by their poset structures, whence the result. □

In the case of spherical Artin groups, standard parabolic subgroups are again Artin groups by construction. The above proposition can be seen as an analogue of this situation for dual braid monoids.

Since all Coxeter elements in W are conjugate, and since $\preccurlyeq_T$ is invariant under conjugacy, Proposition 8.2.2 implies that parabolic Coxeter elements are exactly the conjugates in W of the elements of $I(c)_T$. Since all parabolic subgroups of W admit Coxeter elements, we obtain that every parabolic subgroup of W is conjugate to a parabolic subgroup of W containing a parabolic Coxeter element which belongs to $I(c)_T$ with $c \in W$ fixed. In particular, the standard parabolic subgroupoids of $G(W, c)$ contain all the dual groups whose type is that of a parabolic subgroup of W. In the next section, we will relate the (algebraically defined) parabolic subgroups of $G(W, c)$ with the (topologically defined) parabolic subgroups of $B(W)$ (see Proposition 8.2.24).

8.2.3 Topological interpretation

In this section, we fix a finite dimensional complex vector space V with $n := \dim(V)$, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We denote by h the highest degree of W, and by T the set of reflections of W. We otherwise keep the notation of Chapter 6. We also fix a system f of basic invariants for W such that $\Delta(W, f)$ has the form 8.1.1. As in the last section, we consider the (extended) Lyashko-Looijenga morphism $\overline{\mathrm{LL}} : V/W \to E_n$, along with the associated notions of circular tunnels, labels...

In order to prove Proposition 8.2.9, we used a correspondence between the dual braid monoid and the braid group $B(W)$ obtained in [Bes15, Section 8]. This correspondence uses the notion of tunnel introduced in [Bes15, Section 6], along with the associated notion of label. However, in our study, we prefer working with circular tunnels and the associated cyclic label. By [Bes15, Lemma 11.10] and [Bes15, Corollary 6.18], an element of $B(W)$ is represented by a circular tunnel (i.e. is a simple element) if and only if it is represented by a tunnel in the sense of [Bes15, Definition 6.7]. Thus we obtain the following result:

Theorem 8.2.17. *[Bes15, Proposition 8.5 and Lemma 11.10]*
Let V be a finite dimensional complex vector space with $n := \dim(V)$. Let also $W \subset \mathrm{GL}(V)$ be a well-generated irreducible complex reflection group, with $T := \mathrm{Ref}(W)$, and let f be a system of basic invariants for W such that $\Delta(W, f)$ has the form 8.1.1. Consider the associated Lyashko-Looijenga morphism $\overline{\mathrm{LL}}$, along with the set

$$\mathcal{U} := \{x \in V/W \mid \overline{\mathrm{LL}}(x) \cap i\mathbb{R}_{\geqslant 0} = \varnothing\}.$$

Consider also the homotopy class Δ of the circular tunnel $(x, \frac{2\pi}{h})$ in $B(W) := \pi_1(X/W, \mathcal{U})$ (for any $x \in \mathcal{U}$).
The projection map $B(W) \to W$ sends Δ to a Coxeter element c of W. It induces a one-to-one correspondence between simple elements of $B(W)$ and the interval $I(c)_T$, which extends to an isomorphism between $B(W)$ and the dual group $G(c, W)$.

Using this proposition, we can identify the element $\Delta \in B(W)$ with the Garside element c in the dual group $G(W, c)$. Thus we will sometimes denote this Garside element by Δ (also this notation is more coherent with the first part of the thesis).

Corollary 8.2.18. *Let V be a finite dimensional complex vector space, and let $W \subset \mathrm{GL}(V)$ be a well-generated complex reflection group. Let also $c \in W$ be a Coxeter element, and let $x \in X$. There is an isomorphism $G(W, c) \to B(W) = \pi_1(X/W, W.x)$ which sends the atoms of $M(W, c)$*

to distinguished braided reflections of $B(W)$, and such that the following triangle commutes:

$$\begin{tikzcd} B(W) \arrow[r, two heads] & W \\ G(W,c) \arrow[u, "\simeq"] \arrow[ur, two heads] & \end{tikzcd}$$

where $B(W) \twoheadrightarrow W$ and $G(W,c) \twoheadrightarrow W$ are the natural projection maps.

Proof. We first show the result in the case where W is well-generated. We use the notation of Theorem 8.2.17. Let f be a system of basic invariants for W such that $\Delta(W,f)$ has the form 8.1.1. We consider the category of groups over W: objects are group morphisms $G \to W$, and morphisms $(G \to W) \to (G' \to W)$ are group morphisms from G to G' making the obvious triangle commute. The statement in the theorem is simply that there is an isomorphism of groups over W between $G(W,c)$ and $B(W)$, and that this isomorphism sends the atoms to distinguished braided reflections.

Let $W.u \in \mathcal{U}$, and let $\gamma : x \to u$ be a path in X. The change of basepoint induced by $W.\gamma$ gives an isomorphism of groups over W between $B(W)$ and $\pi_1(X/W,\mathcal{U})$, which preserves the set of braided reflections. Let then $\Delta \in \pi_1(X/W,\mathcal{U})$ defined in Corollary 8.1.15, and let c' be the image of Δ in W. By Theorem 8.2.17, c' is a Coxeter element in W, and we have an isomorphism of groups over W between $\pi_1(X/W,\mathcal{U})$ and $G(W,c')$, such that the atoms of $M(W,c')$ are sent to distinguished braided reflections. Lastly, the isomorphism $G(W,c') \to G(W,c)$ of Lemma 8.2.7 is an isomorphism of groups over W, which preserves the atoms. Composing those various isomorphisms gives the following diagram:

$$\begin{tikzcd} B(W) \arrow[r, "\simeq"] \arrow[drrr, two heads] & \pi_1(X/W,\mathcal{U}) \arrow[r, "\simeq"] & G(W,c') \arrow[r, "\simeq"] & G(W,c) \arrow[d, two heads] \\ & & & W \end{tikzcd}$$

which gives the desired result.

Now, we no longer assume that W is irreducible. Let V be endowed with a W-invariant hermitian scalar product. By Lemma 6.2.26, we can assume that W is essential. Let V be written as an orthogonal sum $V = V_1 \perp \cdots \perp V_k$, and let W be written as a product $W_1 \times \cdots \times W_k$ such that W_i acts as an irreducible complex reflection group on V_i. If $x = (x_1, \ldots, x_k)$, then we have $B(W) \simeq B(W_1) \times \cdots \times B(W_k)$, where $B(W_i) = \pi_1(X_i/W_i, W_i.x_i)$ for $i \in [\![1,k]\!]$. Moreover we have a commutative square:

$$\begin{tikzcd} B(W_1) \times \cdots \times B(W_k) \arrow[r, "\simeq"] \arrow[d, two heads] & B(W) \arrow[d, two heads] \\ W_1 \times \cdots \times W_k \arrow[r, "\simeq"'] & W \end{tikzcd}$$

Each W_i is well-generated, and we can apply the first part of the proof: let $c \in W$ be a Coxeter element. By definition, we can write $c = c_1 \cdots c_k$, where c_i is a Coxeter element in W_i for $i \in [\![1,k]\!]$. We have an isomorphism of groups over W_i between $G(W_i,c_i)$ and $B(W_i)$ which sends atoms to distinguished braided reflections. Taking the products of these isomorphisms yields an isomorphism $G(W_1,c_1) \times \cdots \times G(W_k,c_k) \to B(W_1) \times \cdots \times B(W_k)$ of groups over

$W_1 \times \cdots \times W_k$. Applying the above commutative square, we obtain an isomorphism of groups $G(W_1, c_1) \times \cdots \times G(W_k, c_k) \to B(W)$ over W which sends the atoms to braided reflections.

By Lemma 8.2.10, the interval germ $(I(c)_T, \cdot)$ is naturally isomorphic to the direct product of the interval germs $(I(c_i)_{T\cap W_i}, \cdot)$. We then obtain an isomorphism of groups over W between $G(W, c)$ and the product $G(W_1, c_1) \times \cdots \times G(W_k, c_k)$, which gives the desired result. □

Note that, in the notation of Corollary 8.2.18, the isomorphism $G(W, c) \simeq B(W)$ we constructed in the proof of Corollary 8.2.18 depends on several choices. However, making different choices yields another isomorphism $G(W, c) \simeq B(W)$, which differs from the first by an inner automorphism of $B(W)$.

At the end of Section 6.2.5, among all complex braid groups, only $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{31}), B(G_{33})$ and $B(G_{34})$ were not yet endowed with a weak Garside group structure. Since the associated complex reflection groups are well-generated, we have the following corollary of Theorem 8.2.17:

Corollary 8.2.19. *[Bes15, Theorem 0.4] The complex braid groups $B(G_{24}), B(G_{27}), B(G_{29}), B(G_{33}), B(G_{34})$ are Garside groups.*

Thus, at this stage, only $B(G_{31})$ has not been endowed with a weak Garside group structure. This will be done in Section 9.1.2 using Springer groupoids.

A classical result on dual braid monoids is that they admit a presentation with so-called Hurwitz relations (or dual braid relations). As we will later generalize this result to Springer categories, we recall the classical case here. Let $c \in W$ be a Coxeter element. By construction, the atoms of $M(W, c)$ are given by $T_c := T \cap I(c)_T$. Note that this set equals T exactly when W is a complexified real reflection group (see discussions before [Bes15, Lemma 8.8]).

Proposition-Definition 8.2.20 (Hurwitz relations). *[Bes15, Definition 8.7 and Lemma 8.8] Let $c \in W$ be a Coxeter element. The* Hurwitz relations *with respect to W and c are all the formal relations in $M(W, c)$ of the form*

$$rr' = r'r'',$$

where $r, r', r'' \in T$ are such that $r \neq r'$, $rr' \in I(c)_T$, and the relation $rr' = r'r''$ holds in W. The monoid $M(W, c)$ is presented by $M(W, c) \simeq \langle T_c \mid \text{Hurwitz relations}\rangle^+$.

In particular, combining this with Theorem 8.2.17 yields a presentation of $B(W)$, which we can then manipulate with Tietze transformations in order to obtain more convenient presentations of complex braid groups.

Remark 8.2.21. We can be a little less specific and say that $M(W, c)$ (and $G(W, c)$) is presented by relations of the form $ss' = rr'$ with $s, s', r, r' \in T_c$, and $ss' = rr' \in I(c)_T$. That is $M(W, c)$ is presented by its atoms and the equality between decompositions of simple elements of length 2. It is this precise rephrasing which we will show holds in Springer categories (see Theorem 9.2.18).

Another important result, due to Bessis, is that the Garside structure of the dual braid monoid allows us to describe the center of braid groups of irreducible well-generated complex reflection groups.

Theorem 8.2.22 (Center). *[Bes15, Theorem 12.3]*
Let W be a well-generated irreducible complex reflection group with highest degree h, and let

$c \in W$ be a Coxeter element. The center of $B(W) \simeq G(W,c)$ is cyclic and generated by $\Delta^{h'}$, where $h' := h/Z(W)$.

An immediate corollary of this result is the computation of the center of non irreducible well-generated complex reflection groups using the dual braid monoid.

Corollary 8.2.23. *Let $W = W_1 \times \cdots \times W_r$ be a direct product of irreducible well-generated complex reflection groups, with respective highest degree $h_1, \ldots, h_r$. Let also $h_i' := h_i/Z(W_i)$ for $i \in [\![1,r]\!]$. Let $c_i \in W_i$ be a Coxeter element for $i \in [\![1,r]\!]$, so that $c = (c_1, \ldots, c_r)$ is a Coxeter element in W. The smallest central power of $\Delta \in G(W,c) = G(W_1,c_1) \times \cdots \times G(W_r,c_r)$ is $c^{h_1' \vee \cdots \vee h_r'}$.*

Note that, if $r > 1$ in the above corollary, then the center of $B(W)$ is not cyclic. In fact it is generated by the $c_i^{h_i'}$ for $i \in [\![1,r]\!]$.

Finally the identification between the braid group W and the dual group $G(W)$ identifies the "topological" parabolic subgroups of $B(W)$ (according to Definition 7.1.5) with the "algebraic" parabolic subgroups of $G(W)$ (according to Definition 5.1.20).

Proposition 8.2.24. *[GM22, Proposition 3.14] The isomorphism $B(W) \simeq G(W,c)$ of Theorem 8.2.17 identifies the parabolic subgroups of $B(W)$ with the parabolic subgroups of $G(W,c)$.*

Remark 8.2.25. Actually, as the dual braid monoid is a particular case of the Springer categories we will study in Section 9.2, this result is also a particular case of a more general result (Theorem 9.2.42).

8.3 Noncrossing partitions lattices

In the case where W is a well-generated complex reflection group of the form $G(de,e,n)$ for some integers d,e,n, the lattice of simple elements of the dual group of type W can be understood as a set of noncrossing partitions of some part of the complex plane. This point of view will offer a very convenient visualization tool in Section 8.4, when we prove that dual groups are support-preserving. In this section, we explain how to construct the various sets of noncrossing partitions we are interested in for the study of well-generated complex braid groups. This is essentially taken from [BC06, Section 1] and [DDGKM, Section IX.2.2].

Recall from [BC06, Section 1] and [DDGKM, Section IX.2.2] that a (set-theoretic) partition of a set X is an unordered family $P \subset \mathcal{P}(X) \setminus \{\varnothing\}$ such that any two parts (i.e. elements) of P are disjoint and $\bigcup P = \bigsqcup_{u \in P} u = X$. The set of partitions of X will be denoted by $\mathrm{P}(X)$. Note that taking a partition P of X is equivalent to considering an equivalence relation $\equiv$ on X (where the equivalence classes for $\equiv$ are the parts of P).

The set $\mathrm{P}(X)$ is ordered with the refining order [BC06, Definition 1.1], defined by

$$P \preccurlyeq Q \Leftrightarrow \forall u \in P, \exists v \in Q \mid u \subset v.$$

The poset $(\mathrm{P}(X), \preccurlyeq)$ is a lattice. For P,Q two partitions of X, the meet $P \wedge Q$ is the partitions whose parts are the nonempty intersections of a part of P with a part of Q. The join $P \vee Q$ corresponds to the smallest equivalence relation whose graph contains the graphs of P and Q (seen as equivalence relations). We denote by 1 the discrete partitions $\{\{x\} \mid x \in X\}$, and by Δ the partition $\{X\}$.

Assume that X is finite, or equivalently, that $\mathrm{P}(X)$ is a finite lattice. If $N \subset \mathrm{P}(X)$ is a subset which is stable under $\wedge$ and which contains Δ, then $(N, \preccurlyeq)$ is a lattice [BC06, Lemma 1.2]. The meet of two elements of N is simply their meet in $\mathrm{P}(X)$, and their join is the meet of all their upper bounds lying in N.

If $X \subset \mathbb{C}$ is a finite subset, then for some $u \in P \in \mathrm{P}(X)$, we can consider the convex hull $\mathrm{Conv}(u)$ of u in the complex plane. We can then state the following definition:

Definition 8.3.1 (Noncrossing partition). Let $X \subset \mathbb{C}$ be a finite set. A partition $P \in \mathrm{P}(X)$ is *noncrossing* if we have

$$\forall u, u' \in P,\ \mathrm{Conv}(u) \cap \mathrm{Conv}(u') \neq \varnothing \Rightarrow u = u'.$$

We denote by $\mathrm{NCP}(X)$ the set of noncrossing partitions of X.

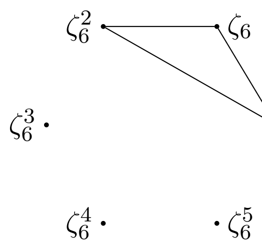
The trivial partition Δ is always noncrossing, as well as the discrete partition 1. Since the meet of two noncrossing partitions is clearly noncrossing, we obtain that $\mathrm{NCP}(X)$ is always a lattice for the refinement order [BC06, Lemma 1.3].

Example 8.3.2. Consider the set $\mu_4 = \{1, i, -1, -i\} \subset \mathbb{C}$. The partitions $r_1 := \{\{1, -1\}, \{i\}, \{-i\}\}$ and $r_2 := \{\{1\}, \{i, -i\}, \{-1\}\}$ are both noncrossing. Their meet 1 is noncrossing, but their join in $\mathrm{P}(X)$ is $\{\{\pm 1\}, \{\pm i\}\}$ which does not lie in $\mathrm{NCP}(\mu_4)$. The join of r_1 and r_2 in $\mathrm{NCP}(\mu_4)$ is actually the partition Δ.

8.3.1 Noncrossing partitions of type $(1, 1, n)$

In this section, we fix $n \geqslant 2$ an integer. We identify the set $[\![0, n-1]\!]$ with the cyclic group $\mathbb{Z}/n\mathbb{Z}$. We also consider the symmetric group $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z}) \simeq \mathfrak{S}_n$, endowed with the generating set T made of the transpositions, along with the n-cycle $c := (0\ 1\ \cdots\ n-1)$.

We denote by $\mathrm{NCP}(1, 1, n)$ the noncrossing partitions of μ_n (i.e. the vertices of a regular n-gon). For $s \in \mathrm{NCP}(1, 1, n)$, we can draw s by drawing the convex hulls of the nonsingleton parts of P. For instance, the partition $s := \{\{1, \zeta_6, \zeta_6^2\}, \{\zeta_6^3\}, \{\zeta_6^4\}, \{\zeta_6^5\}\}$ of μ_6 is drawn as



In order to alleviate notations, we can replace μ_n by the set $\mathbb{Z}/n\mathbb{Z}$ using the usual bijection $i \mapsto \zeta_n^i$. Under this identification, we can identify $\mathrm{NCP}(1, 1, n)$ with the "noncrossing" partitions of $\mathbb{Z}/n\mathbb{Z}$. The partition s considered above is then simply $\{\{0, 1, 2\}, \{3\}, \{4\}, \{5\}\}$.

The set $\mu_n \subset \mathbb{C}$ is endowed with a (counterclockwise) *cyclic ordering*, given by the set of triples (a, b, c) (denoted $a \lhd b \lhd c$) such that the triangle abc is oriented counterclockwise. If X is a finite set, choosing a bijection $X \simeq \mu_n$ (where $n = |X|$) naturally endows X with a cyclic ordering. In particular, the bijection $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$ considered above endows the former with a cyclic ordering such that, for instance, $1 \lhd 2 \lhd n-1$ and $n-1 \lhd 1 \lhd 2$.

The connection between the noncrossing partitions of μ_n and the simple elements of the dual braid monoid is given by the following result:

Proposition 8.3.3. *[DDGKM, Proposition IX.2.7] The map sending a permutation $\sigma \in \mathfrak{S}(\mathbb{Z}/n\mathbb{Z})$ to its set of orbits (acting on $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$) induces an isomorphism of lattices between the interval $I(c)_T$ and the lattice* $\mathrm{NCP}(1,1,n)$.

We have an isomorphism $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z}) \simeq \mathfrak{S}_n \simeq G(1,1,n)$, which identifies the transpositions with the reflections, and which sends c to the element $c(1,1,n)$, given by the matrix

$$c(1,1,n) = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

The element $c(1,1,n)$ is a Coxeter element in $G(1,1,n)$ since it admits (for instance) $(\zeta_n^{n-1}, \zeta_n^{n-2}, \ldots, 1)$ as a ζ_n-eigenvector. Thus the interval $I(c)_T$ of the above proposition is indeed the set of simple of the dual group of type $G(1,1,n)$. Under the isomorphism $I(c)_T \simeq \mathrm{NCP}(1,1,n)$, a cycle $(i_1\ i_2 \cdots i_k) \in \mathfrak{S}(\mathbb{Z}/n\mathbb{Z}) \simeq G(1,1,n)$ with $k \geqslant 3$ is simple if and only if $i_1 \lhd i_2 \lhd \ldots \lhd i_k$.

Since $G(1,1,n)$ is a complexified real reflection group, the reflection length of an element $w \in G(1,1,n)$ is always given by $\ell_T(w) = \mathrm{codim}(\mathrm{Ker}(w-1))$ by [DDGKM, Lemma IX.2.19]. The following standard lemma gives a description of the reflection length which is intrinsic to $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z})$ and does not use the identification with $G(1,1,n)$.

Lemma 8.3.4. *Let $w = c_1 \cdots c_k$ be a product of disjoint cycles in $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z})$. We have*

$$\ell_T(w) = \sum_{i=1}^{k} \ell_T(c_i) = \sum_{i=1}^{k} (\ell(c_i) - 1),$$

where $\ell(c_i)$ denotes the size of the support of c_i (i.e. the length of c_i as a cycle) for $i \in [\![1,k]\!]$.

Using Proposition 8.3.3, we can easily deduce the set of atoms in $\mathrm{NCP}(1,1,n)$.

Lemma 8.3.5 (Atoms of $\mathrm{NCP}(1,1,n)$). *For $x \neq y \in \mathbb{Z}/n\mathbb{Z}$, we denote by $r_{x,y}$ the noncrossing partition whose only nonsingleton part is $\{x,y\}$. The atoms of* $\mathrm{NCP}(1,1,n)$ *are exactly the $r_{x,y}$ for $x \neq y \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. We use the notation of Proposition 8.3.3. The isomorphism between $I(c)_T$ and $\mathrm{NCP}(1,1,n)$ sends atoms to atoms. By construction, the atoms of $I(c)_T$ are exactly the elements of T (i.e. the transpositions) which lie in $I(c)_T$. For $x \neq y \in \mathbb{Z}/n\mathbb{Z}$, the only permutation $\sigma \in \mathfrak{S}(\mathbb{Z}/n\mathbb{Z})$ whose set of orbits is $r_{x,y}$ is the transposition $(x\ y)$, thus the transposition $(x\ y)$ must lie in $I(c)_T$ since its image $r_{x,y}$ lies in $\mathrm{NCP}(1,1,n)$. The atoms of $I(c)_T$ are thus exactly the transpositions, and the result follows. $\square$

Note that we do not need the identification between $\mathrm{NCP}(1,1,n)$ and $I(c)_T$ in order to define atoms of $\mathrm{NCP}(1,1,n)$. Indeed, an *atom* in a lattice can be defined as an element s distinct from the minimum m and such that $t \leqslant s$ implies $t \in \{m, s\}$.

By Theorem 8.2.11 (dual group is Garside), the poset $I(c)_T$ is the germ of simples of a Garside monoid, and the Garside automorphism on $I(c)_T$ is induced by conjugacy by the Coxeter element c. If $\sigma = (i_1\ i_2\ \cdots\ i_k)$ is a cycle such that $i_1 \lhd i_2 \lhd \ldots \lhd i_k$, then $\sigma^c = (i_1 - 1\ i_2 - 1\ \cdots\ i_k - 1)$. Thus the automorphism induced on $\mathrm{NCP}(1,1,n)$ is equal to the automorphism on $\mathrm{NCP}(1,1,n)$ induced by a rotation of angle $\frac{-2\pi}{n}$ on the set μ_n.

The lattice $I(c)_T$ naturally comes equipped with a right-complement operation $\backslash$ (see Section 2.1.4). This operation coincides with the left-complement operation since every element of the germ $I(c)_T$ is balanced, and we can simply talk about the complement operation. If $s, t \in I(c)_T$ are such that $s \preccurlyeq t$ in $I(c)_T$, then the complement is given by $s\backslash t = s^{-1}t \in I(c)_T$. The element of $\mathrm{NCP}(1,1,n)$ associated to $s^{-1}t$ can be defined only using the combinatorics of $\mathrm{NCP}(1,1,n)$ (without resorting to the identification with $I(c)_T$). This construction dates back to the so-called Kreweras complement introduced in [Kre72].

Definition 8.3.6 (Complement on $\mathrm{NCP}(1,1,n)$). Let $s \in \mathrm{NCP}(1,1,n)$.

- If s contains only one nonsingleton part $\{i_1, \dots, i_k\}$ with $i_1 \lhd i_2 \lhd \dots \lhd i_k$, then we define $\overline{s}$ as the partition whose nonsingleton parts are $\{z \in \mathbb{Z}/n\mathbb{Z} \mid i_j \unlhd z \lhd i_{j+1}\}$ for $j = 1, \dots, k$
- If s contains several nonsingleton parts $X_1, \dots, X_k$. For $i \in [\![1, k]\!]$, we can consider the partition s_i whose only nonsingleton part is X_i, and we have $s = s_1 \vee s_2 \vee \dots \vee s_k$. We define
$$\overline{s} = (s_1\backslash\Delta) \wedge (s_2\backslash\Delta) \wedge \dots \wedge (s_k\backslash\Delta).$$

If $s, t \in \mathrm{NCP}(1,1,n)$ are such that $s \preccurlyeq t$, then the *complement* of s in t is defined as $s\backslash t := t \wedge (s\backslash\Delta)$.

Note in particular that, if $s \in \mathrm{NCP}(1,1,n)$, then we have $s\backslash\Delta = \Delta \wedge (\overline{s}) = \overline{s}$, $\overline{(-)}$ is just another notation for $-\backslash\Delta$, like in the lattice of simple elements of a Garside monoid.

We now show that the complements defined on $\mathrm{NCP}(1,1,n)$ and on $I(c)_T$ actually coincide.

Lemma 8.3.7. *Let $f : I(c)_T \to \mathrm{NCP}(1,1,n)$ be the isomorphism of posets of Proposition 8.3.3. For $s, t \in I(c)_T$ such that $s \preccurlyeq t$, we have $f(s\backslash t) = f(s)\backslash f(t)$.*

Proof. First, we show that for all $s \in I(c)_T$, we have $f(s^{-1}c) = f(s\backslash c) = \overline{f(s)}$. Let $s \in I(c)_T$ be a cycle $(i_1 \cdots i_k)$ with $i_1 \lhd i_2 \lhd \dots \lhd i_k$. By construction, $f(s)$ contains $\{i_1, \dots, i_k\}$ as its only nonsingleton part, and $\overline{f(s)}$ is the partition whose nonsingleton parts are $\{z \in \mathbb{Z}/n\mathbb{Z} \mid i_j \unlhd z \lhd i_{j+1}\}$ for $j = 1, \dots, k$. On the other hand, the product $(i_1 \ \cdots \ i_k)^{-1}c$ is given by
$$(i_1 \ \cdots \ i_k)^{-1}c = (i_1 \ i_1+1 \ \cdots \ i_2-1)(i_2 \ i_2+1 \ \cdots \ i_3-1)\cdots(i_k \ i_k+1 \ \cdots i_1-1),$$
where the indices are seen modulo n if need be. We then have $f(s\backslash c) = \overline{f(s)}$ by definition.

Then, if $s \in I(c)_T$ is a product of disjoint cycles, say $s = s_1 \cdots s_i$, then $f(s_i)$ is a partition with only one nonsingleton part. By definition, and by the first part of the proof, $\overline{f(s)}$ is given by
$$\begin{aligned}\overline{f(s)} &= (\overline{f(s_1)}) \wedge \dots \wedge (\overline{f(s_k)})\\ &= f(s_1\backslash c) \wedge \dots \wedge f(s_k\backslash c)\\ &= f((s_1\backslash c) \wedge \dots \wedge (s_k\backslash c))\end{aligned}$$
Since $I(c)_T$ is a Garside germ, $s_i\backslash c = \overline{s_i}$ is simply the complement in the Garside element. We then have
$$(s_1\backslash c) \wedge \dots \wedge (s_k\backslash c) = \overline{s_1} \wedge \dots \wedge \overline{s_k} = \overline{s_1 \vee \dots \vee s_k} = \overline{s},$$
and thus $\overline{f(s)} = f(\overline{s}) = f(s\backslash c)$ as claimed.

Lastly, if $s, t \in I(c)_T$ are such that $s \preccurlyeq t$, then we have $s \cdot (s\backslash t) = t$, and thus $s\backslash t = \overline{s} \wedge t$ (see the discussion after Proposition 8.2.13). We deduce that
$$f(s\backslash t) = f(t \wedge (s\backslash c)) = f(t) \wedge f(s\backslash c) = f(t) \wedge (\overline{f(s)}) = f(s)\backslash f(t).$$

□

The complement operation allows us to define a partial product on $\mathrm{NCP}(1,1,n)$, where $s \cdot t$ is defined if and only if $t \preccurlyeq \overline{s}$, with $s \cdot t = s \vee t$ in this case. By Lemma 8.3.7, this partial product on $\mathrm{NCP}(1,1,n)$ endows it with a germ structure which is isomorphic to that of the interval germ $I(c)_T$.

8.3.2 Noncrossing partitions of type $(r,1,n)$

In this section, we fix two positive integers r, n with $r \geqslant 2$ and we set $m := rn$. We keep the notation of the last section regarding the lattice $\mathrm{NCP}(1,1,m)$, in particular, the identification $\mu_m \simeq \mathbb{Z}/m\mathbb{Z}$, along with the induced cyclic ordering on $\mathbb{Z}/m\mathbb{Z}$.

The group μ_r acts on μ_m by multiplication. This action in turn induces a group action on $\mathrm{NCP}(1,1,m)$, and we can consider the set $\mathrm{NCP}(1,1,m)^{\mu_r}$ of noncrossing partitions which are fixed under the action of μ_r.

Following [BC06, Definition 1.11], we define $\mathrm{NCP}(r,1,n) := \mathrm{NCP}(1,1,rn)^{\mu_r}$. Since μ_r acts on $\mathrm{NCP}(1,1,m)$ by lattice automorphisms, the set $\mathrm{NCP}(r,1,n)$ is a sublattice of $\mathrm{NCP}(1,1,m)$. Furthermore, we easily see that, if $s, t \in \mathrm{NCP}(r,1,n)$ are such that $s \preccurlyeq t$, then $s \backslash t$ also lies in $\mathrm{NCP}(r,1,n)$.

Remark 8.3.8. Let $\mathcal{S}$ be the lattice of simple elements in the dual braid monoid (M, Δ) of type $G(1,1,m)$. In Proposition 8.3.3, we saw that $\mathcal{S}$ is identified as a poset with $\mathrm{NCP}(1,1,m)$. Under this identification, the action of ζ_r on $\mathrm{NCP}(1,1,m)$ is the n-th power of the Garside automorphism ϕ of M. The lattice $\mathrm{NCP}(r,1,n)$ is the the lattice of simple elements of $G(M)^{\phi^n}$, which is a Garside group by Theorem 4.1.11 (Garside groupoid of fixed points).

By Theorem 8.2.17, the group $G(M)$ is identified with the centralizer in the braid group $B_m = B(G(1,1,m))$ of Δ^d. By [BDM02, Proposition 5.2], this centralizer is known to be isomorphic to B_n^*, and thus the germ structure on $\mathrm{NCP}(r,1,n)$ gives a Garside group structure on B_n^*. We will see in the next section that this Garside structure on B_n^* is simply the dual structure.

Elements of $\mathrm{NCP}(r,1,n)$ can be partitioned into two groups: a partition $s \in \mathrm{NCP}(r,1,n)$ is called *long* if one of its parts contains the origin (by noncrossedness, this part is unique, and referred to as the *long part* of s). A partition $s \in \mathrm{NCP}(r,1,n)$ which is not long is called *short*. Note that, since we identified μ_m with $\mathbb{Z}/m\mathbb{Z}$, denoting the origin of the plane by 0 is misleading. In the sequel, we will write $0'$ for the origin of the plane, to distinguish it from $0 \in \mathbb{Z}/m\mathbb{Z}$, which is identified to $1 \in \mu_m \subset \mathbb{C}$.

By [BC06, Lemma 1.14], the complement operation $s \mapsto \overline{s}$ (restricted from $\mathrm{NCP}(1,1,m)$ to $\mathrm{NCP}(r,1,n)$) swaps long and short elements of $\mathrm{NCP}(r,1,n)$. More precisely, if $s, t \in \mathrm{NCP}(r,1,n)$ are such that $s \preccurlyeq t$, then we have

- $s \backslash t$ is short if s, t are both long, or if they are both short.
- $s \backslash t$ is long if s is short and t is long (having s long and t short is impossible since $s \preccurlyeq t$).

It is fairly easy to give a complete list of atoms of the lattice $\mathrm{NCP}(r,1,n)$ (in the lattice-theoretic sense).

Definition 8.3.9 (Elements $u_{x,y}$ and v_p). For $x, y \in \mathbb{Z}/m\mathbb{Z}$ such that $x \lhd y \lhd x+n$, we set $u_{x,y} \in \mathrm{NCP}(r,1,n)$ to be the partition whose nonsingleton parts are

$$\{x,y\}, \{x+n, y+n\}, \ldots, \{x+(r-1)n, y+(r-1)n\}.$$

For $p \in \mathbb{Z}/m\mathbb{Z}$, we set $v_p \in \mathrm{NCP}(r,1,n)$ to be the partition whose only nonsingleton part is

$$\{p, p+n, p+2n, \ldots, p+(r-1)n\}.$$

We may have $u_{x,y} = u_{x',y'}$ for distinct $(x,y), (x',y')$, for instance we always have $u_{x,y} = u_{x+n,y+n}$. The following lemma gives a complete description of this situation, along with a complete description of the atoms of $\mathrm{NCP}(r,1,n)$.

Lemma 8.3.10 (Atoms of $\mathrm{NCP}(r,1,n)$). *(a) Let $x, y, x', y' \in \mathbb{Z}/m\mathbb{Z}$ be such that $x \lhd y \lhd x+n$ and $x' \lhd y' \lhd x'+n$. We have $u_{x,y} = u_{x',y'}$ if and only if there is some integer k such that $(x',y') = (x+kn, y+kn)$ in $(\mathbb{Z}/m\mathbb{Z})^2$. Thus there are $n(n-1)$ distinct elements of the form $u_{p,q}$ in $\mathrm{NCP}(r,1,n)$.*

(b) Let $p, p' \in \mathbb{Z}/m\mathbb{Z}$. We have $v_p = v_{p'}$ if and only if there is some integer k such that $p' = p + kn$ in $\mathbb{Z}/m\mathbb{Z}$. Thus there are n distinct elements of the form v_p in $\mathrm{NCP}(r,1,n)$.

(c) The $n(n-1)$ elements of the form $u_{x,y}$, along with the n elements of the form v_p, is a complete list of the atoms of $\mathrm{NCP}(r,1,n)$.

Proof. (a) First, since we obviously have $u_{x+n,y+n} = u_{x,y}$, the if part is trivial. For the converse, assume that $u_{x,y} = u_{x',y'}$. By definition, $\{x', y'\}$ is a part of $u_{x,y}$. Thus there is some $k \in [\![0, r-1]\!]$ such that $\{x', y'\} = \{x+kn, y+kn\}$. Since $x' \lhd y' \lhd x'+n$ and since $x+kn \lhd y+kn \lhd x+(k+1)n$, we obtain $x' = x+kn$ and $y' = y+kn$. From this we deduce that an element of the form $u_{x,y}$ can be written uniquely as $u_{x',y'}$ with $0 \unlhd x' \lhd n$ and $x' \lhd y' \lhd x'+n$. In particular there are $n(n-1)$ such elements.

(b) Again, the if part is trivial. Conversely, if p, p' are such that $v_p = v_{p'}$, then we have $p' = p + kn$ for some integer k, as it lies in the only nonsingleton part of v_p. From this we deduce that an element of the form v_p can be written uniquely as $v_{p'}$ with $0 \unlhd p' \lhd n$. In particular there are n^2 such elements.

(c) By Remark 8.3.8, we can see $\mathrm{NCP}(r,1,n)$ as a germ of fixed point under some power of the Garside automorphism of a dual braid monoid of type $G(1,1,m)$. In particular we can apply Lemma 4.1.10 (atoms of category of fixed points). Let x, y be distinct elements of $\mathbb{Z}/m\mathbb{Z}$, and consider the atom $r_{x,y}$ of $\mathrm{NCP}(1,1,m)$. By construction, the action of ζ_r sends $r_{x,y}$ to $r_{x+n,y+n}$, we then consider

$$(r_{x,y})^{\#} := \bigvee_{k=0}^{r-1} r_{x+kn,y+kn} \in \mathrm{NCP}(r,1,n).$$

By Lemma 4.1.10, the atoms of $\mathrm{NCP}(r,1,n)$ are precisely the $(r_{x,y})^{\#}$ which are minimal for $\preccurlyeq$. Up to exchanging x and y, we can assume that $x \lhd y \unlhd x + \lfloor m/2 \rfloor$, we then have:

- If $x \lhd y \lhd x+n$, then $(r_{x,y})^{\#} = u_{x,y}$ by construction.
- If $y = x+n$, then $(r_{x,y})^{\#} = v_x$ again by construction.
- If $x+n \lhd y \unlhd x + \lfloor m/2 \rfloor$, then the join $(r_{x,y})^{\#}$ contains a part given by $\{x, y-n, x+n, y, x+2n, \ldots, y-2n\}$, and we have $v_x \prec (r_{x,y})^{\#}$, which is not an atom.

Let A be the list of elements of the form $u_{x,y}$ or v_p. We just showed that the atoms of $\mathrm{NCP}(r,1,n)$ are included in A. Then, considering the definition of $u_{x,y}$ and of v_p, it is clear that if $a, b \in A$ are such that $a \preccurlyeq b$, then $a = b$, and thus every element of A is an atom of $\mathrm{NCP}(r,1,n)$.

□

Relation with the dual braid monoid of type $G(r,1,n)$

If $r = 2$, then the noncrossing partition lattice $\mathrm{NCP}(2,1,n)$ coincides with the noncrossing partitions introduced by Reiner in [Rei97, Section 2]. The lattice $\mathrm{NCP}(2,1,n)$ is known to be isomorphic to the lattice of simple elements of the dual braid monoid of type $G(2,1,n)$ by [BW02, Theorem 4.9]. In this section, we show that this also holds for an arbitrary $r \geqslant 2$ (this is claimed but not proven in [BC06, Remark at the end of Section 7]). More precisely, we want to show is that the germ structure on $\mathrm{NCP}(r,1,n)$ is isomorphic to the interval germ giving the dual braid monoid of type $G(r,1,n)$.

Let $W := G(1,1,m)$, and let $T := \mathrm{Ref}(W)$. Our approach is largely modeled on that of [BW02, Section 3 and Section 4], which covers the case where m is even and $r = 2$. We label the canonical basis of $\mathbb{C}^m$ in the following way:

$$\{e(0,1), e(0,2), \dots, e(0,n), e(1,1), \dots, e(r-1,n)\}.$$

A guiding idea is to think of $e(j,i)$ as $\zeta_r^j e_i$ in the vector space $\mathbb{C}^n$ (with canonical basis $e_1, \dots, e_n$). The Coxeter element $c(1,1,m)$ of W introduced in Section 8.3.1 acts on this basis as the m-cycle $(e(0,1)\ e(0,2)\ \cdots\ e(0,n)\ e(1,1)\ \cdots\ e(r-1,n))$. We then have

$$c(1,1,m)^n = (e(0,1)\ e(1,1)\ \cdots\ e(r-1,1))(e(0,2)\ \cdots\ e(r-1,2)) \cdots (e(0,n)\ \cdots\ e(r-1,n)).$$

As a Coxeter element, $c(1,1,m)$ is an m-regular element, and thus $c(1,1,m)^n$ is $\frac{m}{n} = r$-regular. We explicitly describe the ζ_r-eigenspace of $c(1,1,m)^n$ acting on $\mathbb{C}^m$ in the following lemma:

Lemma 8.3.11. *The eigenspace $V_r := \mathrm{Ker}(c(1,1,m)^n - \zeta_r \mathrm{Id})$ has dimension n and admits a basis $v := \{v_1, \cdots, v_n\}$, where*

$$v_i := \sum_{j=0}^{r-1} \zeta_r^{r-j} e(j,i).$$

The isomorphism of vector spaces $V_r \simeq \mathbb{C}^n$ given by the basis v induces an isomorphism between the centralizer $W_r := C_W(c(1,1,m)^n)$ and $G(r,1,n)$.

Proof. By Theorem 6.1.26 (Springer theory in complex reflection groups), the degrees of W_r acting on V_r are the degrees of $G(1,1,m)$ which are divisible by r, that is $r, 2r, \dots, rn$. In particular, W_r has rank n and V_r has dimension n. As the v_i are linearly independent, v is a basis for V_r.

The groups W_r and $G(r,1,n)$ share the same degrees, and the same cardinality (since the order of a complex reflection group is the product of its degrees). We then only have to show that the image of W_r in $\mathrm{GL}_n(\mathbb{C})$ contains a generating set of $G(r,1,n)$, like the one given in Section 6.1.2. This is a direct check: The cycle $(e(0,1)\ e(1,1) \cdots e(r-1,1))$ acts trivially on v_j for $j \neq 1$ and sends v_1 to $\zeta_r v_1$. Then, for $i \in [\![1, n-1]\!]$, the permutation $(e(0,i)\ e(0,i+1)) \cdots (e(r-1,i)\ e(r-1,i+1))$ swaps v_i and v_{i+1}. □

The action of $c(1,1,m)$ on V_r is given (in the basis v) by the matrix

$$c(r,1,n) := \begin{pmatrix} 0 & & & \zeta_r \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}$$

which is indeed a Coxeter element for the group $G(r,1,n)$, since it admits (for instance) $(\zeta_m^{n-1}, \zeta_m^{n-2}, \dots, 1)$ as a ζ_m-eigenvector. We can then consider the interval germ in W_r associated to the Coxeter element $c(1,1,m)$ and to the reflections of W_r. In this section, we will denote this interval by $I(c(1,1,m))_{W_r}$ (we drop the reference to the generating set of the considered group, which will always be the set of its reflections). By construction, the interval germ associated to $I(c(1,1,m))_{W_r}$ is isomorphic to the interval germ associated to $I(c(r,1,n))_{G(r,1,n)}$.

On the other hand, we can consider the interval $I(c(1,1,m))_W$, which is isomorphic to $\mathrm{NCP}(1,1,m)$ by Proposition 8.3.3, along with $(I(c(1,1,m))_W)^{c(1,1,m)^n} \simeq \mathrm{NCP}(r,1,n)$. Our goal is to show that $I(c(1,1,m))_{W_r} = (I(c(1,1,m))_W)^{c(1,1,m)^n}$ as a poset in W_r. This will in turn induce an isomorphism of germs between $I(c(r,1,n))_{G(r,1,n)}$ and $\mathrm{NCP}(r,1,n)$.

The group $G(r,1,q)$ is endowed with a character χ, sending a monomial matrix to the product of its nonzero entries. The following lemma gives a first restrictive condition for an element of $G(r,1,n)$ to belong in the interval $I(c(r,1,n))_{G(r,1,n)}$.

Lemma 8.3.12. *A nontrivial diagonal matrix $D \in G(r,1,n)$ lies in $I(c(r,1,n))_{G(r,1,n)}$ if and only if D is a diagonal reflection s with $\chi(s) = \zeta_r$. Furthermore, an element $g \in I(c(r,1,n))_{G(r,1,n)}$ is such that $\chi(g) \in \{1, \zeta_r\}$.*

Proof. Let $\ell_{G(r,1,n)}$ denote the reflection length in $G(r,1,n)$. By [Shi07, Theorem 2.1], $\ell_{G(r,1,n)}(D)$ is the number of nontrivial diagonal entries of D. The underlying permutation of $D^{-1}c(r,1,n)$ is an n-cycle. Thus, again by [Shi07, Theorem 2.1], we have

$$\ell_{G(r,1,n)}(D^{-1}c(r,1,n)) = \begin{cases} n-1 & \text{if } \chi(D) = \chi(c(r,1,n)) = \zeta_r, \\ n & \text{otherwise.} \end{cases}$$

If D is non trivial (i.e. if its reflection length is nonzero), we get $D \preccurlyeq c(r,1,n)$ if and only if $\chi(D) = \zeta_r$ and D has exactly one nontrivial diagonal entry. In particular, we obtain that the diagonal reflections belonging to $I(c(r,1,n))_{G(r,1,n)}$ are exactly the $r_{\zeta_r,i}$ for $i \in [\![1,n]\!]$.

Let now $g \in I(c(r,1,n))_{G(r,1,n)}$. If $\chi(g) \neq 1$, then any minimal decomposition of g as a product of reflections contains at least one reflection of the form $r_{\zeta,i}$. If $g \preccurlyeq c(r,1,n)$, then we have $r_{\zeta,i} \preccurlyeq g \preccurlyeq c(r,1,n)$ and $\zeta = \zeta_r$. The element $g' := r_{\zeta_r,i}^{-1} g$ also belongs to $I(c(r,1,n))_{G(r,1,n)}$. Again, if $\chi(g') \neq 1$, then we have $r_{\zeta_r,j} \preccurlyeq g'$ for some j. In this case, we have $r_{\zeta_r,i} r_{\zeta_r,j} \preccurlyeq c(r,1,n)$, which contradicts the first part. □

The isomorphism $W_r \simeq G(r,1,n)$ of Lemma 8.3.11 allows us to define χ on W'. We set ℓ_r the reflection length for elements of W_r relative to the reflections of W_r acting on V_r. Let $c := (e(j_1,i_1) \ \cdots \ e(j_k,i_k)) \in W$ be a cycle. We set

$$c^{(1)} := c^{c(1,1,n)^q} = (e(j_1-1,i_1) \ \cdots \ e(j_k-1,i_k)).$$

More generally, $c^{(k)}$ is defined inductively as $(c^{(k-1)})^{(1)}$.

Proposition 8.3.13. *An element $\sigma \in W$ lies in W_r if and only if can be written as decomposition as a product of disjoint cycles of the form*

$$\sigma = c_1 c_1^{(1)} \cdots c_1^{(r-1)} c_2 c_2^{(1)} \cdots c_2^{(r-1)} \cdots c_a c_a^{(1)} \cdots c_a^{(r-1)} \gamma_1 \cdots \gamma_b.$$

where $\gamma_i^{(1)} = \gamma_i$ for $i \in [\![1, b]\!]$.

Proof. Our proof is an adaptation of the proof of [BW02, Proposition 3.1], which deals with the case $r = 2$. First, it is clear that elements of the given form lie in W_r. Conversely, let $\sigma = c_1 \cdots c_k$ be a product of disjoint cycles in $\mathfrak{S}(\mathbb{Z}/m\mathbb{Z})$. We have that $c(1,1,m)^n$ centralizes σ if and only if $c_1 \cdots c_k = c_1^{(1)} \cdots c_k^{(1)}$. By uniqueness (up to reordering) of cycle decompositions in $\mathfrak{S}(\mathbb{Z}/m\mathbb{Z})$, for each i either $c_i = c_j^{(1)}$ for some $j \neq i$ or else $c_i = c_i^{(1)}$. An immediate induction then gives that σ has the required decomposition. $\square$

We note in particular that the cycles $c_i, \ldots, c_i^{(r-1)}$ in the decomposition of Proposition 8.3.13 are disjoint for all $i \in [\![1, a]\!]$.

Definition 8.3.14 (Saturated and balanced cycles). Let $c \in W$ be a cycle such that $c, c^{(1)}, \ldots, c^{(r-1)}$ are disjoint. The product $cc^{(1)} \cdots c^{(r-1)}$ will be denoted by $\widetilde{c}$ and called a *saturated cycle*. If $\gamma \in W$ is a cycle lying in W_r, we say that γ is a *balanced cycle*.

Proposition 8.3.13 states that elements of W_r are the products of disjoint saturated cycles and disjoint balanced cycles. Using this decomposition, we can compute the reflection length of an element of W_r. We denote by ℓ_W the reflection length in W. Recall from Lemma 8.3.4 that the length of a cycle c in $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z})$ is denoted by $\ell(c)$. Under the isomorphism $\mathfrak{S}(\mathbb{Z}/n\mathbb{Z}) \simeq W$, we have $\ell_W(c) = \ell(c) - 1$ (again by Lemma 8.3.4).

Lemma 8.3.15. *Let $\widetilde{c} = cc^{(1)} \cdots c^{(r-1)}$ be a saturated cycle in W_r. We have $\chi(\widetilde{c}) = 1$ and $\ell_r(\widetilde{c}) = \ell(c) - 1$. Let γ be a balanced cycle in W_r. We have $\chi(\gamma) = \zeta_r$ and $\ell_r(\gamma) = \ell(\gamma)/r$.*

Proof. Let $c = (e(j_1, i_1) \;\cdots\; e(j_k, i_k))$ be a cycle such that $c, \ldots, c^{(r-1)}$ are all disjoint. By assumption, $i_1, \ldots, i_k$ are all distinct. One readily sees that $\widetilde{c}$ acts on V by

$$\begin{cases} \widetilde{c}.v_{i_m} = \zeta_r^{j_{m+1}-j_m} v_{i_{m+1}} & \forall m \in [\![1, k-1]\!], \\ \widetilde{c}.v_{i_k} = \zeta_r^{j_1 - j_k} v_{i_1}. \end{cases}$$

In particular, we have $\chi(\widetilde{c}) = \zeta_r^{j_2-j_1} \cdots \zeta_r^{j_k - j_{k-1}} \zeta_r^{j_1 - j_k} = 1$. The fixed space of $\widetilde{c}$ acting on V is generated by all the v_i with $i \notin \{i_1, \ldots, i_k\}$ and $v_{i_1} + \cdots + v_{i_k}$. Thus $\ell_r(\widetilde{c}) = k - 1 = \ell(c) - 1$ by Lemma 8.3.11 and [Shi07, Theorem 2.1].

Let γ be a balanced cycle. It can be written as

$$\gamma = (e(0, i_1)\; e(j_2, i_2) \;\cdots\; e(j_k, i_k)\; e(1, i_1) \;\cdots\; e(j_k + r - 1, i_k)).$$

The action of γ on V is then given by

$$\begin{cases} \gamma.v_{i_1} = \zeta_r^{j_2} v_{i_2}, \\ \gamma.v_{i_m} = \zeta_r^{j_{m+1}-j_m} v_{i_{m+1}} & \forall m \in [\![2, k-1]\!], \\ \gamma.v_{i_k} = \zeta_r^{1-j_k} v_1. \end{cases}$$

In particular we have $\chi(\gamma) = \zeta_r^{j_2} \cdots \zeta_r^{j_k - j_{k-1}} \zeta_r^{1-k} = \zeta_r$. The fixed space of γ acting on V is generated by all the v_i with $i \notin \{i_1, \ldots, i_k\}$. Thus $\ell_r(\gamma) = k = \ell(\gamma)/r$ again by Lemma 8.3.11 and [Shi07, Theorem 2.1]. $\square$

By combining Lemma 8.3.15 and Lemma 8.3.4, we obtain the following proposition:

Proposition 8.3.16. *Let $\sigma = \widetilde{c_1} \cdots \widetilde{c_a} \gamma_1 \cdots \gamma_b \in W_r$. The reflection length of σ in W and W_r are given by*

$$\ell_W(\sigma) = \sum_{i=1}^{a} r(\ell(c_i) - 1) + \sum_{j=1}^{b} (\ell(\gamma_i) - 1) \text{ and } \ell_r(\sigma) = \sum_{i=1}^{a} (\ell(c_i) - 1) + \sum_{j=1}^{b} \frac{\ell(\gamma_i)}{r}.$$

In particular, we have $\ell_W(\sigma) + b = r\ell_r(\sigma)$.

This relation between reflection lengths in W and in W_r will be the key element in our proof that $I(c(r,1,n))_{G(r,1,n)}$ and $\mathrm{NCP}(r,1,n)$ are isomorphic.

Proposition 8.3.17. *Let $\sigma = \widetilde{c_1} \cdots \widetilde{c_a} \gamma_1 \cdots \gamma_b \in W_r$. If $\sigma \in I(c(1,1,m))_{W_r}$ or if $\sigma \in I(c(1,1,m))_W$, then $b \in \{0,1\}$ and $\chi(\sigma) = \zeta_r^b$.*

Proof. The assumption on $\chi(\sigma)$ is a direct consequence of Lemma 8.3.15.

Let $\gamma = (e(0,i_1)\ e(j_2,i_2)\ \cdots\ e(j_k,i_k)\ e(1,i_1)\ \cdots\ e(j_k+r-1,i_k))$ be a balanced cycle. The diagonal reflection of W_r sending v_i to $\zeta_r v_i$ is given by $s_i := (e(0,i) \cdots e(r-1,i))$. We have that $s_{i_1}^{-1}\gamma = (e(0,i_1) \cdots e(j_k,i_k)) \cdots (e(r-1,i_1) \cdots e(j_k+r-1,i_k))$ is a saturated cycle. Proposition 8.3.16 then gives that $\ell_r(s_{i_1}) + \ell_r(s_{i_1}^{-1}\gamma) = 1 + k - 1 = k = \ell_r(\gamma)$ and $s_{i_1} \preccurlyeq \gamma$ in W_r.

Assume that $\sigma \in I(c(1,1,m))_{W_r}$ and $b > 1$. Since γ_1 and γ_2 have disjoint support, we deduce that there are two diagonal reflections s, s' of W_r such that $s \preccurlyeq \gamma_1, s' \preccurlyeq \gamma_2$, $ss' \neq \mathrm{Id}$ and $\chi(ss') = \zeta_r^2$. We then have $ss' \preccurlyeq c(1,1,m)$ in W_r which contradicts Lemma 8.3.12.

Assume now that $\sigma \in I(c(1,1,m))_W$. By Proposition 8.3.3, the set theoretic partition of the set μ_m induced by σ lies in $\mathrm{NCP}(1,1,m)$. If $b > 1$, then the orbits of γ_1 and γ_2 induce two parts of μ_m whose convex hull contain 0. Thus the partition of μ_m induced by σ is crossing and $\sigma \notin I(c(1,1,n))_W$. □

Proposition 8.3.18. *The two posets $(I(c(1,1,m))_W)^{c(1,1,m)^n}$ and $I(c(1,1,m))_{W_r}$ are equal.*

Proof. First, we show that the two sets $(I(c(1,1,m))_W)^{c(1,1,m)^n}$ and $I(c(1,1,m))_{W_r}$ are equal. Let $\sigma \in I(c(1,1,m))_{W_r}$. By Proposition 8.3.17, we have $\chi(\sigma) \in \{1, \zeta_r\}$ and $\chi(\sigma^{-1}c(1,1,m)) = \chi(\sigma)^{-1}\zeta_r$. Proposition 8.3.16 then gives

$$\begin{aligned}
\ell_W(\sigma) + \ell_W(\sigma^{-1}c(1,1,m)) &= r\ell_r(\sigma) + r\ell_r(\sigma^{-1}c(1,1,m)) - 1 \\
&= r(\ell_r(\sigma) + \ell_r(\sigma^{-1}c(1,1,m))) - 1 \\
&= r\ell_r(c(1,1,m)^n) - 1 \\
&= \ell_W(c(1,1,m)),
\end{aligned}$$

and $\sigma \in I(c(1,1,m))_W$. The same reasoning gives $(I(c(1,1,m))_W)^{c(1,1,m)^n} \subset I(c(1,1,m))_{W_r}$.

Let now $\sigma, \tau \in I(c(1,1,m))_{W_r}$, we have to show that $\sigma \preccurlyeq \tau$ in W_r if and only if $\sigma \preccurlyeq \tau$ in W. We consider four cases

- $\chi(\sigma) = 1$ and $\chi(\tau) = 1$. We have $\ell_W(\sigma) = r\ell_r(\sigma)$ and $\ell_W(\tau) = r\ell_r(\tau)$. If $\sigma \preccurlyeq \tau$ in W (resp. in W_r), then $\sigma^{-1}\tau \in I(c(1,1,m))_{W_r}$ is such that $\chi(\sigma^{-1}\tau) = 1$. We then have $\ell_W(\sigma^{-1}\tau) = r\ell_r(\sigma^{-1}\tau)$ and $\sigma \preccurlyeq \tau$ in W_r (resp. in W).

- $\chi(\sigma) = \zeta_r$ and $\chi(\tau) = 1$. We cannot have $\sigma \preccurlyeq \tau$ in either W or W_r since this would imply that $\sigma^{-1}\tau$ is an element of $I(c(1,1,m))_{W_r}$ with $\chi(\sigma^{-1}\tau) = \zeta_r^{-1}$ (we can assume $\zeta_r \neq -1$ since the case $r = 2$ is known by [BW02, Lemma 4.8]).
- $\chi(\sigma) = 1$ and $\chi(\tau) = \zeta_r$. We have $\ell_W(\sigma) = r\ell_r(\sigma)$ and $\ell_W(\tau) + 1 = r\ell_r(\tau)$. If $\sigma \preccurlyeq \tau$ in W (resp. in W_r), then $\sigma^{-1}\tau \in I(c(1,1,m))_{W_r}$ is such that $\chi(\sigma^{-1}\tau) = \zeta_r$. We then have $\ell_W(\sigma^{-1}\tau) + 1 = r\ell_r(\sigma^{-1}\tau)$ and $\sigma \preccurlyeq \tau$ in W_r (resp. in W).
- $\chi(\sigma) = \zeta_r$ and $\chi(\tau) = \zeta_r$. We have $\ell_W(\sigma) + 1 = r\ell_r(\sigma)$ and $\ell_W(\tau) + 1 = r\ell_r(\tau)$. If $\sigma \preccurlyeq \tau$ in W (resp. in W_r), then $\sigma^{-1}\tau \in I(c(1,1,m))_{W_r}$ is such that $\chi(\sigma^{-1}\tau) = 1$. We then have $\ell_W(\sigma^{-1}\tau) = r\ell_r(\sigma^{-1}\tau)$ and $\sigma \preccurlyeq \tau$ in W_r (resp. in W).

□

We are now equipped to show the identification between the interval germ $I(c(r,1,n))_{G(r,1,n)}$ and $\mathrm{NCP}(r,1,n)$.

Proposition 8.3.19. *Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. The group $G(r,1,n)$ acts on the set*

$$E := \{e_1, \dots, e_n, \zeta_r e_1, \dots, \zeta_r e_n, \dots, \zeta_r^{r-1} e_1, \dots, \zeta_r^{r-1} e_n\} \simeq \mathbb{Z}/m\mathbb{Z}.$$

The map sending $g \in G(r,1,n)$ to its set of orbits acting on $E \simeq \mathbb{Z}/m\mathbb{Z}$ induces an isomorphism of germs between $I(c(r,1,n))_{G(r,1,n)}$ and $\mathrm{NCP}(r,1,n)$.

Proof. We know that the germ isomorphism $I(c(1,1,m))_W \simeq \mathrm{NCP}(1,1,m)$ given in Proposition 8.3.3 induces a commutative square of germs.

$$\begin{tikzcd}
(I(c(1,1,m))_W)^{c(1,1,m)^n} \arrow[r, hook] \arrow[d, "\simeq"'] & I(c(1,1,m))_W \arrow[d, "\simeq"] \\
\mathrm{NCP}(r,1,n) \arrow[r, hook] & \mathrm{NCP}(1,1,m)
\end{tikzcd}$$

By Proposition 8.3.18, the posets $(I(c(1,1,m))_W)^{c(1,1,m)^n}$ and $I(c(1,1,m))_{W_r}$ are equal. Moreover, since the embedding $W_r \to W$ is a group morphism, the isomorphism of posets between $(I(c(1,1,m))_W)^{c(1,1,m)^n}$ and $I(c(1,1,m))_{W_r}$ preserves the complement to $c(1,1,m)$. Since the germ structure on these posets depends only on the poset structure and on the complement, we obtain that the germs structures on $(I(c(1,1,m))_W)^{c(1,1,m)^n}$ and $I(c(1,1,m))_{W_r}$ are equal.

We obtain that the map $I(c(1,1,m))_{W_r} \to \mathrm{NCP}(r,1,n)$ sending g to its orbit acting on the canonical basis of $\mathbb{C}^n$ is an isomorphism of germs. Under the identification $G(r,1,n) \simeq W_r$, the natural bijection $E \simeq \{e(0,1), e(0,2), \dots, e(r-1,n)\}$ is an isomorphism of $G(r,1,n)$-set. The identification $I(c(r,1,n))_{G(r,1,n)} \simeq I(c(1,1,m))_{W_r}$ then gives the desired result. □

In particular, since the groups $G(r,1,n)$ and $G(2,1,n)$ are isodiscriminantal for $r \geqslant 2$, Proposition 8.2.9 then implies that the lattices $\mathrm{NCP}(2,1,n)$ and $\mathrm{NCP}(r,1,n)$ are isomorphic for $r \geqslant 2$. However, considering the latter instead of the former will be useful in defining a noncrossing partition lattice for the group $G(r,r,n+1)$ in the next section.

We saw in Lemma 8.3.12 that an element $g \in I(c(r,1,n))_{G(r,1,n)}$ is such that $\chi(g) \in \{1, \zeta_r\}$. The following lemma shows that this alternative corresponds (under the isomorphism of Proposition 8.3.19) to the alternative between long and short elements of $\mathrm{NCP}(r,1,n)$.

Corollary 8.3.20. *Let $g \in I(c(r,1,n))_{G(r,1,n)}$. The image of g under the isomorphism $I(c(r,1,n))_{G(r,1,n)} \simeq \mathrm{NCP}(r,1,n)$ of Proposition 8.3.19 is long if and only if $\chi(g) = \zeta_r$.*

Proof. Let E be the set given in Proposition 8.3.19, and let φ be the isomorphism given in Proposition 8.3.19. For $i \in [\![1,n]\!]$, the action of $r_{\zeta_r,i}$ on E is given by the cycle $(e_i\ \zeta_r e_i\ \cdots\ \zeta_r^{r-1} e_i)$ and $\varphi(r_{\zeta_r,i}) = v_{i-1}$. We saw in the proof of Lemma 8.3.12 that an element $g \in I(c(r,1,n))_{G(r,1,n)}$ is such that $\chi(g) = \zeta_r$ if and only if there is some $i \in [\![1,n]\!]$ such that $r_{\zeta_r,i}$. Applying the isomorphism φ yields that $\chi(g) = \zeta_r$ if and only if $v_p \preccurlyeq \varphi(g)$ for some $p \in [\![0,n-1]\!]$, which is equivalent to stating that $\varphi(g)$ is long (see for instance the proof of Lemma 8.3.10). □

8.3.3 Noncrossing partitions of type $(e,e,n+1)$

Let $e, n \geqslant 2$ be integers, and let $m := en$. The natural embedding $\mu_m \to \mu_m \sqcup \{0\}$ induces a map $(-)^\flat : \mathrm{NCP}(\mu_m \sqcup \{0\}) \to \mathrm{NCP}(\mu_m) = \mathrm{NCP}(1,1,m)$ which simply consists in "forgetting 0". For $s \in \mathrm{NCP}(\mu_m \sqcup \{0\})$, we denote by $s^\flat$ the image of s in $\mathrm{NCP}(1,1,m)$ under the map $(-)^\flat$. As in the last section, we use the identification $\mu_m \simeq \mathbb{Z}/m\mathbb{Z}$, and we denote by $0'$ the origin of the complex plane.

Definition 8.3.21. [BC06, Definition 1.15] The set of noncrossing partitions of type $(e,e,n+1)$ is defined as

$$\mathrm{NCP}(e,e,n+1) := \{s \in \mathrm{NCP}(\mu_{en} \cup \{0'\}) \mid s^\flat \in \mathrm{NCP}(e,1,n)\}.$$

An element $s \in \mathrm{NCP}(e,e,n+1)$ is said to be

- *short symmetric* if $s^\flat \in \mathrm{NCP}(e,1,n)$ is short, and if $\{0'\}$ is a part of s.
- *long symmetric* if $s^\flat \in \mathrm{NCP}(e,1,n)$ is long. The noncrossing condition then implies that $a \cup \{0'\}$ is a part of s, where X is the long part of $s^\flat$. The part $X \cup \{0'\}$ is called the *long part* of s.
- *asymmetric* if $s^\flat \in \mathrm{NCP}(e,1,n)$ is short, and if $\{0'\}$ is not a part of s. There is then a unique part X of $s^\flat$ such that $X \cup \{0'\}$ is a part of s. The part $X \cup \{0'\} \in s$ is called the *asymmetric part* of s.

By construction, these cases are mutually exclusive and they cover all elements of $\mathrm{NCP}(e,e,n+1)$.

If s is short symmetric, t asymmetric, and w long symmetric, then one cannot have $t \preccurlyeq s$ nor $w \preccurlyeq t$ nor $w \preccurlyeq s$. We can relate the poset $\mathrm{NCP}(e,e,n+1)$ with the lattice $\mathrm{NCP}(e,1,n)$ using the map $(-)^\flat$. This is done by constructing maps $(-)_* : \mathrm{NCP}(e,1,n) \to \mathrm{NCP}(e,e,n+1)$ and $(-)^\# : \mathrm{NCP}(e,e,n+1) \to \mathrm{NCP}(e,1,n)$ in the following way:

First, let $t \in \mathrm{NCP}(e,1,n)$. The partition $t_* \in \mathrm{NCP}(e,e,n+1)$ is defined as follows:

- If t is short, then we set $t_* := s \cup \{\{0'\}\}$. This identifies short elements in $\mathrm{NCP}(e,1,n)$ with short symmetric elements in $\mathrm{NCP}(e,e,n+1)$.
- If t is long and if X is the long part of t, the partition t_* is obtained from t by replacing X by $X \cup \{0'\}$. This identifies long elements in $\mathrm{NCP}(e,1,n)$ with long symmetric elements in $\mathrm{NCP}(e,e,n+1)$.

Now, let $s \in \mathrm{NCP}(e,e,n+1)$. The partition $s^\# \in \mathrm{NCP}(e,e,n+1)$ is defined as follows.

- If s is symmetric, then $s^\# := s^\flat$.

- If s is asymmetric, then let X be the part containing $0'$. Let $\widetilde{X} := \left(\bigcup_{\zeta\in\mu_e} \zeta X\right) \setminus \{0\}$. We set $s^{\#}$ to be the element of $\mathrm{NCP}(e,1,n)$ containing $\widetilde{X}$ as a part, and apart from that made of parts in s.

From these definitions, we easily obtain that, for $s \in \mathrm{NCP}(e,e,n+1)$ and $t \in \mathrm{NCP}(e,1,n)$, we have

$$(t_*)^\flat = t \text{ and } (t_*)^{\#} = t,$$

$$(s^\flat)_* \preccurlyeq s \preccurlyeq (s^{\#})_*,$$

$$(s^\flat)_* = s \Leftrightarrow (s^\flat)_* = s \Leftrightarrow s \text{ is symmetric.}$$

Not only are $(-)^\flat$ and $(-)^{\#}$ retractions of $(-)_*$, but they also satisfy an adjointness property by the following lemma:

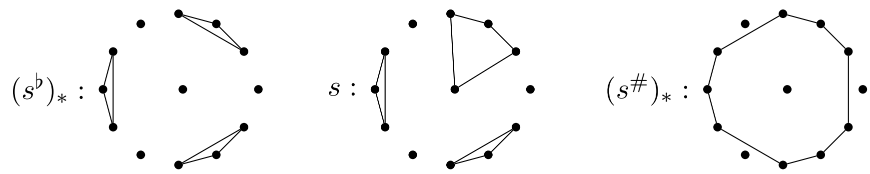
Lemma 8.3.22. *[BC06, Lemma 1.16] Let $s \in \mathrm{NCP}(e,e,n+1)$ and $t \in \mathrm{NCP}(e,1,n)$. We have*

$$s \preccurlyeq t_* \Leftrightarrow s^{\#} \preccurlyeq t \text{ and } t_* \preccurlyeq s \Leftrightarrow t \preccurlyeq s^\flat.$$

In other words, we have the following adjunction diagram:

$$\begin{tikzcd}
\mathrm{NCP}(e,1,n) \arrow[r, hook, "*"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{NCP}(e,e,n+1) \arrow[l, bend right=40, two heads, "\#"'] \arrow[l, bend left=40, two heads, "\flat"]
\end{tikzcd}$$

Example 8.3.23. Let $s \in \mathrm{NCP}(e,e,n+1)$ be asymmetric. By construction, $(s^\flat)_*$ is the biggest element (for $\preccurlyeq$) which is symmetric and such that $(s^\flat)_* \preccurlyeq s$. Likewise, $(s^{\#})_*$ is the smallest symmetric element such that $s \preccurlyeq (s^{\#})_*$. In the example below, we have $e = 3$ and $n = 4$.



Using Lemma 8.3.22, it is easy to prove that $\mathrm{NCP}(e,e,n+1)$ is a lattice [BC06, Lemma 1.17]. Moreover, the map $(-)_* : \mathrm{NCP}(e,1,n) \to \mathrm{NCP}(e,e,n+1)$ has both a left- and a right- adjoint, and as such it is a morphism of lattices, which induces an isomorphism of lattices between $\mathrm{NCP}(e,1,n)$ and the sublattice of $\mathrm{NCP}(e,e,n+1)$ made of symmetric elements.

A description of the atoms of the lattice $\mathrm{NCP}(e,e,n+1)$ is given in [BC06, Section 3]. On the one hand, elements of the form $(u_{p,q})_*$ are clearly atoms of $\mathrm{NCP}(e,e,n+1)$, since elements of the form $u_{p,q}$ are atoms in $\mathrm{NCP}(e,1,n)$, and since asymmetric elements of $\mathrm{NCP}(e,e,n+1)$ cannot be smaller than short symmetric elements. On the other hand, $(v_p)_*$ is not an atom, since it is for instance greater than the noncrossing partition whose only nonsingleton part is $\{0', p\}$.

Definition 8.3.24 (Elements $u_{x,y}$ and a_p). For $p \in \mathbb{Z}/m\mathbb{Z}$, we set $a_p \in \mathrm{NCP}(e,e,n+1)$ to be the partition whose only nonsingleton part is $\{0', p\}$. By abuse of notation, for $x, y \in \mathbb{Z}/m\mathbb{Z}$ such that $x \lhd y \lhd x+n$, we denote by $u_{x,y}$ the element $(u_{x,y})_* \in \mathrm{NCP}(e,e,n+1)$.

Lemma 8.3.25 (Atoms of $\mathrm{NCP}(e,e,n+1)$**).** *[BC06, Lemma 3.2]*
The $n(n-1)$ elements of the form $u_{p,q}$ (the symmetric atoms*), along with the en elements of the form a_p (the* asymmetric atoms*), form a complete list of atoms of* $\mathrm{NCP}(e,e,n+1)$*.*

Now, we give the construction of the complement operation $\backslash$ on $\mathrm{NCP}(e,e,n+1)$. It will be more intricate than in the case of $\mathrm{NCP}(r,1,n)$, and we need to consider different cases (namely, symmetric and asymmetric elements).

First, for $p \in \mathbb{Z}/m\mathbb{Z}$, we set $\alpha_p := \{p-1, p-2, \ldots, p-n\} \subset \mathbb{Z}/m\mathbb{Z}$. We then consider the asymmetric element M_p, whose nonsingleton parts are $\{0'\} \cup \alpha_p, \alpha_p - n, \alpha_p - 2n, \ldots, \alpha_p - (e-1)n$.

Lemma 8.3.26. *[BC06, Lemma 1.19] Let $p \in \mathbb{Z}/m\mathbb{Z}$. Since $\alpha_p \subset \{0'\} \cup \mathbb{Z}/m\mathbb{Z}$, we can consider the restriction morphism res_p from the lattice of partitions of $\{0'\} \cup \mathbb{Z}/m\mathbb{Z}$ to the lattice of partitions of α_p. Since α_p is the vertex set of a strictly convex $(n+1)$-gon, res_p sends $\mathrm{NCP}(e,e,n+1)$ to $\mathrm{NCP}(\alpha_p) \simeq \mathrm{NCP}(1,1,n+1)$.*

(a) The map res_p induces a poset isomorphism

$$\varphi_p : (\{s \in \mathrm{NCP}(e,e,n+1) \mid s \preccurlyeq M_p\}, \preccurlyeq) \longrightarrow (\mathrm{NCP}(\alpha_p), \preccurlyeq).$$

(b) The map res_p induces a poset isomorphism

$$\psi_p : (\{s \in \mathrm{NCP}(e,e,n+1) \mid a_p \preccurlyeq s\}, \preccurlyeq) \longrightarrow (\mathrm{NCP}(\alpha_p), \preccurlyeq).$$

(c) The composition $\psi_p^{-1}\varphi_p$ is the map $s \mapsto a_p \vee s$.

Definition 8.3.27 (Complement). [BC06, Definition 1.20]
Let $s \in \mathrm{NCP}(e,e,n+1)$, the element $\overline{s} := s\backslash\Delta$ is defined in the following way:

(a) If s is symmetric, then $\overline{s} := \left(\overline{s^\flat}\right)_*$.

(b) If s is asymmetric, we choose $p \in \mathbb{Z}/m\mathbb{Z}$ such that $a_p \preccurlyeq s$. We define $\overline{s} := \varphi_p^{-1}(\overline{\psi_p(s)})$, where the complement operation is defined on $\mathrm{NCP}(\alpha_p)$ via a standard identification with $\mathrm{NCP}(1,1,n+1)$.

In general, if $s, t \in \mathrm{NCP}(e,e,n+1)$ are such that $s \preccurlyeq t$, then the complement $s\backslash t$ of s in t is defined by $s\backslash t := t \wedge \overline{s}$.

Note that the definition of $\overline{s}$ when s is asymmetric does not depend on the choice of an asymmetric atom smaller than s. By definition, we have $\overline{a_p} = M_p$. Moreover, also by definition, the map $(-)_* : \mathrm{NCP}(e,1,n) \to \mathrm{NCP}(e,e,n+1)$ preserve the complement.

This notion of complement induces a germ structure on $\mathrm{NCP}(e,e,n+1)$, where $s \cdot t$ is defined whenever $t \preccurlyeq \overline{s}$, in which case it is equal to $s \vee t$. In other words, we have $s \cdot s\backslash u = u$ whenever $s \preccurlyeq u$ in $\mathrm{NCP}(e,e,n+1)$.

Relation with the dual braid monoid of type $G(e,e,n+1)$

As in Section 8.3.2, we want to prove that the germ structure on $\mathrm{NCP}(e,e,n+1)$ is isomorphic to the interval germ giving the dual group of type $G(e,e,n+1)$. This is done in [BC06, Theorem 4.6], but the isomorphism given there is described rather abstractly. We would like to have a description in the spirit of Propositions 8.3.3 and Proposition 8.3.19.

Consider the group $G(e,1,n)$. The character $\chi : G(e,1,n) \to \mathbb{C}^*$ giving the product of the nonzero entries allows us to define an injective group morphism

$$\begin{array}{rccc} i: & G(e,1,n) & \longrightarrow & G(e,e,n+1) \\ & g & \longmapsto & \begin{pmatrix} \chi(g)^{-1} & 0 \\ 0 & g \end{pmatrix} \end{array}$$

Consider the element $c(e,e,n+1) := i(c(e,1,n))$ (where $c(e,1,n) \in G(e,1,n)$ is the element defined in Section 8.3.2). The element $c(e,e,n+1)$ is a Coxeter element for $G(e,e,n+1)$, since it admits (for instance) $(0, \zeta_{en}^{n-1}e_2, \ldots, \zeta_{en}e_{n-1}, e_n)$ as a ζ_{en}-eigenvector.

In this section, we will denote by $I(c(e,1,n))$ (resp. $I(c(e,e,n+1))$) the interval in $G(e,1,n)$ (resp. $G(e,e,n+1)$) induced by the generating set of all reflections of $G(e,1,n)$ (resp. $G(e,e,n+1)$). We will also denote by ℓ_1 (resp. ℓ_e) the reflection length in $G(e,1,n)$ (resp. in $G(e,e,n+1)$).

In order to simplify notation, we label the canonical basis of $\mathbb{C}^{n+1}$ as $v_{0'}, v_0, \ldots, v_{n-1}$. We also denote by x_i the coordinate associated to the vector v_i. Under this notation, the reflections of $G(e,e,n+1)$ are the $s_{\zeta,i,j}$ for $\zeta \in \mu_e$ and $0 \leqslant i < j < n$, along with the $s_{\zeta,0',i}$ for $\zeta \in \mu_e$ and $0 \leqslant i < n$ (see Section 6.1.2). First, we are able to describe the of reflections of $G(e,e,n+1)$ which belong to $I(c(e,e,n+1))$.

Lemma 8.3.28. *The reflections of $G(e,e,n+1)$ which belong to $I(c(e,e,n+1))$ are the reflections of the form $s_{1,x,y}$ and $s_{\zeta_e,x,y}$ for $0 \leqslant x < y < n$, and the reflections of the form $s_{\zeta_e^k,0',r}$ for $0 \leqslant r < n$ and $0 \leqslant k < e$.*

Proof. The map $\mathrm{NCP}(e,e,n+1) \to G(e,e,n+1)$ defined in [BC06, Definition 2.3] is known to induce an isomorphism of posets between $\mathrm{NCP}(e,e,n+1)$ and $I(c(e,e,n+1))$ by [BC06, Theorem 4.6]. The images under this map of the atoms of $\mathrm{NCP}(e,e,n+1)$ are then the atoms of the lattice $I(c(e,e,n+1))$, that is, the reflections of $G(e,e,n+1)$ lying in $I(c(e,e,n+1))$. The images of the atoms of $\mathrm{NCP}(e,e,n+1)$ are described in [BC06, Lemma 3.7], and they coincide with the given set. ☐

Using this description of the reflections lying in $I(c(e,e,n+1))$, it is possible to describe an isomorphism between $I(c(e,e,n+1))$ and $\mathrm{NCP}(e,e,n+1)$.

Theorem 8.3.29. *[BC06, Theorem 4.6] Let $e,n \geqslant 2$ be integers. There is a well-defined map $\varphi_1 : I(c(e,e,n+1)) \to \mathrm{NCP}(e,e,n+1)$ given by*

- *If $g \in I(c(e,e,n+1))$ is a reflection, then we set either*

$$\begin{cases} \varphi_1(g) = u_{x,y} & \text{if } g = s_{1,x,y} \text{ for some } 0 \leqslant x < y < n, \\ \varphi_1(g) = u_{y,x+n} & \text{if } g = s_{\zeta_e,x,y} \text{ for some } 0 \leqslant x < y < n, \\ \varphi_1(g) = a_{nk+r} & \text{if } g = s_{\zeta_e^k,0',r} \text{ for some } 0 \leqslant r < n \text{ and } 0 \leqslant k < e. \end{cases}$$

  (this covers all the cases by Lemma 8.3.28).

- *If $g \in I(c(e,e,n+1))$ admits a decomposition $g = t_1 \cdots t_m$ as a product of reflections where $m = \ell_e(g)$, then $\varphi_1(g) := \varphi_1(t_1) \vee \cdots \vee \varphi_1(t_m) \in \mathrm{NCP}(e,e,n+1)$ (it does not depend on the choice of $t_1, \ldots, t_m$).*

Furthermore, the map φ_1 is an isomorphism of germs between $I(c(e,e,n+1))$ and $\mathrm{NCP}(e,e,n+1)$.

Proof. The map φ_1 is well-defined by [BC06, Lemma 4.5], and it is an isomorphism of posets by [BC06, Theorem 4.6]. It remains to show that φ_1 is an isomorphism of germs.

Let $s, t \in \mathrm{NCP}(e, e, n+1)$ such that $s \cdot t$ is defined, we have $t = s\backslash(s \vee t)$. Let $g := \varphi_1^{-1}(s)$ and $h := \varphi_1^{-1}(t)$. By [BC06, Lemma 4.5 and Proposition 2.4], we have $gh = \varphi_1^{-1}(s \vee t)$. Since $s \preccurlyeq s \vee t$ implies $g \preccurlyeq \varphi_1^{-1}(s \vee t)$, we obtain that the product $g \cdot (g^{-1}\varphi_1^{-1}(s \vee t)) = g \cdot h$ is defined and equal to $\varphi_1^{-1}(s \vee t)$.

Conversely, let $g, h, k \in I(c(e, e, n+1))$ be such that $g \cdot h = k$. We have $g \preccurlyeq k$ and thus $\varphi_1(g) \preccurlyeq \varphi_1(k)$. The complement $t := \varphi_1(g)\backslash\varphi_1(k)$ is then such that $\varphi_1(g) \cdot t = \varphi_1(k)$. Again by [BC06, Proposition 2.4], we have $t = \varphi_1(h)$, which finishes the proof. □

As we said, our goal is now to give another (hopefully more convenient) description of the isomorphism φ_1.

We first show that the embedding i sends the interval $I(c(e, 1, n))$ in $I(c(e, e, n+1))$. In order to do so, we show how, for some $g \in I(c(e, 1, n))$, the length $\ell_e(i(g))$ can be computed using $\ell_1(g)$. Recall from Lemma 8.3.12 that all elements $g \in I(c(e, 1, n))$ are such that $\chi(g) \in \{1, \zeta_e\}$.

Lemma 8.3.30. *Let $g \in I(c(e, 1, n))$. If $\chi(g) = 1$, then $\ell_e(i(g)) = \ell_1(g)$. If $\chi(g) = \zeta_e$, then $\ell_e(i(g)) = 1 + \ell_1(g)$.*

Proof. First, for any $g \in G(e, 1, n)$, let $c_1 := \operatorname{codim}_{\mathbb{C}^n} \operatorname{Ker}(g-1)$, and let $c_e := \operatorname{codim}_{\mathbb{C}^{n+1}} \operatorname{Ker}(i(g)-1)$. We have $c_e = c_1$ is $\chi(g) = 1$ and $c_e = c_1 + 1$ otherwise. By [Shi07, Theorem 2.1], we have $\ell_1(g) = c_1$ for all $g \in G(e, 1, n)$. We also have $\ell_e(i(g)) \geqslant c_e$ for all $g \in G(e, 1, n)$ (this is a general fact, see for instance the proof of [Rip10, Proposition 1.3]). Let $g \in I(c(e, 1, n))$. We saw in the proof of Lemma 8.3.12 that $\chi(g) = \zeta_e$ if and only if there is a diagonal reflection $r = r_{\zeta_e, i}$ such that $r \preccurlyeq g$.

If $\chi(g) = 1$, then g can be written as a product of $\ell_1(g)$ nondiagonal reflections. If s is such a reflection, then $i(s)$ is clearly also a reflection in $G(e, e, n+1)$, thus we have $\ell_e(i(g)) \leqslant \ell_1(g)$. Conversely, we have $\ell_e(i(g)) \geqslant c_e = c_1 = \ell_1(g)$, and thus $\ell_e(i(g)) = \ell_1(g)$.

If $\chi(g) = \zeta_e$, then let r be a diagonal reflection such that $r \preccurlyeq g$, and let $g' := r^{-1}g$. Since $i(r)$ can be written as a product of 2 reflections of $G(e, e, n+1)$, we have

$$\ell_e(i(g)) \leqslant \ell_e(i(r)) + \ell_e(i(g')) \geqslant 2 + \ell_1(g') = 1 + \ell_1(g).$$

Conversely, we have $\ell_e(i(g)) \geqslant c_e = c_1 + 1 = \ell_1(g) + 1$, and thus $\ell_e(i(g)) = 1 + \ell_1(g)$ as claimed. □

Proposition 8.3.31. *The embedding i induces an isomorphism of posets between $I(c(e, 1, n))$ and $I(c(e, e, n+1)) \cap i(G(e, 1, n))$, which preserves the complement operation.*

Proof. First, we show that, for $g \in G(e, 1, n)$, we have $i(g) \in I(c(e, e, n+1))$ if and only if $g \in I(c(e, 1, n))$.

Let $g \in G(e, 1, n)$, and let $g' := g^{-1}c(e, 1, n)$, so that $i(g)^{-1}c(e, e, n+1) = i(g')$. Let c_1 and c_1' be the codimensions in $\mathbb{C}^n$ of the fixed spaces of g and g', respectively. Likewise, let c_e and c_e' be the codimensions in $\mathbb{C}^{n+1}$ of the fixed spaces of $i(g)$ and $i(g')$, respectively. By [Shi07, Theorem 2.1], we have $\ell_1(g) = c_1$ and $\ell_1(g') = c_1'$. We have $c_e = c_1$ if $\chi(g) = 1$ and $c_e = c_1 + 1$ otherwise.

Assume that $i(g) \in I(c(e,e,n+1))$. We first show that $\chi(g) \in \{1, \zeta_e\}$. If this is not the case, then $\chi(g') \notin \{1, \zeta_e\}$ since $\chi(c(e,1,n)) = \zeta_e$. Since $i(g) \preccurlyeq c(e,e,n+1)$, we have $\ell_e(i(g)) = c_e$ by Proposition 8.2.3, and likewise, $\ell_e(i(g')) = c'_e$. We then have

$$n = \ell_1(c(e,1,n)) \leqslant \ell_1(g) + \ell_1(g') = c_1 + c'_1 = c_e + c'_e - 2 = \ell_e(c(e,e,n+1)) - 2 = n-1,$$

which is a contradiction. Up to exchanging g and g', we can then assume that $\chi(g) = 1$ and $\chi(g') = \zeta_e$. If $i(g) \in I(c(e,e,n+1))$, then we have

$$n = \ell_1(c(e,1,n)) \leqslant \ell_1(g) + \ell_1(g') = c_1 + c'_1 = c_e + c'_e - 1 = \ell_e(c(e,e,n+1)) - 1 = n.$$

Since the left- and right-hand side of this chain of inequalities are equal, we obtain that $n = \ell_1(g) + \ell_1(g')$ and thus $g \preccurlyeq c(e,1,n)$.

Conversely, assume that $g \in I(c(e,1,n))$. If $\chi(g) = 1$ then $\chi(g^{-1}c(e,1,n)) = \zeta_e$ since $\chi(c(e,1,n)) = \zeta_e$. Conversely, if $\chi(g) = \zeta_e$, then $\chi(g^{-1}c(e,1,n)) = 1$. In any case, we have

$$\begin{aligned}\ell_e(i(g)) + \ell_e(i(g)^{-1}c(e,e,n+1)) &= \ell_e(i(g)) + \ell_e(i(g^{-1}c(e,1,n))) \\ &= \ell_1(g) + \ell_1(g^{-1}c(e,1,n)) + 1 \\ &= \ell_1(c(e,1,n)) + 1 = \ell_e(c(e,e,n+1)),\end{aligned}$$

and thus $i(g) \preccurlyeq c(e,e,n+1)$ and $i(g) \in I(c(e,e,n+1))$.

Now, let $g, h \in I(c(e,1,n))$. If $g \preccurlyeq h$ or if $i(g) \preccurlyeq i(h)$, then $\chi(g^{-1}h) \in \{1, \zeta_e\}$. We then either have $\chi(g) = 1$ or $\chi(h) = \zeta_e$. Assume that $\chi(g) = 1$ and $\chi(h) = 1$. If $g \preccurlyeq h$ in $I(c(e,1,n))$, then $g^{-1}h \in I(c(e,1,n))$ is such that $\chi(g^{-1}h) = 1$. By Lemma 8.3.30, we have

$$\ell_e(i(h)) = \ell_1(h) = \ell_1(g) + \ell_1(g^{-1}h) = \ell_e(i(g)) + \ell_e(i(g)^{-1}i(h)),$$

and thus $i(g) \preccurlyeq i(h)$ in $I(c(e,e,n+1))$. Conversely, if $i(g) \preccurlyeq i(h)$, then $g^{-1}h \in I(c(e,1,n))$ by the first part of the proof. Lemma 8.3.30 again gives that $g \preccurlyeq h$. The other cases are dealt with similarly.

Lastly, the embedding i preserves the complement since it is the restriction of a group morphism: for $g \in I(c(e,1,n))$, the complement of g is $g^{-1}c(e,1,n)$, which is sent to $i(g)^{-1}c(e,e,n+1)$. □

The decomposition of $I(c(e,e,n+1))$ between $i(I(c(e,1,n)))$ and $I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$ will be useful in describing our alternative form of the morphism φ_1. Consider the set

$$E := \{v_{0'}, \zeta_e v_{0'}, \dots, \zeta_e^{e-1}, v_0, v_1, \dots, v_{n-1}, \zeta_e v_0, \dots, \zeta_e v_{n-1}, \dots, \zeta_e^{e-1} v_{n-1}\},$$

along with the sets $E' := E \setminus \{\zeta_e v_{0'}, \dots, \zeta_e^{e-1} v_{0'}\}$ and $E'' := E' \setminus \{v_{0'}\}$. There is a natural bijection $E' \simeq \{0'\} \cup_Z /m\mathbb{Z}$ sending $v_{0'}$ to $0'$ and $\zeta_e^i v_j$ to $in + j$. Since E' is a subset of E, any partition of E can be restricted to a partition of E'.

For $g \in I(c(e,e,n+1))$, we define a partition $\varphi_2(g)$ of $\mu_{en} \cup \{0'\}$ in the following way:

- If $g = i(g') \in i(I(c(e,1,n)))$, then $\varphi_2(g) := (\psi(g'))_*$, where $\psi : I(c(e,1,n)) \to \mathrm{NCP}(e,1,n)$ is the isomorphism described in Proposition 8.3.19, which sends g' to its set of orbits when acting on the set E''.
- If $g \notin i(I(c(e,1,n)))$, then $\varphi_2(g)$ is the restriction to E' of the partition of E given by the orbits of g.

The remainder of this section is devoted to the proof of the following proposition:

Proposition 8.3.32. *For all $g \in I(c(e,e,n+1))$, we have $\varphi_1(g) = \varphi_2(g)$. In particular, the map $\varphi_2 : I(c(e,e,n+1)) \to \mathrm{NCP}(e,e,n+1)$ is an isomorphism of germs. Furthermore, the map $\varphi_2 = \varphi_1$ identifies the elements of $i(I(c(e,1,n))$ (resp. of $I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$ with the symmetric (resp. asymmetric) elements of $\mathrm{NCP}(e,e,n+1)$.*

We begin by showing that φ_1 and φ_2 coincide on the set of reflections of $G(e,e,n+1)$ which belong to $I(c(e,e,n+1))$ (i.e. the atoms of the lattice $I(c(e,e,n+1))$).

Lemma 8.3.33. *Let $s \in I(c(e,e,n+1))$ be a reflection of $G(e,e,n+1)$. We have $\varphi_1(s) = \varphi_2(s)$.*

Proof. We use the description of reflections which belongs to $I(c(e,e,n+1))$ given in Lemma 8.3.28. First, let $0 \leqslant x < y < n$. By construction, the reflection $s_{1,x,y}$, which belongs to $i(G(e,1,n))$, swaps v_x and v_y. Its set of nonsingleton orbits when acting on E'' is then given by

$$\bigsqcup_{\zeta \in \mu_e} \{\zeta v_x, \zeta v_y\},$$

and we obtain that $\varphi_2(s_{1,x,y}) = \varphi_1(s_{1,x,y}) = u_{x,y}$. Likewise, the nonsingleton orbits of $s_{\zeta_e,x,y}$ acting on E'' are given by

$$\bigsqcup_{\zeta \in \mu_e} \{\zeta\zeta_e v_x, \zeta v_y\},$$

and $\varphi_2(s_{\zeta_e,x,y}) = \varphi_1(s_{\zeta_e,x,y}) = u_{y,x+n}$. Then, let $0 \leqslant r < n$ and let $0 \leqslant k < e$. The reflection $s_{\zeta_e^k,0',r}$ does not preserve the line generated by $v_{0'}$ globally, and thus it does not belong to $i(G(e,1,n))$. Its set of nonsingleton orbits when acting on E is given by

$$\bigsqcup_{\zeta \in \mu_e} \{\zeta v_{0'}, \zeta\zeta_e^k v_r\}.$$

The only nonsingleton orbit included in E' is $\{v_{0'}, \zeta_e^k v_r\}$, which is then the unique nonsingleton part of $\varphi_2(s_{\zeta_e^k,0',r})$, which is thus equal to $\varphi_1(s_{\zeta_e^k,0',r}) = a_{nk+r}$. $\square$

By construction, we have a commutative square of poset morphisms

$$\begin{tikzcd} I(c(e,1,n)) \arrow[r, hook, "i"] \arrow[d, "\psi"', "\simeq"] & I(c(e,e,n+1)) \arrow[d, "\varphi_2"] \\ \mathrm{NCP}(e,1,n) \arrow[r, hook, "(-)_*"'] & \mathrm{NCP}(e,e,n+1) \end{tikzcd}$$

which shows that φ_2 restricted to $i(G(e,1,n))$ preserves both meet and join. Since it coincides with φ_1 on the atoms, and since every element of $i(G(e,1,n))$ is the join of the atoms smaller than it, we obtain that φ_2 and φ_1 coincide on $i(G(e,1,n))$. Once we have showed that φ_2 is a bijection, the above commutative square will also prove that φ_2 identifies the elements of $i(I(c(e,1,n))$ (resp. of $I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$ with the symmetric (resp. asymmetric) elements of $\mathrm{NCP}(e,e,n+1)$.

It remains to show that φ_1 and φ_2 coincide on the set $I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$. In order to do so, we first prove an easy lemma on asymmetric elements of $\mathrm{NCP}(e,e,n+1)$:

Lemma 8.3.34. *Let $s \in \mathrm{NCP}(e,e,n+1)$ be an asymmetric element. There is an asymmetric atom a_p such that $a_p \preccurlyeq s$ and such that $a_p \backslash s$ is short symmetric.*

Proof. First, if $p \in [\![0, en-1]\!]$ is such that $a_p \preccurlyeq s$, then $a_{p-n} \not\preccurlyeq s$. Otherwise, we would have $a_p \vee a_{p-n} = (v_p)_* \preccurlyeq s$, which implies that s is long symmetric. We can then consider p such that $a_p \preccurlyeq s$ and such that for all $p' \in [\![0, en-1]\!]$ distinct from p and such that $a_{p'} \preccurlyeq s$, we have $p \preccurlyeq p' \preccurlyeq p+n$.

We claim that $a_p \backslash s = \overline{a_p} \wedge s$ is short symmetric. Since s is asymmetric, $a_p \backslash s$ is either asymmetric or short symmetric, and we just have to show that no asymmetric atom are inferior to $a_p \backslash s$. Let $a_{p'}$ be an asymmetric atom. If $a_{p'} \preccurlyeq \overline{a_p}$, we have $p' \preccurlyeq p \preccurlyeq p'+n$ by construction of $\overline{a_p}$. Since we have $p \preccurlyeq p' \preccurlyeq p+n$ for all p' such that $a_{p'} \preccurlyeq s$, we obtain that $a_p \not\preccurlyeq s$ and $a_p \not\preccurlyeq a_p \backslash s$, which is then short symmetric. ☐

Corollary 8.3.35. *Let $g \in I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$. We have $\varphi_2(g) = \varphi_1(g)$.*

Proof. First, we show that $\varphi_1(g)$ is asymmetric. If $g \in i(I(c(e,1,n)))$, then we know that $\varphi_1(g) = \varphi_2(g)$ is symmetric. Since the cardinality of $i(I(c(e,1,n))$ is equal to that of the set of symmetric elements of $\mathrm{NCP}(e,e,n+1)$, and since φ_1 is a bijection, we obtain that φ_1 sends $I(c(e,e,n+1)) \setminus i(I(c(e,1,n)))$ to asymmetric elements in $\mathrm{NCP}(e,e,n+1)$.

Let $s := \varphi_1(g)$, and let by Lemma 8.3.34 a_p be an asymmetric atom in $\mathrm{NCP}(e,e,n+1)$ such that $a_p \preccurlyeq s$ and $a_p \backslash s$ is short symmetric. By construction, $s = a_p \vee a_p \backslash s$ is obtained from s by replacing the part X of s containing p with $X \sqcup \{0'\}$. Let $p = kn + r$ be the euclidean division of p by n. Since $a_p \backslash s \preccurlyeq \overline{a_p}$ and since $a_p \backslash s$ is symmetric, we have $a_p \backslash s \preccurlyeq (\overline{a_p}^\flat)_*$. The nonsingleton parts of the latter all have the form $\zeta_e^j P$, where $P = \{\zeta_e^k v_r, \dots, \zeta_e^k v_{n-1}, \zeta_e^{k+1} v_0, \dots \zeta_e^{k+1} v_{r-1}\}$.

Since $\varphi_2(g') = a_p \backslash s$, we know that the orbits of g' acting on E' are the parts of $a_p \backslash s$. Let X be the orbit of g' on E containing $\zeta_e^k v_r$. Since $a_p \backslash s \preccurlyeq (\overline{a_p}^\flat)_*$, we have $X \subset P$, and thus the $\zeta_e^j X$ are all disjoint for $i \in [\![0, e-1]\!]$. The orbits of tg' acting on E are the same as those of g', except orbits of the form $\zeta_e^j X$, which are replaced with $\{\zeta_e^j v_{0'}\} \cup \zeta_e^j X$. By construction, we obtain that $\varphi_2(tg') = \varphi_2(g) = s$ as claimed. ☐

Along with the preceding discussion, this finishes the proof of Proposition 8.3.32.

8.4 Support-preservingness

In this section, we study the support-preservingness of dual groups attached to well-generated complex reflection groups. The result is based on a stronger theorem giving a complete description of minimal positive conjugators in dual groups. The proof of this theorem is rather long and uses a case-by-case approach in the irreducible case, in particular making use of noncrossing partition lattices.

Theorem 8.4.1 (Minimal positive conjugators in dual braid monoids). *Let W be a well-generated complex reflection group, and let $x \in M(W)$ have standard parabolic closure $\mathrm{SPC}(x) = G(W)_s$. If a is an atom of $M(W)$, then we either have*

- *a is a left-divisor of $\overline{s}$ in $M(W)$, and $\rho_a(x) = a$ is a minimal positive conjugator.*
- *a is a left-divisor of s in $M(W)$, and $\rho_a(x) \in M(W)_s$.*
- *None of the above, and $\rho_a(x)$ is not a minimal positive conjugator.*

Proving this result in the case of an irreducible group is sufficient since, by Lemma 8.2.10, the dual group of a direct product is the direct product of the dual group of the factors.

Note that if W and W' are isodicriminantal well-generated groups, then they share the same dual structure and we have to prove Theorem 8.4.1 only once. In particular, proving this theorem for the dual groups associated to well-generated groups in the infinite series will also prove it for the dual groups associated to the exceptional groups which are isodiscriminantal to members of the infinite series (this covers in particular all well-generated exceptional complex reflection groups of rank 2, along with G_{25}, G_{26} and G_{32}).

For now, we examine consequences of this theorem on the structure of parabolic subgroups of well-generated braid groups. By following the proof of [GM22, Proposition 5.25], we deduce from Theorem 8.4.1 that elements of well-generated braid groups admit a parabolic closure.

Corollary 8.4.2. *Let W be a well-generated complex reflection group. The shoal of all standard parabolic subgroups of the dual group $(G(W), M(W), \Delta)$ is support-preserving. In particular, any element $x \in B(W)$ admits a parabolic closure* $\mathrm{PC}(x)$*.*

Proof. By Proposition 8.2.24, the isomorphism $B(W) \simeq G(W)$ of Corollary 8.2.18 identifies the collections of parabolic subgroups. The second statement is then an immediate consequence of the first, which implies that elements of $G(W)$ admit a parabolic closure in $G(W)$ [GM22, Theorem 4.31].

By Proposition 5.2.20, it is sufficient to check that, for $x \in M(W)$ and ρ a minimal positive conjugator of x, we have $\mathrm{SPC}(x)^\rho = \mathrm{SPC}(x^\rho)$. Let $G(W)_s := \mathrm{SPC}(x)$. By Theorem 8.4.1, we either have $\rho \in M(W)_s$, or $\rho \preccurlyeq \overline{s}$ is an atom.

If $\rho \preccurlyeq \overline{s}$ is an atom of $M(W)_s$, then for every $t \preccurlyeq s$ (possibly $t = s$), we have $s \succcurlyeq t$ since every simple is balanced. Thus, $t\rho$ is simple, and $\rho \preccurlyeq t\rho$. Thus, ρ conjugates t to some simple $\psi(t)$. If $s \succcurlyeq t$, then $s\rho = \rho\psi(s) \succcurlyeq \rho\psi(t)$, and thus $\rho\psi(t) \preccurlyeq \rho\psi(s)$ and $\psi(t) \preccurlyeq \psi(s)$. Actually, conjugacy by ρ induces a poset isomorphism between the divisors of s and those of $\psi(s)$.

Let $x = s_1 \cdots s_r$ be the greedy normal form of x in $M(W)$. We know that s is the lcm of all the s_i by Lemma 8.2.15. We deduce that $\psi(s)$ is the lcm of all the $\psi(s_i)$, and thus $\mathrm{SPC}(x^\rho) = G(W)_{\psi(s)} = (G(W)_s)^\rho$ as claimed.

Now, assume that $\rho \in G(W)_s$, we have $x^\rho \in G(W)_s$, and thus $\mathrm{SPC}(x^\rho) \subset G(W)_s$. If $s' \in \mathcal{S}$ is such that $\mathrm{SPC}(x^\rho) = G(W)_{s'}$, then we have $s' \preccurlyeq s$. In particular, we have $\ell(s') \leqslant \ell(s)$ (recall that dual braid monoids, as interval monoids, are homogeneous). Moreover, if $\rho \in \mathcal{A}(\overline{s})$, then the element $s' \in \mathcal{S}$ such that $\mathrm{SPC}(x^\rho) = G(W)_{s'}$ is s^ρ, and we have $\ell(s) = \ell(s')$. Thus, conjugating by a minimal positive conjugator cannot increase the length of the simple element giving the standard parabolic closure.

Let $m > 0$ be a big enough integer such that $\Delta^m \in Z(G(W))$, we have $\rho' := \rho^{-1}\Delta^m$ is a conjugator of x^ρ to x, which is positive and can be decomposed as a product of minimal positive conjugators. Since conjugating by a minimal positive conjugator cannot increase the length of the simple element giving the standard parabolic closure, we obtain $\ell(s) \leqslant \ell(s')$, thus $\ell(s) = \ell(s')$ and $s = s'$ since we already know that $s' \preccurlyeq s$. We then have $\mathrm{SPC}(x^\rho) = G(W)_s = \mathrm{SPC}(x) = \mathrm{SPC}(x)^\rho$, which finishes the proof. □

Let W be a well-generated complex reflection group. Since we know that simple elements of $M(W)$ are squarefree by Lemma 8.2.12, support-preservingness of the shoal of all standard parabolic subgroups of $G(W)$ allows us to apply Proposition 5.2.12 to obtain [DMM11, Theorem 1.4] for well-generated complex reflection groups.

Theorem 8.4.3. *Let W be a well-generated complex reflection group. If $U \subset B(W)$ is a finite index subgroup, then we have $Z(U) \subset Z(B(W))$.*

Remark 8.4.4. Let W be a well-generated complex reflection group, and let $\rho \in B(W)$. If the centralizer $U := C_{B(W)}(\rho)$ has finite index in $B(W)$, then $\rho \in Z(U) \subset Z(B(W))$ by the above theorem, and thus $U = B(W)$. In other words, the centralizer of an element in $B(W)$ never has finite index, unless it is equal to the whole group $B(W)$.

Let W be a well-generated complex reflection group. The fact that any element of $G(W)$ admits a parabolic closure was already obtained by Cumplido, Gebhardt, González-Meneses, Wiest in [CGGW19] in the case of spherical Artin groups, and by González-Meneses and Marin [GM22, Theorem 1.1] for other well-generated complex reflection groups. However, they considered different Garside structures (for instance, the Artin-Tits monoid for complexified real reflections groups, and the parachute monoid for groups of the form $G(e, e, n)$). Likewise, the following result, which is a corollary of Theorem 5.2.33 (intersection of parabolic subgroups), was also already obtained by Cumplido, Gebhardt, González-Meneses, Wiest in [CGGW19] in the case of spherical Artin groups, and by González-Meneses and Marin [GM22, Theorem 1.1] for other well-generated complex reflection groups.

Theorem 8.4.5. *Let W be a well-generated complex reflection group. Every intersection of a family of parabolic subgroups of $B(W)$ is a parabolic subgroup of $B(W)$.*

Lastly, a consequence of Proposition 5.2.27 is that we have a system of conjugacy representatives for the parabolic subgroups of dual groups:

Proposition 8.4.6. *[GM22, Proposition 6.2] Let W be a well-generated complex reflection group, and let $\mathcal{S}$ be the set of simple elements in the dual group $(G(W), M(W), \Delta)$ of type W. For $s \in \mathcal{S}$, we define $z_s := s^k$, where k is the smallest positive integer such that s^k is central in $G(W)_s$. The set $\{z_s \mid s \in \mathcal{S}\}$ is a system of conjugacy representatives for the standard parabolic subgroups of $(G(W), M(W), \Delta)$.*

Remark 8.4.7. We keep the notation of the above proposition. Let $s \in \mathcal{S}$, and let c_0 be the image of s in W. By Proposition 8.2.16, we know that $(G(W)_s, M(W)_s s)$ is isomorphic to the dual group associated to c_0 in the parabolic subgroup of W fixing the flat $\mathrm{Ker}(c_0 - 1)$. In particular, we can apply Theorem 8.2.22 (center) to obtain that, if $G(W)_s$ is irreducible, then z_s is the unique positive generator of the center of $G(W)_s$.

8.4.1 Reductions for the proof

In this section, we fix a well-generated complex reflection group W. We denote by $\mathcal{S}$ the set of simple elements in the dual group $(G(W), M(W), \Delta)$, and by $\mathcal{A}$ its set of atoms. We also write ϕ for the Garside automorphism of $(G(W), M(W), \Delta)$. Following [GM22, Definition 4.25], for $s \in \mathcal{S}$, we define the *support* $\mathcal{A}(s)$ of s as the atoms of $M(W)$ which left- (or right-)divide s. These are exactly the atoms of the standard parabolic submonoid $M(W)_s$ attached to s. Notice that, as an element s of $\mathcal{S}$ is the lcm of the elements of $\mathcal{A}(s)$, we have

$$\forall s, t \in \mathcal{S},\ \mathcal{A}(s) \subset \mathcal{A}(t) \Leftrightarrow s \preccurlyeq t.$$

In theory, proving Theorem 8.4.1 requires that we compute minimal positive conjugators for all elements of $M(W)$, which is of course impossible in finite time. However, in [GM22, Section 5], a clever induction argument is used in order to show Theorem 8.4.1 (in the few cases the authors consider here). It essentially boils down to constructing, for each simple $s \in \mathcal{S}$, a convenient enumeration of the atoms in $\mathcal{A} \setminus \mathcal{A}(s)$. In [GM22, Lemma 5.24], the existence of such an enumeration is proved by direct computations on the few groups considered there. We give an abstract version of this procedure in this section.

First, we have a very easy reduction of Theorem 8.4.1:

Lemma 8.4.8. *The following statements are equivalent:*

(i) Theorem 8.4.1 holds for $M(W)$.

(ii) For $x \in M(W)$ with standard parabolic closure $\mathrm{SPC}(x) = G(W)_s$*, and a an atom of $M(W)$ which divides neither s nor $\overline{s}$, the positive conjugator $\rho_a(x)$ is not minimal.*

Proof. $(i) \Rightarrow (ii)$ is simply the third case in the statement of Theorem 8.4.1. For $(ii) \Rightarrow (i)$, Let $x \in M(W)$ be a positive element, and let $a \in \mathcal{A}$.

If $a \preccurlyeq \overline{s}$, then a positively conjugates all the elements of $\mathcal{A}(s)$. Indeed, for $b \in \mathcal{A}(s)$, ba is a simple element, and we have $ba = b \vee a = ax$ by Proposition 8.2.13 for some $x \in M(W)$. Since x is a product of elements of $\mathcal{A}(s)$, we obtain that a is a positive conjugator for x. Since a is an atom, we have $a = \rho_a(x)$ is a minimal positive conjugator and the first case of Theorem 8.4.1 holds.

If $a \in M(W)_s$, then the sequence of converging prefixes $(c_i)_{i \geqslant 0}$, defined by $c_0 = a$ and $c_{i+1} = x \backslash c_i \vee c_i$ belongs to $M(W)_s$, as $M(W)_s$ is stable under $\backslash$ and $\vee$. Since there is some $m \geqslant 0$ such that $c_m = \rho_a(x)$ by Lemma 5.2.21, we obtain that $\rho_a(x) \in M(W)$ and the second case of Theorem 8.4.1. □

In particular, this lemma proves that we need not consider the case where $\mathrm{SPC}(x) = G(W)$, since in this case, condition (ii) is empty: there are no atoms of $M(W)$ which do not divide $s = \Delta$.

If $s \in \mathcal{S}$ is a simple element different from Δ, then the set $\mathcal{A} \setminus \mathcal{A}(s)$ is nonempty. The concept of reachable atoms give an enumeration of (a priori a subset of) $\mathcal{A} \setminus \mathcal{A}(s)$.

Definition 8.4.9 (Reachable atoms). Let $s \in \mathcal{S}$ be a simple element. For $i \geqslant 0$, we define the set of *i-reachable atoms* for s in the following way:

- $R_0(s) := \mathcal{A}(\overline{s})$ is the set of atoms a such that $sa \in \mathcal{S}$.
- For $i \geqslant 1$, $R_i(s)$ is defined by

$$R_i(s) := \{a \in \mathcal{A} \setminus \mathcal{A}(s) \mid \forall b \in \mathcal{A}(s) \setminus \mathcal{A}(a^*),\ \mathcal{A}(b\backslash a) \cap R_{i-1}(s) \neq \varnothing\}.$$

We obtain a sequence $R_0(s) \subset R_1(s) \subset R_2(s) \subset \cdots \subset \mathcal{A} \setminus \mathcal{A}(s)$. The union $\bigcup_{i=0}^{\infty} R_i(s)$ is the set of *reachable atoms* for s. When there is no confusion, we may say that an atom in $\mathcal{A} \setminus \mathcal{A}(s)$ is s-reachable instead of reachable for s.

Let $s \in \mathcal{S}$. For $a \in R_0(s)$ we have $sa \in \mathcal{S}$ by definition, thus $a^* \succcurlyeq s$ and $\mathcal{A}(s) \subset \mathcal{A}(a^*)$. The condition for a to belong to $R_1(s)$ is then empty and $R_0(s) \subset R_1(s)$. The fact that $R_i(s) \subset R_{i+1}(s)$ for $i \geqslant 1$ is then obtained by an immediate induction.

The definition of $R_i(s)$ is hard to understand at first glance, let us detail it a bit more. Let $a \in \mathcal{A} \setminus \mathcal{A}(s)$ be an atom of $M(W)$ not dividing s. For $b \in \mathcal{A}(s)$, we have $ba \in \mathcal{S}$ if and only if $b \in \mathcal{A}(a^*)$. In this case, $b\backslash a = a$ and a conjugates b to another atom of $M(W)$. For

$b \in \mathcal{A}(s) \setminus \mathcal{A}(a^*)$, $b\backslash a \vee a$ is the first converging prefix of the positive conjugator $\rho_a(b)$. Thus, if a is i-reachable for s, we obtain that $b\backslash a \preccurlyeq \rho_a(b)$ is divisible by some $(i-1)$-reachable atom. This will be important in order to use induction arguments in Proposition 8.4.11.

Note that, since $\mathcal{A}$ is finite, the sequence $(R_i(s))_{i\geqslant 0}$ is stationary for every $s \in \mathcal{S}$. Furthermore, for $s \in \mathcal{S}$, an atom $a \in \mathcal{A} \setminus \mathcal{A}(s)$ is reachable for s if and only if, for all $b \in \mathcal{A}(s) \setminus \mathcal{A}(a^*)$, the support of $b\backslash a$ contains a reachable atom.

The notion of reachable atom allows us to imitate the proof of [GM22, Proposition 5.25], which states that if, for all $s \in \mathcal{S}$, the set of reachable atoms for s is equal to $\mathcal{A} \setminus \mathcal{A}(s)$, then Theorem 8.4.1 holds. To simplify statements in the sequel, we introduce an intermediate definition.

Definition 8.4.10 (Full reachability). We say that $\mathcal{S}$ is *fully reachable* if, for all $s \in \mathcal{S}$, the set of reachable atoms for s is equal to $\mathcal{A} \setminus \mathcal{A}(s)$.

Note that the notion of reachability depends both on the lcm of atoms, that is, on the lattice structure of $\mathcal{S}$, and on the right-complement, that is, on the germ structure of $\mathcal{S}$.

Using the notion of full reachability, we can rephrase [GM22, Proposition 5.25] in our context.

Proposition 8.4.11. *If $\mathcal{S}$ is fully reachable, then Theorem 8.4.1 holds for $M(W)$.*

Proof. Following Lemma 8.4.8, we consider $x \in M(W)$ such that $\mathrm{SPC}(x) = G(W)_s \neq G(W)$, along with $a \notin \mathcal{A}(s) \sqcup \mathcal{A}(s^*)$ (the union is disjoint by Lemma 8.2.12). By assumption, there is some $i \geqslant 1$ such that a is i-reachable. We show that $\rho_a(x)$ is not a minimal positive conjugator by induction on i.

First, assume that $i = 1$, and let us write $x = b_1 \cdots b_k$ as a product of atoms. Since $x \in M(W)_s$, which is stable under factor as a standard parabolic submonoid, we obtain that $b_i \in \mathcal{A}(s)$ for all $i \in [\![1, k]\!]$. We claim that $b_j a$ is not simple for some $i \in [\![1, k]\!]$. By Corollary 8.2.15, s is given by the lcm of the b_i. If $b_i a$ is always simple, then (since all simple elements are balanced) we then have

$$sa = (b_1 \vee \cdots \vee b_k)a = (b_1 \vee_L \cdots \vee_L b_k)a = (b_1 a) \vee_L \cdots \vee_L (b_k a),$$

which is a simple element as a right-lcm of simple elements. This contradicts the assumption that $a \notin \mathcal{A}(\overline{s})$.

Let then j be the first index such that $b_j a$ is not simple. We compute the pre-minimal conjugators (see Section 5.2.2), starting with $s_{0,0} = a$. Since $b_i a$ is simple for $i < j$, we have $b_i a = a \vee b_i = a b_i'$ by Lemma 8.2.13 for some atom b_i'. By an immediate induction, we have $s_{0,i} = a$ for $i < j$ and $a = s_{0,j-1}$. Since $b_j \in \mathcal{A}(s)$ and since $b_j a$ is not simple by assumption, there is some $d \in R_0(s) = \mathcal{A}(\overline{s})$ which divides $b_j\backslash a = s_{0,j}$. Since d left-divides $\overline{s}$, it positively conjugates all the elements of $\mathcal{A}(s)$, and thus all subsequent pre-minimal conjugators will admit d as a prefix. Therefore, d left-divides the first converging prefix c_1 of $\rho_a(x)$. Since $c_1 \preccurlyeq \rho_a(x)$ by construction, we have $d \preccurlyeq \rho_a(x)$, which proves that $\rho_a(x)$ is not minimal since $d = \rho_d(x)$ is already a positive conjugator of x (see Lemma 8.4.8).

Now, suppose that $i > 1$ and that the claim holds for smaller values of i. Let $a \in R_i(s)\backslash R_{i-1}(s)$. Using the above argument, we know that some pre-minimal conjugator for a will admit some $d \in R_{i-1}(s)$ as a prefix. Again we obtain that all subsequent pre-minimal conjugators will left-divided by an element of $R_{i-1}(s)$. Hence, some converging prefix for $\rho_a(x)$ admits some $d' \in R_{i-1}(s)$ as a left-divisor. It follows that $\rho_{d'}(x) \preccurlyeq \rho_a(x)$, and $\rho_a(x)$ is not minimal (either

$d' \in \mathcal{A}(\overline{s})$ and $\rho_{d'}(x) = d'$ as before, or $d' \in R_i(s) \setminus R_0(s)$, and we already know that $\rho_{d'}(s)$ is not minimal). $\square$

We are now able to prove Theorem 8.4.1 by showing that $\mathcal{S}$ is fully reachable. For a given group W, this is checkable in finite time, provided that we know how to construct the set of reachable atoms for a given simple $s \in \mathcal{S}$. We will give in Section 8.4.2 an algorithm to do so.

For now, we give some intermediate result which will help us simplify proofs of full reachability in the sequel. First, since the Garside automorphism ϕ induces an automorphism of the germ $\mathcal{S}$, it induces an automorphism of lattices which preserves the right-complement. In particular, for $s \in \mathcal{S}$, the image under ϕ of the set $R_i(s)$ of i-reachable atoms for s is the set $R_i(\phi(s))$ of i-reachable atoms for $\phi(s)$. We deduce the following lemma, which allows us to check full reachability by only considering representatives of ϕ-orbits.

Lemma 8.4.12. *Let $s \in \mathcal{S}$. If every element of $\mathcal{A} \setminus \mathcal{A}(s)$ is reachable for s, then every element of $\mathcal{A} \setminus \mathcal{A}(\phi(s))$ is reachable for $\phi(s)$.*

The following result is less immediate, but it will prove crucial in order to do induction arguments in the sequel, using standard parabolic submonoids.

Lemma 8.4.13. *Let $s, s' \in \mathcal{S}$ be such that $s \preccurlyeq s'$. If every element in $\mathcal{A}(s') \setminus \mathcal{A}(s)$ is reachable for s, and if every element of $\mathcal{A} \setminus \mathcal{A}(s')$ is reachable for s', then every element of $\mathcal{A} \setminus \mathcal{A}(s)$ is reachable for s.*

Proof. Let $a \in \mathcal{A} \setminus \mathcal{A}(s)$. If $a \preccurlyeq s'$, then $a \in \mathcal{A}(s') \setminus \mathcal{A}(s)$ and a is reachable for s. If $a \not\preccurlyeq s'$, then $a \in \mathcal{A} \setminus \mathcal{A}(s')$ and a is reachable for s'. It remains to show that elements of $\mathcal{A} \setminus \mathcal{A}(s')$ are then also reachable for s.

Let $R_i(s)$ (resp. $R_i(s')$) denote the set of i-reachable atoms for s (resp. for s'). We show by induction that $R_i(s') \subset R_i(s)$ for all $i \geqslant 0$. First, $R_0(s') = \mathcal{A}(\overline{s'}) \subset \mathcal{A}(\overline{s}) = R_0(s)$ since $s \preccurlyeq s'$. Then, assume that $R_i(s') \subset R_i(s)$ for some $i \geqslant 0$ and let $a \in R_{i+1}(s')$. If $b \preccurlyeq s$ is an atom such that ba is not simple, then we also have $b \preccurlyeq s'$, and thus $b \backslash a$ is divided by some atom in $R_i(s')$ by definition of $R_{i+1}(s')$. Since $R_i(s') \subset R_{i+1}(s')$ by induction hypothesis, we obtain that $a \in R_{i+1}(s)$ by definition.

By assumption, there is some k such that $R_k(s') = \mathcal{A} \setminus \mathcal{A}(s')$. We then have that $\mathcal{A} \setminus \mathcal{A}(s') \subset R_k(s)$ consists of reachable atoms for s. $\square$

Proposition 8.4.14. *For $s \in \mathcal{S}$, let c_0 denote the image of s in W. Let also W_0 denote the parabolic subgroup of W associated to c_0, and let T_0 denote its set of reflections. Assume that*

- *For every maximal element $s \in \mathcal{S} \setminus \{\Delta\}$, the germ $I(c_0)_{T_0} = \mathcal{S}_s$ is fully reachable.*
- *For every maximal element $s \in \mathcal{S} \setminus \{\Delta\}$, the elements of $\mathcal{A} \setminus \mathcal{A}(s)$ are all s-reachable.*

Then $\mathcal{S}$ is fully reachable.

Proof. Let $s \in \mathcal{S}$. By Proposition 8.2.16(a) and (b), we have an inclusion map $I(c_0)_{T_0} \to I(c)_T = \mathcal{S}$ which is also a germ morphism. This inclusion map identifies $I(c_0)_{T_0}$ with the parabolic subgerm $\mathcal{S}_s$. In particular, if $s \in \mathcal{S} \setminus \{\Delta\}$ is maximal, then the parabolic subgerm $\mathcal{S}_s$ is fully reachable by assumption.

Let now $s \in \mathcal{S}$, and let $s' \in \mathcal{S} \setminus \{\Delta\}$ be maximal and such that $s \preccurlyeq s'$. The first assumption proves that every element in $\mathcal{A}(s') \setminus \mathcal{A}(s)$ is reachable for s. The second assumption proves that every element of $\mathcal{A} \setminus \mathcal{A}(s')$ is reachable for s'. We conclude by Lemma 8.4.13 that every element of $\mathcal{A} \setminus \mathcal{A}(s)$ is reachable for s. Since this is true for all $s \in \mathcal{S}$, we obtain that $\mathcal{S}$ is fully reachable. □

Remark 8.4.15. Of course, if W_1, W_2 are two well-generated complex braid groups, and if $\mathcal{S}_1, \mathcal{S}_2$ are the respective germs of simples of their associated dual braid monoids, then $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is fully reachable if both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fully reachable. In particular, we can replace the first assumption in Proposition 8.4.14 by "for every maximal element $s \in \mathcal{S} \setminus \{\Delta\}$ such that W_0 is irreducible, the germ $I(c_0)_{T_0} = \mathcal{S}_s$ is fully reachable".

Lastly, the following lemma will help us to deal with cases where a simple element can be decomposed as a product of two commuting simple elements.

Lemma 8.4.16. *Let $s \in \mathcal{S}$ be written as $s = s_1 s_2 = s_2 s_1$ for some nontrivial $s_1, s_2 \in \mathcal{S}$. If the parabolic subgerm $\mathcal{S}_{\overline{s_1}}$ of divisors of $\overline{s_1}$ is fully reachable, then every atom of $\mathcal{A}(\overline{s_1}) \setminus \mathcal{A}(s_2)$ is reachable for s.*

Proof. First, note that $s_1 s_2 \in \mathcal{S}$ implies that $s_2 \preccurlyeq \overline{s_1}$. In particular we have $\mathcal{A}(s_2) \subset \mathcal{A}(\overline{s_1})$ (in fact, $\mathcal{A}(s_2) = \mathcal{A}(s) \cap \mathcal{A}(\overline{s_1})$).

By assumption, every atom in $\mathcal{A}(\overline{s_1})$ which does not divide s_2 in $\mathcal{S}_{\overline{s_1}}$ is reachable for s_2. In other words, if $\widetilde{R}_i$ denotes the set of i-reachable atoms for s_2 in $\mathcal{S}_{\overline{s_1}}$, then we have

$$\mathcal{A}(\overline{s_2}) \cap \mathcal{A}(\overline{s_1}) = \widetilde{R}_0 \subset \widetilde{R}_1 \subset \cdots \subset \widetilde{R}_k = \mathcal{A}(\overline{s_1}) \setminus \mathcal{A}(s_2).$$

for some integer $k \geqslant 0$. We show by induction on i that the elements of $\widetilde{R}_i$ are s-reachable in $\mathcal{S}$. First, elements of $\widetilde{R}_0 = \mathcal{A}(\overline{s_1}) \cap \mathcal{A}(\overline{s_2}) = \mathcal{A}(\overline{s})$ are reachable by definition. Then, assume that the elements of $\widetilde{R}_i$ are s-reachable for some integer i, and let $a \in \widetilde{R}_{i+1}$. Let $b \in \mathcal{A}(s)$. Since s_1 and s_2 commute, we have $\mathcal{A}(s_1 s_2) = \mathcal{A}(s_1) \sqcup \mathcal{A}(s_2)$ (the union is disjoint by Lemma 8.2.12). If $b \in \mathcal{A}(s_1)$, then ba is simple since $a \in \mathcal{A}(\overline{s_1})$ by definition. If $b \in \mathcal{A}(s_2)$ and if ba is not simple, then the complement $b \backslash a$ (which is the same in $\mathcal{S}_{\overline{s_1}}$ and in $\mathcal{S}$) divided by some $d \in \widetilde{R}_i$, which is s-reachable by induction hypothesis. We obtain that $\widetilde{R}_k = \mathcal{A}(\overline{s_1}) \setminus \mathcal{A}(s_2)$ is s-reachable, which is what we wanted to show. □

We saw in Proposition 8.2.16 that standard parabolic subgroups of dual groups are again dual groups. The following proposition states that we can prove Theorem 8.4.1 for the dual group of type W by embedding it as a standard parabolic subgroups in another dual group which we know that Theorem 8.4.1 holds.

Proposition 8.4.17. *Let $s \in \mathcal{S}$ have image c_0 in W, and let W_0 be the parabolic subgroup of W attached to the flat* $\mathrm{Ker}(c_0 - 1)$. *If Theorem 8.4.1 holds for $M(W)$, then it also holds for $M(W)_s \simeq M(W_0, c_0)$*

Proof. If $\mathrm{SPC}(x)$ is the standard parabolic closure of x in $G(W)$, then $\mathrm{SPC}(x) \cap G(W)_s$ is a standard parabolic subgroup of $G(W)$, which also contains s by definition. We then have $\mathrm{SPC}(x) \subset G(W)_s$. In other words, if $\mathrm{SPC}(x) = G(W)_t$, then we have $t \preccurlyeq s$.

Let $a \in M(W)_s$ be an atom, since the embedding $M(W)_s \subset M(W)$ preserves lcms and complements, the converging prefixes of $\rho_a(x)$ computed in $M(W)_s$ or in $M(W)$ are equal, and thus the minimal positive conjugators of x associated to a, computed in $M(W)_s$ or in $M(W)$, are equal, and we denote them both by $\rho_a(x)$. Let a be an atom of $M(W)_s$ (in particular, it is an atom of $M(W)$), if $a \not\preccurlyeq t$ in $M(W)_s$, then $a \not\preccurlyeq t$ in $M(W)$ by Proposition 5.1.10. Since $a \preccurlyeq s$, we have $a \preccurlyeq \overline{t}$ if and only if $a \preccurlyeq \overline{t} \wedge s$, which is the right-complement of t in s. Thus, if a divides neither t nor its right-complement in s, then a divides neither t nor its right-complement in Δ. In this case, we know by assumption that $\rho_a(x)$ is not minimal in $M(W)$: there is some strict left-divisor ρ of $\rho_a(x)$ which conjugates x to a positive element. By Proposition 5.1.10, ρ belongs to $M(W)_s$, and it is a positive conjugator for x, thus $\rho_a(x)$ is not minimal and we conclude by Lemma 8.4.8. □

In the infinite series, we deduce that if Theorem 8.4.1 holds for instance for the dual braid monoid associated to $G(e,e,n)$, then it also holds for $G(e,e,m)$ with $m < n$. Of course this is not very helpful, as we would like to prove for Theorem 8.4.1 for every irreducible complex reflection group, and not to be bounded by the rank n. Among exceptional groups, we have embeddings (as parabolic subgroups)

$$G_{23} \hookrightarrow G_{30}, \quad G_{33} \hookrightarrow G_{34}, \quad G_{35}, G_{36} \hookrightarrow G_{37},$$

which means that proving Theorem 8.4.1 for G_{30}, G_{34}, G_{37} also proves it for $G_{23}, G_{33}, G_{35}, G_{36}$.

8.4.2 Exceptional well-generated groups

In this section, we fix a well-generated complex reflection group W, and we keep the notation from Section 8.4.1.

By Proposition 8.4.11, we can prove Theorem 8.4.1 by proving that the germ $\mathcal{S}$ of simple elements is fully reachable. In practice, this can be done explicitly using the algorithm below, which computes the sets of reachable atoms for a simple $s \in \mathcal{S}$. We can then prove Theorem 8.4.1 for a given dual braid group directly by proving that, for all $s \in \mathcal{S}$, the set of reachable atoms for s is equal to $\mathcal{A} \setminus \mathcal{A}(s)$.

Input: A simple element $s \in \mathcal{S}$.
Output: **true** If all atoms of $\mathcal{A} \setminus \mathcal{A}(s)$ are reachable. **false** otherwise.
compute $R_0 := \{a \in \mathcal{A} \mid a \preccurlyeq \overline{s}\} = \mathcal{A}(\overline{s})$
compute $\mathcal{A}(s) := \{a \in A \mid a \preccurlyeq s\}$
put $R_\bullet := \varnothing$
put $R_+ := R_0$
while $R_\bullet \subsetneq R_\bullet \cup R_+$ and $R_\bullet \cup R_+ \cup \mathcal{A}(s) \subsetneq \mathcal{A}$ **do**
  put $R_\bullet := R_\bullet \cup R_+$
  put $R_+ := \varnothing$
  for $a \in \mathcal{A} \setminus (R_\bullet \cup \mathcal{A}(s))$ **do**
    put $X := \{b \in \mathcal{A}(s) \mid ba \notin \mathcal{S}\}$
    if $\forall b \in X,\ \mathcal{A}(b \backslash a) \cap R_\bullet = \varnothing$ **then**
      add a to R_+
    end if
  end for
end while
if $R_\bullet \cup R_+ \cup \mathcal{A}(s) = \mathcal{A}$ **then**
  return **true**
else

return false
end if

Note that this algorithm can theoretically be used on every well-generated complex reflection group W. However, it is of course impossible to run it in finite time on all the well-generated groups in the infinite series of irreducible complex reflection groups.

Furthermore, the reductions we gave above prove that we only need to use this method on the exceptional groups $G_{24}, G_{27}, G_{28}, G_{29}, G_{30}, G_{34}, G_{37}$.

Remark 8.4.18. By [Bes15, Section 13], for $W = G_{37}$, we have $|\mathcal{S}| = 25080$. Rather than apply the above algorithm on all the simple elements, we can use Lemma 8.4.12, and only apply the above algorithm to representatives of the ϕ-orbits (there are 1680 such orbits). Furthermore, if we already know that the parabolic subgerms of $\mathcal{S}$ are fully reachable, then we can use Proposition 8.4.14 and we only have to use the algorithm on the 120 maximal proper simple elements (or even simpler, on representatives of the 8 ϕ-orbits of such elements).

8.4.3 The case of $\widetilde{G}(1,1,n)$

We fix an integer $n \geqslant 1$. This section is devoted to the proof of Theorem 8.4.1 in the case where $W = \widetilde{G}(1,1,n)$. As we said in Section 9.2.2, the dual braid monoid associated to W is equal to the dual braid monoid associated to $G(1,1,n)$, and we can restrict our attention to the later.

We will use the results of Section 8.3.1 a lot. Recall from Proposition 8.3.3, we know that the germ of simple elements of $M(G(1,1,n))$ is isomorphic to $\mathrm{NCP}(1,1,n)$, and we can restrict our attention to this latter germ. Using the germ structure on $\mathrm{NCP}(1,1,n)$, we see the order $\preccurlyeq$ as a divisibility order (in particular, we talk about lcms instead of joints,...).

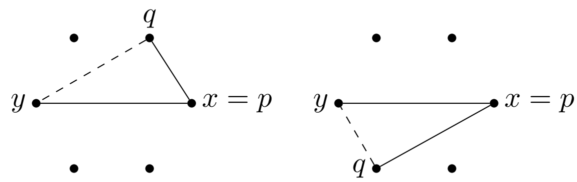
Computation of lcms

We begin by computing the lcms of the atoms of $\mathrm{NCP}(1,1,n)$, along with the associated complements. By Lemma 8.3.5, the atoms of $\mathrm{NCP}(1,1,n)$ all have the form $r_{x,y}$ for $x \neq y \in \mathbb{Z}/n\mathbb{Z}$. Let then $p \neq y, x \neq y \in \mathbb{Z}/n\mathbb{Z}$. If $\{p,q\} = \{x,y\}$, then $r_{p,q,} = r_{x,y} = r_{p,q} \vee r_{x,y}$.

If $\#(\{p,q\} \cap \{x,y\}) = 1$ (where $\#(-)$ denotes the cardinality of a set), then up to exchanging x with y and p with q, we can assume that $p = x$. We then either have:

- $x = p \lhd q \lhd y$, in which case the lcm is given by $r_{x,y} \vee r_{p,q} = r_{p,q} r_{q,y} = r_{q,y} r_{x,y} = r_{x,y} r_{p,q}$.
- $x = p \lhd y \lhd q$, in which case the lcm is given by $r_{x,y} \vee r_{p,q} = r_{p,q} r_{x,y} = r_{x,y} r_{y,q} = r_{y,q} r_{p,q}$.

These two situations are summarized in the diagrams below, where $n = 6$.



Lastly, if $\{x,y\}$ and $\{p,q\}$ are disjoints, then we can assume (up to exchanging p and q) that $x \lhd p \lhd q$. We then either have:

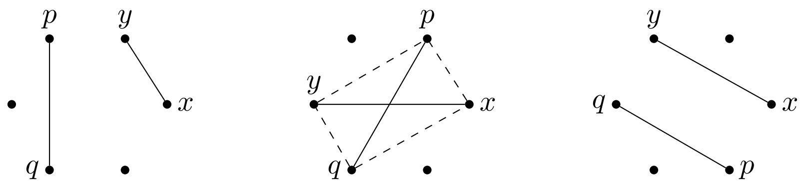
- $x \lhd y \lhd p \lhd q$, in which case $r_{x,y}$ and $r_{p,q}$ commute with one another.

- $x \lhd p \lhd y \lhd q$, in which case the lcm is given by $r_{x,y} \vee r_{p,q} = r_{x,p} r_{p,y} r_{y,q}$. The right-complements are given by

$$r_{p,q} \backslash r_{x,y} = r_{p,y} r_{q,x} = r_{q,x} r_{p,y} \text{ and } r_{x,y} \backslash r_{p,q} = r_{x,p} r_{y,q} = r_{y,q} r_{x,p}.$$

- $x \lhd p \lhd q \lhd y$, in which case $r_{x,y}$ and $r_{p,q}$ commute with one another.

These three situations are summarized in the diagrams below, where $n = 6$.



Proof of Theorem 8.4.1

We are going to show that the germ $\mathrm{NCP}(1,1,n)$ is fully reachable, which is sufficient to prove Theorem 8.4.1 by Proposition 8.4.11. We do so by induction on the integer n.

The case $n = 2$ is obvious as $\mathrm{NCP}(1,1,2) = \{1, \Delta\}$. The case $n = 3$ is also quite easy: we have $\mathrm{NCP}(1,1,3) = \{1, a, b, c, \Delta\}$ with $ab = bc = ca = \Delta$. The only proper simples are a, b, c, which are in the same ϕ-orbit. We have $R_0(a) = \{b\}$, and $R_1(a) = \{c\}$, thus all atoms are reachable for a, and $\mathrm{NCP}(1,1,3)$ is fully reachable.

Now, let $n \geqslant 3$ and assume that $\mathrm{NCP}(1,1,m)$ is fully reachable for $m < n$. We will show that $\mathrm{NCP}(1,1,n)$ is fully reachable using Proposition 8.4.14.

First, let $s \in \mathrm{NCP}(1,1,n)$. Identifying the nonsingleton parts of s with regular polygons yields an identification between the parabolic subgerm $\mathrm{NCP}(1,1,n)_s$ and a product of the form

$$\mathrm{NCP}(1,1,m_1) \times \cdots \times \mathrm{NCP}(1,1,m_r),$$

with $m_1 + \cdots + m_r \leqslant n$. By induction hypothesis (and by Remark 8.4.15), we know that such a product of germs is fully reachable. In particular, the first assumption of Proposition 8.4.14 is satisfied.

Now, we need to show that, if $s \in \mathrm{NCP}(1,1,n) \setminus \{\Delta\}$ is maximal, then every atom in $\mathcal{A} \setminus \mathcal{A}(s)$ is reachable for s. If $s \in \mathrm{NCP}(1,1,n) \setminus \{\Delta\}$ is maximal, then s^* is an atom of $\mathrm{NCP}(1,1,n)$. Since $s = \overline{s^*}$, we have that maximal proper elements are the complements in Δ of the atoms.

Let $r = r_{\ell,k}$ be an atom of $\mathrm{NCP}(1,1,n)$. Since Δ acts on noncrossing partitions by a rotation (of angle $\frac{2\pi}{n}$), we can rotate $r_{\ell,k}$ and assume that $0 = \ell \lhd k \lhd \lfloor \frac{n}{2} \rfloor$ (this is sufficient by Lemma 8.4.12). By [BC06, Definition 1.6], the right-complement of $r_{0,k}$ is easily computable. It is equal to the noncrossing partition s_k whose parts are $[\![0, k-1]\!]$ and $[\![k, n-1]\!]$.

We only have to show that, for all $k \in [\![1, \lfloor \frac{n}{2} \rfloor]\!]$, all elements of $\mathcal{A} \setminus \mathcal{A}(s_k)$ are reachable for s_k. We saw that $s_k = \{[\![0, k-1]\!], [\![k, n-1]\!]\}$, thus an atom $r_{x,y}$ does not belong to $\mathcal{A}(s_k)$ if and only if we have $x \in [\![0, k-1]\!]$ and $y \in [\![k, n-1]\!]$ (up to exchanging x and y). For such an atom $r_{x,y}$, we can define

$$d_1(x,y) := \#(\{z \in [\![0, k-1]\!] \mid x \lhd z \lhd k\}) \text{ and } d_2(x,y) := \#(\{z \in [\![k, n-1]\!] \mid y \lhd z \lhd 0\}).$$

By construction, $d_1(x,y) = 0$ if and only if $x = k-1$, and $d_2(x,y) = n-1$ if and only if $y = n-1$. In particular, the only atom $r_{x,y}$ with $d_1(x,y) = 0 = d_2(x,y)$ is $r_{n-1,k-1} = \overline{s_k}$, which is reachable. We then prove that every element of $\mathcal{A} \setminus \mathcal{A}(s_k)$ is reachable for s_k by an induction using d_1 and d_2.

Lemma 8.4.19. *Let $k \in [\![1, \lfloor \frac{n}{2} \rfloor]\!]$. All elements of $\mathcal{A} \setminus \mathcal{A}(s_k)$ are reachable for s_k.*

Proof. We order the atoms in $\mathcal{A} \setminus \mathcal{A}(s_k)$ with the lexicographic order on d_1 and d_2. That is, for $r_{x,y}, r_{x',y'} \in \mathcal{A} \setminus \mathcal{A}(s_k)$, we have $r_{x,y} \leqslant r_{x',y'}$ if either $d_1(x,y) \leqslant d_1(x',y')$ or if $d_1(x,y) = d_1(x',y')$ and $d_2(x,y) \leqslant d_2(x',y')$. This induces a well-ordering of $\mathcal{A} \setminus \mathcal{A}(s_k)$, on which we can perform an induction. First, the only atom $r_{x,y}$ with $d_1(x,y) = 0 = d_2(x,y)$ is $r_{n-1,k-1} = \overline{s_k}$, which is reachable by definition.

Now, let $r_{x,y} \in \mathcal{A} \setminus \mathcal{A}(s_k)$, and assume that all atoms in $\mathcal{A} \setminus \mathcal{A}(s_k)$ which are strictly lower than $r_{x,y}$ in our ordering are reachable. Let $b = r_{p,q} \in \mathcal{A}(s_k)$.

We first assume that $1 \trianglelefteq p, q \triangleleft k$. If $\#(\{p,q\} \cap \{x,y\}) = 1$, then $\{p,q\} \cap \{x,y\} = \{x\}$ since $y \notin [\![1, k-1]\!]$. Up to exchanging p and q, we can assume that $p = x$. In this case, we know that $br_{x,y}$ is not simple if and only if $p = x \triangleleft q \triangleleft y$, in which case $r_{p,q} \backslash r_{x,y} = r_{q,y}$. If $\{p,q\}$ and $\{x,y\}$ are disjoint, then $br_{x,y}$ is not simple if and only if $x \triangleleft q \triangleleft y \triangleleft p$ (up to exchanging p and q). In this case, $r_{p,q} \backslash r_{x,y}$ is divisible by $r_{q,y}$. In both cases, we have $x \triangleleft q \triangleleft y$, which forces q to be in $[\![x, k-1]\!]$. We then have $d_1(q,y) < d_1(x,y)$ and $r_{q,y} \preccurlyeq r_{p,q} \backslash r_{x,y}$ is reachable by induction hypothesis.

Now, assume that $k \trianglelefteq p, q \triangleleft n$. If $\#(\{p,q\} \cap \{x,y\}) = 1$, then $\{p,q\} \cap \{x,y\} = \{y\}$ since $x \notin [\![k, n-1]\!]$. Up to exchanging p and q, we can assume that $p = y$. In this case, we know that $br_{x,y}$ is not simple if and only if $p = y \triangleleft q \triangleleft x$, in which case $r_{p,q} \backslash r_{x,y} = r_{q,x}$. If $\{p,q\}$ and $\{x,y\}$ are disjoint, then $br_{x,y}$ is not simple if and only if $x \triangleleft p \triangleleft y \triangleleft q$ (up to exchanging p and q). In this case, $r_{p,q} \backslash r_{x,y}$ is divisible by $r_{q,x}$. In both cases, we have $y \triangleleft q \triangleleft x$, which forces q to be in $[\![y, n-1]\!]$. We then have $d_1(x,q) = d_1(x,q)$ and $d_2(x,q) < d_2(x,q)$, and thus $r_{q,y} \preccurlyeq r_{p,q} \backslash r_{x,y}$ is reachable by induction hypothesis.

In every case, we have that $r_{p,q} \backslash r_{x,y}$ is divisible by a reachable atom, thus $r_{x,y}$ is reachable and we have the desired result. ☐

As we said, Lemma 8.4.19 is sufficient to prove that $\mathrm{NCP}(1,1,n)$ is fully reachable by Proposition 8.4.14 and Lemma 8.4.12. In turn this is enough to prove that Theorem 8.4.1 for the dual braid monoid attached to $G(1,1,n)$ by Proposition 8.4.11.

8.4.4 The case of $G(r,1,n)$

We fix two integers $r, n \geqslant 2$ and we set $m := rn$. This short section is devoted to the proof of Theorem 8.4.1 in the case where $W = G(r,1,n)$. Having Theorem 8.4.1 for $W = G(1,1,m)$ is actually sufficient to show Theorem 8.4.1 for $W = G(r,1,n)$.

Proposition 8.4.20. *Theorem 8.4.1 holds for the dual group of type $G(r,1,n)$.*

Proof. Let (G, M, Δ) be the dual group of type $G(1,1,m)$. Let $\mathcal{S}$ denote its set of simple elements, and let ϕ be its Garside automorphism. Let also $M' := M^{\phi^n}$, $G' := G^{\phi^n}$ and $\mathcal{S}' := \mathcal{S}^{\phi^n}$.

By Proposition 8.3.18, the monoid M' is isomorphic to the dual braid monoid of type $G(r,1,n)$ in a way which preserves the germs of simple elements. Thus it is sufficient to show that Theorem 8.4.1 holds for M'.

Let $x \in M'$ have greedy normal form $x = s_1 \cdots s_r$ in M'. By Theorem 4.1.11 (Garside groupoid of fixed points), we know that $s_1 \cdots s_r$ is also the greedy normal form of x in M. By Corollary 8.2.15, the standard parabolic closure of x in G is given by G_s where $s = s_1 \vee \cdots \vee s_r$. Likewise (since M' is also a dual braid monoid by Proposition 8.3.18), the standard parabolic closure of x in G' is given by $G'_{s'}$, where $s' = s_1 \vee \cdots \vee s_r$ (where the lcm is taken in M'). By Lemma 4.1.9, we obtain that $s = s'$.

Now, let $a \in \mathcal{S}'$ be an atom. By Lemma 4.1.10 (atoms in category of fixed points), we can write $a = \alpha^{\#} = \bigvee_{i \geqslant 0} \phi^{qi}(\alpha)$ for some atom α of M. By Lemma 4.1.8 (adjointness), we have $a \preccurlyeq s$ if and only if $\alpha \preccurlyeq s$ and $a \preccurlyeq \overline{s}$ if and only if $\alpha \preccurlyeq \overline{s}$. Assume that we have neither $a \preccurlyeq s$ nor $a \preccurlyeq \overline{s}$. In this case, by Theorem 8.4.1 applied to M, the positive conjugator $\rho_\alpha(x)$ of x in G is not minimal. It is then a strict right-multiple of some minimal positive conjugator ρ in G. We show that $\rho^{\#} := \bigvee_{i \geqslant 0} \phi^{iq}(\rho)$ is a positive conjugator for x. For $i \geqslant 0$, we have

$$\phi^{iq}(\rho) \preccurlyeq \phi^{iq}(x\rho) = x\phi^{iq}(\rho) \preccurlyeq x \bigvee_{i \geqslant 0} \phi^{iq}(\rho),$$

since $\phi^q(x) = x$ by definition. We deduce that $\rho^{\#} \preccurlyeq x\rho^{\#}$, that is $\rho^{\#}$ is a positive conjugator of x as claimed. Now, applying Theorem 8.4.1 in M proves that we either have

- $\rho \in M_s$, in which case $\rho^{\#} \in M'_s$ as a lcm of elements of M'_s. Since $a \not\preccurlyeq s$, we have $\rho_a(x) \notin M_s$, and thus $\rho^{\#}$ is a strict right divisor of $\rho_a(x)$, which is not a minimal positive conjugator.
- $\rho \preccurlyeq \overline{s}$ is an atom, in which case $\rho^{\#} \preccurlyeq \overline{s}$ as a lcm of divisors of $\overline{s}$. Again, since $a \not\preccurlyeq \overline{s}$ by assumption, having $a \preccurlyeq \rho_a(x)$ and $\rho^{\#} \preccurlyeq \rho_a(x)$ implies that $\rho^{\#} \neq \rho_a(x)$, and the latter is not a minimal positive conjugator.

We conclude by Lemma 8.4.8. □

8.4.5 The case of $G(e,e,n+1)$

We fix two integers $e \geqslant 2, n \geqslant 1$. This section is devoted to the proof of Theorem 8.4.1 in the case where $W = G(e,e,n+1)$.

We will use the results of Section 8.3.3 a lot. Recall from Proposition 8.3.32, we know that the germ of simple elements of $M(G(e,e,n+1))$ is isomorphic to $\mathrm{NCP}(e,e,n+1)$, and we can restrict our attention to this latter germ. Using the germ structure on $\mathrm{NCP}(e,e,n+1)$, we see the order $\preccurlyeq$ as a divisibility order (in particular, we talk about lcms instead of joints,...).

As in the case of $\mathrm{NCP}(1,1,n)$, we start by computing the lcms of the atoms of $\mathrm{NCP}(e,e,n+1)$, along with the associated complements. However, in the case of $\mathrm{NCP}(e,e,n+1)$, we have many more cases to consider. By Lemma 8.3.25, the atoms of $\mathrm{NCP}(e,e,n+1)$ are separated in two groups. On the one hand the symmetric atoms of the form $u_{p,q}$ for $p \lhd q \lhd p + nin\mathbb{Z}/en\mathbb{Z}$, and on the other hand the asymmetric atoms a_p for $p \in \mathbb{Z}/en\mathbb{Z}$.

Lcm of two asymmetric atoms

Let $x \neq p \in \mathbb{Z}/en\mathbb{Z}$, and let us consider the asymmetric atoms a_x and a_p.

- $x \lhd p \lhd x+n$, in which case the lcm is given by

$$a_x \vee a_p = a_p a_x = a_x u_{x,p} = u_{x,p} a_p.$$

- $p \in \{x+n, x+2n, \dots, x+(e-1)n\}$, in which case the lcm is given by $a_x \vee a_p = a_{x+n} a_0$. The right-complements are given by

$$a_p \backslash a_x = a_{p-n} \text{ and } a_x \backslash a_p = a_{x-n}.$$

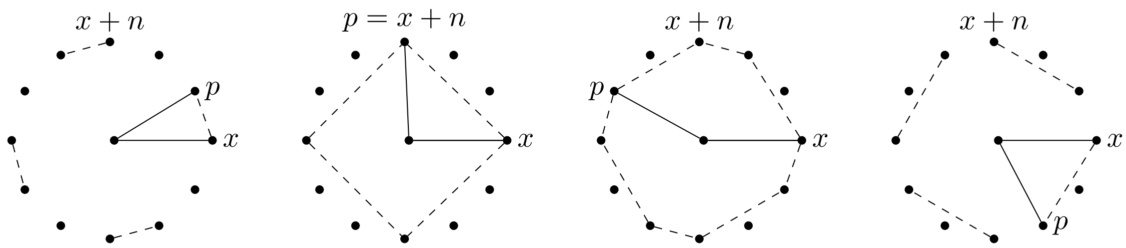
- $x + kn \lhd p \lhd x + (k+1)n$ for some $k \in [\![1, e-2]\!]$. We set $r := p - kn$, and we have $x \lhd r \lhd x+n$. The lcm is given by $a_x \vee a_p = a_{x+n} a_r a_x$. The right-complements are given by

$$\begin{aligned} a_p \backslash a_x &= a_{x+nk} a_{p-n} = a_{p-n} u_{p-n,x+nk} = u_{p-n,x+nk} a_{x+nk} \text{ and} \\ a_x \backslash a_p &= a_{r-n} a_{x-n} = a_{x-n} u_{x-n,r-n} = u_{x-n,r-n} a_{r-n}. \end{aligned}$$

- $x + (e-1)n \lhd p \lhd x$, in which case the lcm is given by

$$a_x \vee a_p = a_p u_{p,x} = a_x a_p = u_{p,x} a_x.$$

These four situations are summarized in the diagrams below, where $e = 4$ and $n = 3$.



Lcm of a symmetric atom with an asymmetric atom

Let $x \in \mathbb{Z}/en\mathbb{Z}$, and let $p, q \in \mathbb{Z}/en\mathbb{Z}$ be such that $p \lhd q \lhd p+n$. We compute the lcm of the asymmetric atom a_x with the symmetric atom $u_{p,q}$. Up to changing p, q, we can assume that $x \lhd p \lhd x+n, q$.

- $x = p \lhd q \lhd x+n$, in which case the lcm is given by

$$a_x \vee u_{p,q} = u_{p,q} a_q = a_x u_{x,q} = a_q a_x.$$

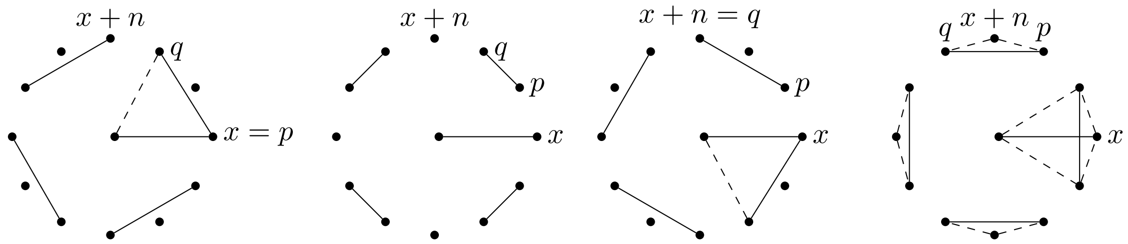
- $x \lhd p \lhd q \lhd x+n$, in which case a_x and $u_{p,q}$ commute.
- $x \lhd p \lhd q = x+n$, in which case the lcm is given by

$$a_x \vee u_{p,q} = u_{p,q} a_x = a_x a_{p-n}.$$

- $x \lhd p \lhd x+n \lhd q$, in which case the lcm is given by $a_x \vee u_{p,q} = a_{p-n} u_{p,x+n} u_{x+n,q}$. The right-complements are given by

$$u_{p,q} \backslash a_x = a_{q-n} u_{p,x+} = u_{p,x+n} a_{q-n} \text{ and } a_x \backslash u_{p,q} = u_{x,q-n} a_{p-n} = a_{p-n} u_{x,q-n}.$$

These four situations are summarized in the diagrams below, where $e = 4$ and $n = 3$.



Lcm of two symmetric atoms

Let $x, y, p, q \in \mathbb{Z}/en\mathbb{Z}$ be such that $x \lhd y \lhd x+n$ and $p \lhd q \lhd p+n$. We compute the lcms of the symmetric atoms $u_{x,y}$ and $u_{p,q}$. Up to changing p, q, we can assume that $x \unlhd p \lhd x+n$. We distinguish several cases.

If $p = x$, then we either have:

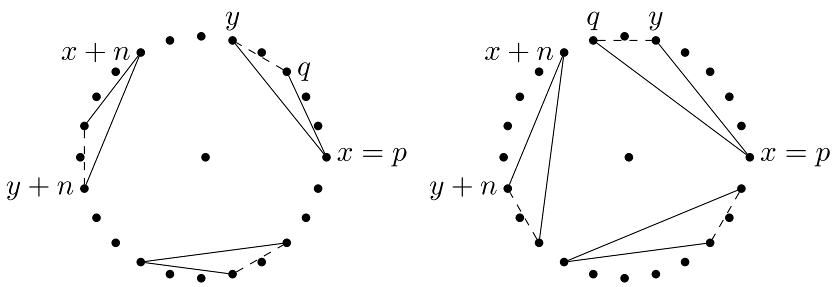
- $x = p \lhd q \lhd y$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{q,y} = u_{x,y}u_{p,q} = u_{q,y}u_{x,y}.$$

- $x = p,\ q = y$, in which case $u_{0,x} = u_{p,q}$.
- $x = p \lhd y \lhd q$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{x,y} = u_{x,y}u_{y,q} = u_{y,q}u_{p,q}.$$

The two situations where $u_{x,y} \neq u_{p,q}$ are summarized in the diagrams below, where $e = 3$ and $n = 8$.



If $x \lhd p \lhd y$, then we either have:

- $x \lhd p \lhd q \lhd y$, in which case $u_{x,y}$ and $u_{p,q}$ commute with one another.
- $x \lhd p \lhd q = y$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{x,y} = u_{x,y}u_{x,p} = u_{x,p}u_{p,q}.$$

- $x \lhd p \lhd y \lhd q \lhd x+n$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = u_{x,p}u_{p,y}u_{y,q}$. The right-complements are given by

$$u_{p,q}\backslash u_{x,y} = u_{p,y}u_{x,q} = u_{x,q}u_{p,y} \text{ and } u_{x,y}\backslash u_{p,q} = u_{x,p}u_{y,q} = u_{y,q}u_{x,p}.$$

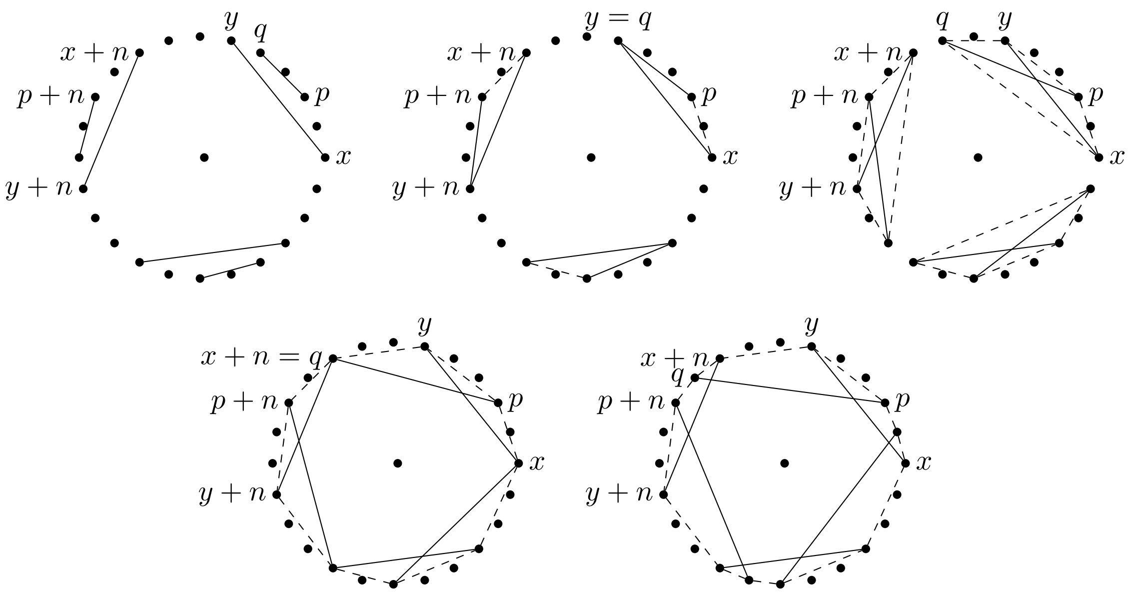
- $x \lhd p \lhd y \lhd q = x+n$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = a_{x+n}a_y a_p a_x$. The right-complements are given by

$$u_{p,q} \backslash u_{x,y} = u_{p,y}(a_{x+n}a_x) \text{ and } u_{x,y} \backslash u_{p,q} = u_{x,p}(a_{y+n}a_y).$$

- $x \lhd p \lhd y \lhd x+n \lhd q \lhd p+n$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = a_{x+n}a_y a_p a_{q-n} a_x$. The right-complements are given by

$$u_{p,q} \backslash u_{x,y} = (u_{p,y}u_{y,x+n})(a_{q+n}a_q) \text{ and } u_{x,y} \backslash u_{p,q} = (u_{x,q-n}u_{q-n,p})(a_{y+n}a_y).$$

These five situations are summarized in the diagrams below, where $e = 3$ and $n = 8$.



If $p = y$, then we either have:

- $y = p \lhd q \lhd x+n$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{x,q} = u_{x,y}u_{p,q} = u_{x,q}u_{x,y}.$$

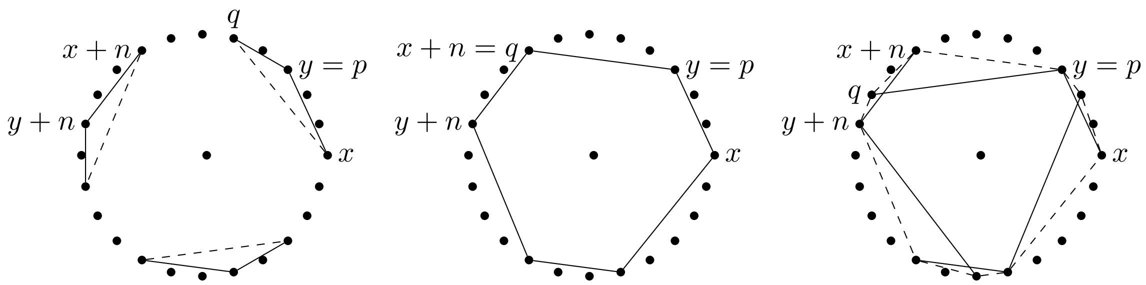
- $y = p,\ q = x+n$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = u_{x,y}a_{y+n}a_y$. The right-complements are given by

$$u_{p,q} \backslash u_{x,y} = a_{x+n}a_x \text{ and } u_{x,y} \backslash u_{p,q} = a_{y+n}a_y.$$

- $y = p \lhd x+n \lhd q$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = u_{x,q-n}u_{q-n,p}a_{y+n}a_y$. The right-complements are given by

$$u_{p,q} \backslash u_{x,y} = u_{p,x+n}a_{q+n}a_q \text{ and } u_{x,y} \backslash u_{p,q} = u_{x,q-n}a_{y+n}a_y.$$

These three situations are summarized in the diagrams below, where $e = 3$ and $n = 8$.



Lastly, if $x \lhd y \lhd p$, then we either have:

- $y \lhd p \lhd q \lhd x+n$, in which case $u_{x,y}$ and $u_{p,q}$ commute with one another.
- $y \lhd p \lhd q = x+n$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{x,y} = u_{x,y}u_{p-n,y} = u_{p-n,y}u_{p,q}.$$

- $y \lhd p \lhd x+n \lhd q \lhd y+n$, in which case the lcm is given by $u_{x,y} \vee u_{p,q} = u_{p,x+n}u_{x+n,q}u_{q,y+n}$. The right-complements are given by

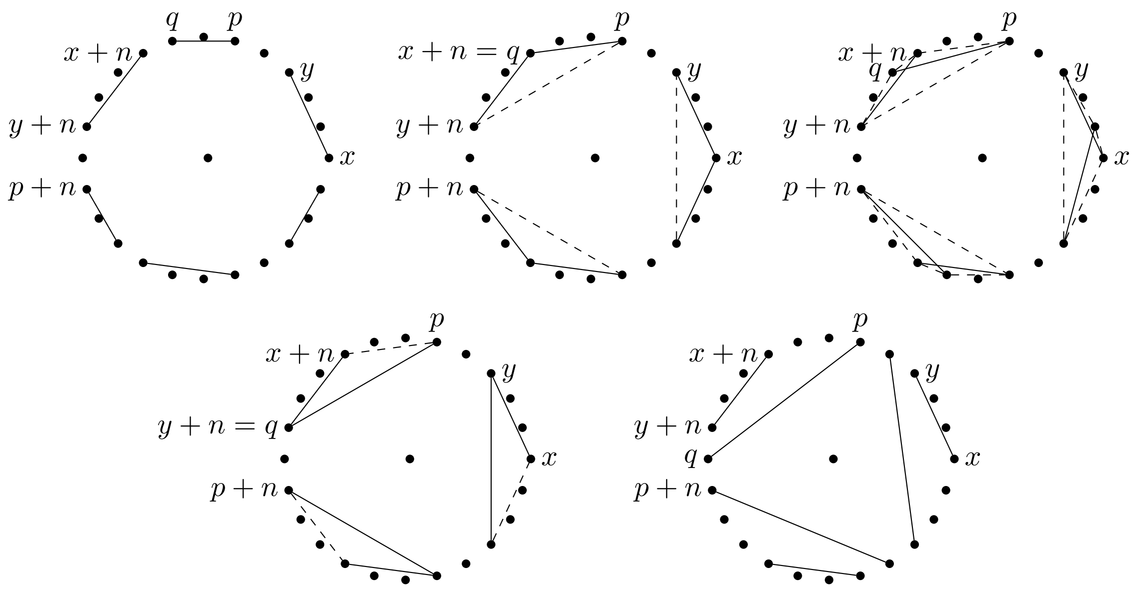
$$u_{p,q}\backslash u_{x,y} = u_{p,x+n}u_{q,y+n} = u_{q,y+n}u_{p,x+n} \text{ and } u_{x,y}\backslash u_{p,q} = u_{x,q-n}u_{p-n,y} = u_{p-n,y}u_{x,q-n}.$$

- $y \lhd p \lhd x+n \lhd q = y+n$, in which case the lcm is given by

$$u_{x,y} \vee u_{p,q} = u_{p,q}u_{p,x+n} = u_{x,y}u_{p,q} = u_{p,x+n}u_{x,y}.$$

- $y \lhd p \lhd x+n \lhd y+n \lhd q \lhd p+n$, in which case $u_{x,y}$ and $u_{p,q}$ commute with one another.

These five situations are summarized in the diagrams below, where $e = 3$ and $n = 8$.



Proof of Theorem 8.4.1

We are going to show that $\mathrm{NCP}(e, e, n+1)$ is fully reachable, which is sufficient to prove Theorem 8.4.1 by Proposition 8.4.11. We do so by induction on the integer $n \geqslant 1$.

Let us first consider the case where $n = 1$. In Section 8.3.3, we saw that $\mathrm{NCP}(e, e, 2)$ is defined as the set of partitions of $\mu_e \sqcup \{0\} \simeq \mathbb{Z}/e\mathbb{Z} \sqcup \{0'\}$ such that the partition obtained by removing $0'$ is an element of $\mathrm{NCP}(e, 1, 1) = \mathrm{NCP}(1, 1, e)^{\mu_e} = \{1, \Delta\}$. We obtain that $\mathrm{NCP}(e, e, 2)$ contains only partitions $1, \Delta$, along with asymmetric atoms a_p for $p \in \mathbb{Z}/e\mathbb{Z}$. The lattice $\mathrm{NCP}(e, e, 2)$ then has height 2, and we have $a_p \vee a_{p'} = \Delta$ for all $p \neq p'$ in $\mathbb{Z}/e\mathbb{Z}$. The construction of the complement in $\mathrm{NCP}(e, e, 2)$ gives that $\overline{a_p} = a_{p-1}$. In particular, for $p' \neq p$, we have $a_p \backslash a_{p'} = a_{p-1}$. Thus every element in $\mathcal{A} \setminus \mathcal{A}(a_p)$ is reachable for a_p, and $\mathrm{NCP}(e, e, 2)$ is fully reachable.

Now, let $n \geqslant 2$ and assume that $\mathrm{NCP}(e, e, m + 1)$ is fully reachable for $m < n$. We will show that $\mathrm{NCP}(e, e, n + 1)$ is fully reachable using Proposition 8.4.14. We need to compute the maximal elements distinct from Δ. If $s \in \mathrm{NCP}(e, e, m + 1) \setminus \{\Delta\}$, then s^* is an atom of $\mathrm{NCP}(e, e, m+1)$. Since $s = \overline{s^*}$, we have that maximal proper elements are the right-complements in Δ of the atoms.

The rotation of $\mu_{en} \sqcup \{0\}$ by an angle of $\frac{2\pi}{en}$ induces an automorphism ρ of $\mathrm{NCP}(e, e, n + 1)$ which preserves the right-complement. In particular, if $a \in \mathrm{NCP}(e, e, n + 1)$ is an atom such that all elements of $\mathcal{A} \setminus \mathcal{A}(\overline{a})$ are reachable elements of $\overline{a}$, then all elements of $\mathcal{A} \setminus \mathcal{A}(\overline{\rho(a)})$ are reachable for $\overline{\rho(a)}$. We can then restrict our attention to representatives of the ρ-orbits of the atoms of $\mathrm{NCP}(e, e, n + 1)$. By Lemma 8.3.25 (atoms of $\mathrm{NCP}(e, e, n + 1)$), a family of such representatives is given by $a_n, u_{0,1}, \ldots, u_{0,n-1}$. We can compute the right-complement to Δ of these atoms using Definition 8.3.27 (complement).

- The right-complement of a_n is the partition M_0, given by

$$M_0 := \left\{\{0'\} \sqcup K, K + n, K + 2n, \ldots, K + (e - 1)n\right\},$$

  where $K = [\![0, n - 1]\!]$.

- Let $k \in [\![1, n-1]\!]$, and let $K := [\![0, k - 1]\!]$. The right complement of $u_{0,k}$ is the partition L_k whose parts are $K, K + n, \ldots, K + (e - 1)n$, along with $\{0', k, k + 1, \ldots, n - 1, k + n, k + n + 1, \ldots, 2n - 1, \ldots, en - 1\}$.

By Remark 8.4.15, it is sufficient to check the first assumption of Proposition 8.4.14 on an irreducible element. For $k \in [\![2, n - 1]\!]$, $[\![0, k - 1]\!]$ is not a singleton, and L_k can be written as a product of two commuting elements of $\mathrm{NCP}(e, e, n+1)$, thus the irreducible maximal elements of $\mathrm{NCP}(e, e, n+1)$ distinct from Δ are M_0 and L_0. Identifying $[\![0, en-1]\!] \setminus \{0, n, \ldots, (e-1)n\}$ with $[\![0, e(n-1)-1]\!]$ yields an identification between the subgerm $\mathrm{NCP}(e, e, n+1)_{L_0}$ and $\mathrm{NCP}(e, e, n)$. By [BC06, Lemma 1.19], the subgerm $\mathrm{NCP}(e, e, n + 1)_{M_0}$ is isomorphic to $\mathrm{NCP}(1, 1, n + 1)$. Since $\mathrm{NCP}(e, e, n)$ is fully reachable by induction hypothesis, and since we already showed that $\mathrm{NCP}(1, 1, n + 1)$ is fully reachable, we obtain that the first assumption of Proposition 8.4.14 is satisfied.

It remains to show that all elements of $\mathcal{A} \setminus \mathcal{A}(M_0)$ are reachable for M_0, and that all elements of $\mathcal{A} \setminus \mathcal{A}(L_k)$ are reachable for L_k, for $k \in [\![1, n - 1]\!]$.

Proposition 8.4.21. *Let $k \in [\![1, n - 1]\!]$. All elements of $\mathcal{A} \setminus \mathcal{A}(L_k)$ are reachable for L_k.*

Let $K := [\![0, k - 1]\!]$, and let $L := \{k, k + 1, \ldots, n - 1, k + n, \ldots, en - 1\}$. By definition, the parts of L_k are the $K + in$ for $i \in [\![0, e - 1]\!]$ and $L \sqcup \{0'\}$. The elements of $\mathcal{A} \setminus \mathcal{A}(L_k)$ can be partitioned into three groups

$$\begin{aligned}
E_1 &:= \{u_{x,y} \mid x \in [\![(e-1)n+k, en-1]\!] \subset L, y \in K\},\\
E_2 &:= \{a_p \mid p \in K \sqcup K+n \sqcup \cdots \sqcup K+(e-1)n\},\\
E_3 &:= \{u_{x,y} \mid x \in K, y \in [\![k, n-1]\!] \subset L\}.
\end{aligned}$$

We will show that the elements of each of these groups are reachable for L_k.

Lemma 8.4.22. *The elements of E_1 are reachable for L_k.*

Proof. For $u_{x,y} \in E_1$, we define

$$d_1(u_{x,y}) = \#(\{z \in L \mid x \lhd z \lhd 0\}) \text{ and } d_2(u_{x,y}) = \#(\{z \in K \mid y \lhd z \lhd k\}).$$

As in the proof of Lemma 8.4.19, we endow E_1 with the lexicographic order $\leqslant$ on d_1 and d_2. That is, for $u_{x,y}, u_{x',y'} \in E_1$, we have $u_{x,y} \leqslant u_{x',y'}$ if either $d_1(u_{x,y}) \leqslant d_1(u_{x',y'})$ or if $d_1(u_{x,y}) = d_1(u_{x',y'})$ and $d_2(u_{x,y}) \leqslant d_2(u_{x',y'})$. We show that the elements of E_1 are reachable for L_k by induction on $\leqslant$. First, the only atom $u_{x,y} \in E_1$ such that $d_1(u_{x,y}) = d_2(u_{x,y}) = 0$ is $u_{en-1,k-1}$. By [BC06, Lemma 1.22], we have $u_{en-1,k-1} = \overline{\overline{u_{0,k}}} = \overline{L_k}$, which is reachable by definition.

Now, let $u_{x,y} \in E_1$, and assume that all $u_{x',y'} \in E_1$ such that $u_{x',y'} < u_{x,y}$ are reachable for L_k. Let a_p be an asymmetric atom. We know that $a_p u_{x,y}$ is not simple if and only if $x+in \lhd p \unlhd y+in$ for some $i \in [\![0, e-1]\!]$. If we further assume that $a_p \lhd L_k$, i.e. that $p \in L$, then we have $x+in \lhd p \lhd k+in$. In this case, the complement $a_p \backslash u_{x,y}$ is divided by $u_{p,y+in} = u_{p-in,y}$, which is reachable by induction hypothesis as $d_1(u_{p-in,y}) < d_1(u_{x,y})$.

Let $u_{p,q}$ be a symmetric atom with $x \unlhd p \lhd x+n$. If $u_{p,q} \preccurlyeq L_k$, then we either have $p, q \in L$ or $p, q \in K$. We assume at first that $p, q \in L$, in which case neither p nor q can be equal to either y or $y+n$. In this case, the product $u_{p,q}u_{x,y}$ is not simple if and only if we either have

- $x = p \lhd q \lhd y$, in which case $u_{p,q} \backslash u_{x,y} = u_{q,y}$. Since $q \in L$, we have $u_{q,y} \in E_1$, with $d_1(u_{q,y}) < d_1(u_{x,y})$, thus $u_{q,y}$ is reachable by induction hypothesis.
- $x \lhd p \lhd y \lhd q$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{p,y}$. Since $p \in L$, we have $u_{p,y} \in E_1$, with $d_1(u_{p,y}) < d_1(u_{x,y})$, thus $u_{p,y}$ is reachable by induction hypothesis.
- $x \lhd y \lhd p \lhd x+n \lhd q \lhd y+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{q,y+n} = u_{q-n,y}$. Since $q, q-n \in L$, we have $u_{q-n,y} \in E_1$, with $d_1(u_{q-n,y}) < d_1(u_{x,y})$, thus $u_{q-n,y}$ is reachable by induction hypothesis.

In each case, $u_{p,q} \backslash u_{x,y}$ is divided by a reachable atom. Assume now that $p, q \in K$, in which case neither p nor q can be equal to either x or $x+n$, and $p \lhd x+n \lhd q$ is impossible. In this case, the product $u_{p,q}u_{x,y}$ is not simple if and only if we either have $x \lhd p \lhd y \lhd q \lhd x+n$ or $y = p \lhd q \lhd x+n$. In both cases, $u_{p,q} \backslash u_{x,y}$ is divided by $u_{x,q}$. Since $q \in K$, we have $u_{x,q} \in E_1$ with $d_1(u_{x,q}) = d_1(u_{x,q})$ and $d_2(u_{x,q}) < d_2(u_{x,y})$, and thus $u_{x,y}$ is reachable for L_k.

We obtain that $u_{x,y}$ is reachable for L_k, which finishes the proof. □

Lemma 8.4.23. *The elements of E_2 are reachable for L_k.*

Proof. Let ρ be the automorphism of $\mathrm{NCP}(e, e, n+1)$ induced by the rotation of $\mu_{en} \sqcup \{0'\}$ of angle $\frac{2\pi}{en}$. We have $\rho^n(L_0) = L_0$, and since ρ preserves the right-complement, we obtain that the set of reachable atoms for L_0 is stable under ρ^n. In order to prove that every element of E_2 is reachable for L_k, it is then sufficient to prove that a_x is reachable for all $x \in K = [\![0, k-1]\!]$.

If $k = 1$, then $K = \{0\}$, and we only have to show that a_0 is reachable for L_1. Let a_p be an asymmetric atom. If $a_p \preccurlyeq L_1$, then p cannot be a multiple of n and the product $a_p a_0$ is not simple if and only if $n \lhd p \lhd 0$ with $p \notin \{2n, \ldots, (e-1)n\}$. In every case, the complement $a_p \backslash a_0$ is divided by some atom $u_{x,0}$ with $(e-1)n \lhd x \lhd 0$. By Lemma 8.4.22, such atoms are reachable. Let $u_{p,q}$ be a symmetric atom, say with $0 \unlhd p \lhd n$. If $u_{p,q} \preccurlyeq L_0$, then neither p nor q can be equal to either 0 or n. In this case, the product $u_{p,q} a_0$ is not simple if and only if $0 \lhd p \lhd n \lhd q$, in which case the right-complement $u_{p,q} \backslash a_0$ is divided by $u_{p,n} = u_{p-n,0}$, which is reachable again by Lemma 8.4.22. We obtain that a_0 is reachable for L_1.

Now, assume that $k > 1$. We prove that a_x is reachable for all $x \in K$ by descending induction on x. First, let $L_{k,1}$ be the partition whose only nonsingleton part is $L \sqcup \{0'\}$, and let $L_{k,2}$ be the partition whose nonsingleton part are $K, K+n, \ldots, K+(e-1)n$. We have $L_k = L_{k,1} L_{k,2} = L_{k,2} L_{k,1}$, and $a_{k-1} \preccurlyeq \overline{L_{k,2}}$. Since $k > 1$, both $L_{k,1}$ and $L_{k,2}$ are non trivial. By Lemma 8.4.16, a_{k-1} is then reachable for L_k.

Let now $x \in [\![0, k-2]\!]$, and assume that $a_{x'}$ is reachable for $x < x' \leqslant k-1$. Let a_p be an asymmetric atom. If $a_p \preccurlyeq L_k$, then $p \in L$ cannot have the form $x + in$ for some $i \in [\![0, e-1]\!]$. The product $a_p a_x$ is not simple if and only if $x + n \lhd p \lhd x$ with $p \in L$. In every case, the complement $a_p \backslash a_x$ is divided by some atom $u_{z,0}$ with $(e-1)n + x \lhd z \lhd x$. By Lemma 8.4.22, such atoms are reachable. Let $u_{p,q} \preccurlyeq L_k$ be a symmetric atom. If $p, q \in L$, then the only way for $u_{p,q} a_x$ not to be simple is if $p \lhd x \lhd q$ (up to changing p and q). In this case, $u_{p,q} \backslash a_x$ is divided by $u_{p,x} \in E_1$, which is reachable by Lemma 8.4.22. If $p, q \in K$ and if $u_{p,q} a_x$ is not simple, then we have $p \unlhd x \lhd p$, and $u_{p,q} \backslash a_x$ is divided by a_q, which is reachable by induction hypothesis. We obtain that a_x is reachable for L_k. □

Lemma 8.4.24. *The elements of E_3 are reachable for L_k.*

Proof. For $u_{x,y} \in E_3$, we define

$$d_1(u_{x,y}) = \#(\{z \in K \mid x \lhd z \lhd k\} \text{ and } d_2(u_{x,y}) = \#(\{z \in L \mid y \lhd z \lhd n\}.$$

As in the proof of Lemma 8.4.22, we endow E_3 with the lexicographic order $\leqslant$ on d_1 and d_2, and we show that the elements of E_3 are reachable for L_k by induction on $\leqslant$. First, we show that $u_{k-1,n-1}$ (the only atom with $d_1(u_{k-1,n-1}) = d_2(u_{k-1,n-2}) = 0$) is reachable.

If $k = 0$, then we have to show that $u_{0,n-1}$ is reachable for L_1. Let $a_p \preccurlyeq L_1$ be an asymmetric atom. We have $p \notin \{0, n, \ldots, (e-1)n\}$ and the product $a_p u_{0,n-1}$ is not simple if and only if $in \lhd p \lhd (i+1)n-1$ for some $i \in [\![0, e-1]\!]$. In each case, the complement $a_p \backslash u_{0,n-1} = a_p \backslash u_{in,(i+1)n-1}$ is divided by a_{in}, which is reachable by Lemma 8.4.23. Let $u_{p,q}$ be a symmetric atom, say with $0 \unlhd p \lhd n$. If $u_{p,q} \preccurlyeq L_1$, then neither p nor q is equal to either 0 or n, and $n-1 \unlhd p, q \lhd n$ is impossible. In this case, the product $u_{p,q} u_{x,y}$ is not simple if and only if we either have

- $0 \lhd p \lhd n-1 \lhd n \lhd q \lhd p+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{n-1,n} = u_{en-1,0}$. Since $en-1 \in L$ and $0 \in K$, we have that $u_{en-1,0} \in E_1$ is reachable by Lemma 8.4.22.
- $n-1 = p \lhd n \lhd q$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{p,n} = u_{p-n,0}$. Since $p - n \in L$ and $0 \in K$, we have that $u_{p-n,0} \in E_1$ is reachable by Lemma 8.4.22.

We obtain that $u_{0,n-1}$ is reachable for L_1.

If $k > 1$, then as in the proof of Lemma 8.4.23, we consider the partition $L_{k,1}$ whose only nonsingleton part is $L \sqcup \{0'\}$, along with the partition $L_{k,2}$ whose nonsingleton part are $K, K+n, \ldots, K+(e-1)n$. We have $L_k = L_{k,1}L_{k,2} = L_{k,2}L_{k,1}$, and $a_{k-1} \preccurlyeq \overline{L_{k,2}}$. Since $k > 1$, both $L_{k,1}$ and $L_{k,2}$ are non trivial. By Lemma 8.4.16, $u_{k-1,n-1} \preccurlyeq \overline{L_{k,2}}$ is then reachable for L_k.

Now, let $u_{x,y} \in E_3$, and assume that all $u_{x',y'} \in E_3$ such that $u_{x',y'} < u_{x,y}$ are reachable for L_k. Let a_p be an asymmetric atom. We know that $a_p u_{x,y}$ is not simple if and only if $x + in \lhd p \trianglelefteq y + in$ for some $i \in [\![0, e-1]\!]$. In this case, $a_p \backslash u_{x,y}$ is divided by a_{x+in}, which is reachable by Lemma 8.4.23 since $x \in K$.

Let $u_{p,q}$ be a symmetric atom with $x \trianglelefteq p \lhd x+n$. If $u_{p,q} \preccurlyeq L_k$, then p, q belong to the same part of L_k. We then either have $p, q \in K$, $p, q \in L$, or $p, q \in K+n$. If $p, q \in K$, then $x \trianglelefteq p \lhd q \lhd k \lhd y$. In this case, $u_{p,q}u_{x,y}$ is not simple if and only if $p = x$. We then have $u_{p,q} \backslash u_{x,y} = u_{q,y}\ d_1(u_{q,y}) < d_1(u_{x,y})$. If $p, q \in K+n$, then $x \lhd y \lhd n \trianglelefteq p$. In this case, as $q = y+n$ is impossible, $u_{p,q}u_{x,y}$ is not simple if and only if $y \lhd p \lhd x+n \lhd q \lhd y+n$. We then have $u_{p,q} \backslash u_{x,y} = u_{p,x+n}u_{q,y+n}\ d_1(u_{q-n,y}) < d_1(u_{x,y})$.

Assume now that $p, q \in L$, in which case neither p nor q can be equal to either x or $x+n$. In this case, the product $u_{p,q}u_{x,y}$ is not simple if and only if we either have

- $x \lhd p \lhd y \lhd q \lhd x+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{x,q}$. Since $q \in L$, we have $u_{x,q} \in E_3$, with $d_1(u_{x,q}) = d_1(u_{x,y})$ and $d_2(u_{x,q}) < d_2(u_{x,y})$. Thus $u_{x,q}$ is reachable by induction hypothesis.
- $x \lhd p \lhd y \lhd x+n \lhd q \lhd p+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{y,x+n}$. Since $y \in L$ and $x+n \in K+n$, we have that $u_{y,x+n} \in E_1$ is reachable by Lemma 8.4.22.
- $y = p \lhd q \lhd x+n$, in which case $u_{p,q} \backslash u_{x,y} = u_{x,q}$. Since $q \in L$, we have $u_{x,q} \in E_3$, with $d_1(u_{x,q}) = d_1(u_{x,y})$ and $d_2(u_{x,q}) < d_2(u_{x,y})$. Thus $u_{x,q}$ is reachable by induction hypothesis.
- $y = p \lhd x+n \lhd q$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{p,x+n}$. Since $p \in L$ and $x+n \in K+n$, we have that $u_{p,x+n} \in E_1$ is reachable by Lemma 8.4.22.
- $y \lhd p \lhd x+n \lhd q \trianglelefteq y+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{p,x+n}$. Since $p \in L$ and $x+n \in K+n$, we have that $u_{p,x+n} \in E_1$ is reachable by Lemma 8.4.22.

In each case, $u_{p,q} \backslash u_{x,y}$ is divided by a reachable atom.

We obtain that $u_{x,y}$ is reachable for L_k, which finishes the proof. □

Combining Lemma 8.4.22, 8.4.23 and 8.4.24 proves Proposition 8.4.21. We now turn to M_0. We distinguish between asymmetric and asymmetric atoms in $\mathcal{A} \setminus \mathcal{A}(M_0)$.

Lemma 8.4.25. *Asymmetric atoms in $\mathcal{A} \setminus \mathcal{A}(M_0)$ are reachable for M_0.*

Proof. By definition, the asymmetric atoms not dividing M_0 are the a_x for $x \notin [\![0, k-1]\!]$. We first show by descending induction on $x \in [\![(e-1)n, en-1]\!]$ that a_x is reachable. By [BC06, Lemma 1.22], we have that $a_{en-1} = \overline{\overline{a_n}} = \overline{M_0}$ is reachable for M_0. Let then $x \in [\![(e-1)n, en-2]\!]$, and assume that $a_{x'}$ is reachable for $x < x' \leqslant en-1$.

Let a_p be an asymmetric atom, with $a_p \preccurlyeq M_0$ (i.e. with $p \in [\![0, n-1]\!]$). The product $a_p a_x$ is not simple if and only if $x+n \lhd p \lhd n$, in which case $a_p \backslash a_x$ is divided by a_{p-n}, which is reachable by induction hypothesis since $x \lhd p-n \lhd 0$. Let $u_{p,q}$ be an asymetric atom, with $(e-1)n \trianglelefteq p \lhd q$. If $u_{p,q} \preccurlyeq M_0$, then we have $(e-1)n \trianglelefteq p \lhd q \lhd 0$. The product $u_{p,q}a_x$ is not simple if and only if

$p \trianglelefteq x \triangleleft q$, in which case $u_{p,q}\backslash u_{x,y}$ is divided by a_q, which is reachable by induction hypothesis. We obtain that a_x is reachable for M_0.

Now, let $i \in [\![1, e-2]\!]$. We show by descending induction that a_x is reachable for $x \in [\![in, (i+1)n-1]\!]$. First, we show that $a := a_{(i+1)n-1}$ is reachable. Let $a_p \preccurlyeq M_0$ be an asymmetric atom. The product $a_p a$ is not simple, and the right-complement $a_p\backslash a$ is divided by a_{p-n}, which we know to be reachable. Let $u_{p,q} \preccurlyeq M_0$ be a symmetric atom, with $in \trianglelefteq p \triangleleft q \triangleleft (i+1)n$. We have that $u_{p,q}a$ is simple, and thus a is reachable. Let then $x \in [\![in, (i+1)n-1]\!]$, and assume that $a_{x'}$ is reachable for $x < x' \leqslant (i+1)n-1$.

Let $a_p \preccurlyeq M_0$ be an asymmetric atom. The product $a_p a_x$ is not simple and the right-complement $a_p\backslash a_x$ is divided by a_{p-n}, which we know to be reachable. Let $u_{p,q} \preccurlyeq M_0$ be a symmetric atom, with $in \trianglelefteq p \triangleleft q \triangleleft (i+1)n$. We have that $u_{p,q}a_x$ is not simple if and only if $p \trianglelefteq x \triangleleft q$, in which case $u_{p,q}\backslash a_x$ is divided by a_q, which is reachable by induction hypothesis. We obtain that a_x is reachable for M_0. □

Lemma 8.4.26. *Symmetric atoms in $\mathcal{A} \setminus \mathcal{A}(M_0)$ are reachable for M_0.*

Proof. Let $u_{x,y} \in \mathcal{A} \setminus \mathcal{A}(M_0)$ be a symmetric atom. We can assume that $0 \trianglelefteq p \triangleleft n$. Since $u_{p,q} \not\preccurlyeq M_0$, we have $n \trianglelefteq q \triangleleft 2n$. We define

$$d_1(u_{x,y}) := \#(\{z \in [\![0, n-1]\!] \mid x \triangleleft z \triangleleft n\}) \text{ and } d_2(u_{x,y}) := \#(\{z \in [\![n, 2n-1]\!] \mid y \triangleleft z \triangleleft 2n\}).$$

As in previous proofs, we endow the set of symmetric atoms in $\mathcal{A} \setminus \mathcal{A}(M_0)$ with the lexicographic order $\leqslant$ on d_1 and d_2. And we show that the symmetric atoms in $\mathcal{A} \setminus \mathcal{A}(M_0)$ are reachable for M_0 by induction on $\leqslant$.

First, the minimal element for $\leqslant$ is $u_{n-1,2n-2}$, and we have $d_1(u_{n-1,2n-2}) = 0, d_2(u_{n-1,2n-2}) = 1$. The asymmetric atoms $a_p \leqslant M_0$ such that $a_p u_{n-1,2n-2}$ is not simple are $a_0, \ldots, a_{n-2}$. In each case, the right-complement $a_p\backslash u_{n-1,2n-2}$ is divided by a_{en-1}, which is reachable by Lemma 8.4.25. Let $u_{p,q} \preccurlyeq M_0$ be a symmetric atom, with $p, q \in [\![n, 2n-1]\!]$. If $q < 2n-2$, then we have $n-1 \triangleleft p \triangleleft q \triangleleft 2n-2$ and $u_{p,q}u_{x,y}$ is simple. If $q = 2n-2$, then we have $n-1 \triangleleft p \triangleleft q = 2n-2$ and $u_{p,q}u_{x,y}$ is simple. Thus the product $u_{p,q}u_{n-1,2n-2}$ is not simple if and only if we have $q = 2n-1$. In this case, we have $n-1 \triangleleft p \triangleleft 2n-2 \triangleleft q$, and the complement $u_{p,q}\backslash u_{x,y}$ is divided by a_{2n-1}, which is reachable by Lemma 8.4.25.

Let now $u_{x,y} \in \mathcal{A} \setminus \mathcal{A}(M_0)$ with $0 \trianglelefteq x \triangleleft n$, and assume that all $u_{x',y'} \in \mathcal{A} \setminus \mathcal{A}(M_0)$ such that $u_{x',y'} < u_{x,y}$ are reachable for M_0. Let $a_p \preccurlyeq M_0$ be an asymmetric atom. We have $p \in [\![0, n-1]\!]$, and the product $a_p u_{x,y}$ is not simple if and only if we have $x \triangleleft p \triangleleft n$. In this case, the right-complement $a_p\backslash u_{x,y}$ is divided by $u_{p,y}$. Since we have $p \in [\![0, n-1]\!]$, $y \in [\![n, 2n-1]\!]$, $d_1(u_{p,y}) < d_1(u_{x,y})$, we have that $u_{p,y} < u_{x,y}$ is reachable by induction hypothesis.

Let $u_{p,q} \preccurlyeq M_0$ be a symmetric atom, say with $x \triangleleft p \triangleleft x+n$. Since $u_{p,q} \preccurlyeq M_0$, we either have $x \trianglelefteq p \triangleleft q \triangleleft n$ or $n \trianglelefteq p \triangleleft x+n, q \triangleleft 2n$. The product $u_{p,q}u_{x,y}$ is not simple if and only if we either have

- $x \triangleleft p \triangleleft y \triangleleft q \triangleleft x+n$, in which case $u_{p,q}\backslash u_{x,y}$ is divided by $u_{x,q}$. Since $d_1(u_{x,q}) = d_1(u_{x,y})$ and $d_2(u_{x,q}) < d_2(u_{x,y})$, we have that $u_{x,q} < u_{x,y}$ is reachable by induction hypothesis.
- $x \triangleleft p \triangleleft y \triangleleft q = x+n$, in which case $u_{p,q}\backslash u_{x,y}$ is divided by $a_{x+n} = a_q$, which is reachable by Lemma 8.4.25.

- $x \lhd p \lhd y \lhd x+n \lhd q$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by a_q, which is reachable by Lemma 8.4.25.
- $y = p \lhd q \lhd x+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{x,q}$. Since $d_1(u_{x,q}) = d_1(u_{x,y})$ and $d_2(u_{x,q}) < d_2(u_{x,y})$, we have that $u_{x,q} < u_{x,y}$ is reachable by induction hypothesis.
- $y = p, q = x+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $a_{x+n} = a_q$, which is reachable by Lemma 8.4.25.
- $y = p \lhd x+n \lhd q$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by a_q, which is reachable by Lemma 8.4.25.
- $y \lhd p \lhd x+n \lhd q \lhd y+n$, in which case $u_{p,q} \backslash u_{x,y}$ is divided by $u_{q,y+n} = u_{q-n,y}$. Since $d_1(u_{q-n,y}) < d_1(u_{q-n,y})$, we have that $u_{q-n,y} < u_{x,y}$ is reachable by induction hypothesis.

We obtain that $u_{x,y}$ is reachable for M_0, which finishes the proof. □

As we said, combining Proposition 8.4.21 with Lemma 8.4.25, and Lemma 8.4.26 is sufficient to prove that $\mathrm{NCP}(e, e, n+1)$ is fully reachable by Proposition 8.4.14 and Lemma 8.4.12. In turn this is enough to prove that Theorem 8.4.1 for the dual braid monoid attached to $G(e, e, n+1)$ by Proposition 8.4.11.

As this was the last case to cover, this finishes the proof of Theorem 8.4.1 for all well-generated complex reflection groups.

8.5 Generalization to $G(de, e, n)$

In this section, we fix three integers $d, e, n > 1$, and we set $r := de$. We will combine the results we obtained on parabolic subgroups of well-generated braid groups with the results of Section 7.2, in order to obtain a description of the parabolic subgroups of the complex braid group $B_n^*(e) \simeq B(G(de, e, n))$. This was done first in [GM22, Section 3.2], here we use the results of Section 7.2, along with the groupoid of cosets construction.

The group $W := G(r, 1, n)$ is well-generated, thus we can consider the Coxeter element $c := c(r, 1, n)$ introduced in Section 8.3.2. Exceptionally in this section, we will denote the dual structure attached to W, c by (G, M, δ). We denote by $\mathcal{S}$ the set of simple elements of (G, M, δ). Assume that we have a fixed isomorphism $B(W) = B_n^* \simeq G$, as in Corollary 8.2.18.

The complex reflection group $W(e) := G(de, e, n)$ is a finite index subgroup of W. We use the notation and the results of Section 6.2.4. The braid group $B_n^*(e) \simeq B(W(e))$ is a finite index subgroup of B_n^* and we have a commutative diagram

$$\begin{tikzcd} B_n^*(e) \arrow[r, hook] \arrow[d, two heads] & B_n^* \arrow[r, two heads] \arrow[d, two heads] & B_n^*/B_n^*(e) \arrow[d, equal] \\ W(e) \arrow[r, hook] & W \arrow[r, two heads] & W/W(e) \end{tikzcd}$$

where the horizontal arrows are short exact sequences.

The top exact sequence is induced by the covering map $X_n(r)/W(e) \twoheadrightarrow X_n(r)/W$, induced by the action of $W/W(e) \simeq \mathbb{Z}/e\mathbb{Z}$. We know the groups $W, W(e)$ acts isometrically on $\mathbb{C}^n$ (for the distance induced by the usual hermitian scalar product). The group $W/W(e)$ acts isometrically on $\mathbb{C}^n/W(e)$, and the elements of $\mathbb{C}^n/W(e)$ with nontrivial stabilizer in $W/W(e)$

are exactly the $W(e)$-orbits of points of $\mathbb{C}^n$ with nontrivial stabilizer in W. Applying Corollary 7.2.9 (pseudoparabolic subgroups and group acting by isometries) then yields the following result:

Proposition 8.5.1. *The parabolic subgroups of $B_n^*(e)$ are exactly the intersections of $B_n^*(e)$ with the parabolic subgroups of B_n^*. Furthermore, a parabolic subgroup B_0 of B_n^* is irreducible if and only if $B_0 \cap B_n^*(e)$ is irreducible.*

Proof. We already proved the first statement. For the second statement, we know by Corollary 6.1.18 that the map $H \mapsto H \cap W(e)$ induces a bijection between the irreducible parabolic subgroups of W and those of $W(e)$. If $B_0 \subset B_n^*$ is a parabolic subgroup, then the image of $B_0 \cap B_n^*(e)$ in $W(e)$ is the intersection with $W(e)$ of the image of B_0 in W, whence the result. □

This is the first part of [GM22, Proposition 3.2]. Note that this implies immediately that parabolic subgroups of $B_n^*(e)$ are stable under intersection. Using a Garside-theoretic approach, we can show that the map $B_0 \mapsto B_0 \cap B_n^*(e)$ is not only a surjection, but also a bijection. Since $B_n^*(e)$ has finite index in $B_n^* \simeq G$, we can consider the groupoid of cosets of $B_n^*(e)$ in G, which is a Garside groupoid, as in Section 4.2. We are able to give a combinatorial description of the germ of simples of this groupoid of cosets.

We denote by $\mathcal{S}_e$ the graph whose objects is a set $\{u_0, \dots, u_e\}$ (the indices are seen in $\mathbb{Z}/e\mathbb{Z}$), and such that for $i, j \in \mathbb{Z}/e\mathbb{Z}$, we have

$$\mathcal{S}(u_i, u_j) := \begin{cases} \{s \in \mathrm{NCP}(2,1,n) \mid s \text{ is short}\} & \text{if } j = i, \\ \{s \in \mathrm{NCP}(2,1,n) \mid s \text{ is long}\} & \text{if } j = i+1, \\ \varnothing & \text{otherwise.} \end{cases}$$

For $s \in \mathrm{NCP}(2,1,n)$, we denote by s_i the unique arrow in $\mathcal{S}$ whose source is u_i and whose underlying element of $\mathrm{NCP}(2,1,n)$ is s.

The graph $\mathcal{S}_e$ is endowed with a germ structure, where $s_i \cdot t_j$ is defined if and only if j is the target of s_i (i.e. i if s is short, and j if s is long) and if $s \cdot t$ is defined in the germ $\mathrm{NCP}(2,1,n)$. In this case we have $s_i \cdot t_j = (s \cdot t)_j$, where $s \cdot t$ denotes the product of s and t in $\mathrm{NCP}(2,1,n)$.

Lemma 8.5.2. *The germ $\mathcal{S}_e$ is isomorphic to the germ of simples of the groupoid of cosets $G_{B_n^*(e)}$ defined in Section 4.2.*

Proof. Let $r := r_{\zeta_r,1} \in W$. We know that $\{1, r, \dots, r^{e-1}\}$ is a system of coset representatives for $W(e)$ in W. By Lemma 8.3.12, r belongs to $\mathcal{S}$, and $\{1, r, \dots, r^{e-1}\}$ is also a system of coset representatives for $B_n^*(e)$ in B_n^*. For $b \in B_n^*$, we write $[b]$ for the associated coset.

By construction, the object set of the groupoid of cosets $G_{B_n^*(e)}$ is $\{[1], [r], \dots, [r^{e-1}]\}$, and each simple element $s \in \mathcal{S}$ induces for each $i \in [\![0, e-1]\!]$ a simple morphism $s_{[r^i]}$ starting from $[r^i]$. We compute the target of $s_{[r^i]}$, i.e. the coset $[r^i s]$.

Consider the character χ on W introduced in Section 8.3.2. By definition, $W(e)$ is made of the elements w of W such that $\chi(x) \in \mu_d$. The isomorphism $W/W(e) \simeq \mathbb{Z}/e\mathbb{Z}$ can be realized by composing the character χ with the quotient map $\mu_r \to \mu_r/\mu_d \simeq \mathbb{Z}/e\mathbb{Z}$. By Lemma 8.3.12, we have either $\chi(s) = 1$ or $\chi(s) = \zeta_r$. In the first case, we have $\chi(r^i s) = \zeta_r^i$, and thus $[r^i s] = [r^i]$. In the second case, we have $\chi(r^i s) = \zeta_r^{i+1}$, and thus $[r^i s] = [r^{i+1}]$.

By Proposition 8.2.9, the germ $\mathcal{S}$ is isomorphic to the germ of simple elements of the dual braid monoid of type $G(2,1,n)$. By Proposition 8.3.19, this germ is isomorphic to the germ structure on $\mathrm{NCP}(2,1,n)$ (for the case $r=2$, this was first proven in [BW02, Proposition 5.2]). Under the isomorphism $\mathcal{S} \simeq \mathrm{NCP}(r,1,n) \simeq \mathrm{NCP}(2,1,n)$, an element s is sent to a long element if and only if $\chi(s) = \zeta_r$ by Corollary 8.3.20.

After the above computation of the sources and targets in the graph of simples of the groupoid of cosets $G_{B_n^*(e)}$, we deduce that the isomorphism $\mathcal{S} \simeq \mathrm{NCP}(2,1,n)$ then induces an isomorphism of oriented graphs between the graph of simples of the groupoid of cosets $G_{B_n^*(e)}$ and the graph $\mathcal{S}_e$. By construction of the germ structures, this isomorphism is also an isomorphism of germs. □

In particular, since the germ of simples of the groupoid of cosets $G_{B_n^*(e)}$ is a Garside germ, we obtain that $\mathcal{S}_e$ is a Garside germ. We can in particular give the following definition:

Definition 8.5.3 (Dual groupoid for $B_n^*(e)$). The *dual braid category* of type $B_n^*(e)$ is the Garside category $(\mathcal{C}_e^*, \Delta_e)$ induced by the Garside germ $\mathcal{S}_e$. The *dual groupoid* for $B_n^*(e)$ is the enveloping groupoid $\mathcal{G}_e^*$ of $\mathcal{C}_e^*$.

By construction, the map Δ_e sends an object u_i to Δ_i, where Δ is the maximum in $\mathrm{NCP}(2,1,n)$.

Combining Lemma 8.5.2 and Proposition 7.2.2 (topological groupoid of cosets) yields the following results:

Proposition 8.5.4. *Let p denote the covering map $X_n(r)/W(e) \to X_n(r)/W$, and let $* \in X_n(r)$ be a basepoint defining $B(W) \simeq B_n^*$. The following groupoids are isomorphic:*

- *The fundamental groupoid $\pi_1(X_n(r)/W(e), p^{-1}(*))$.*
- *The groupoid of cosets of $B_n^*(e)$ in B_n^*.*
- *The dual groupoid $\mathcal{G}_e^*$.*

Furthermore, the isomorphism between the groupoid of cosets of $B_n^*(e)$ in B_n^* and $\mathcal{G}_e^*$ induces an isomorphism between the underlying Garside categories.

For readability purposes, we identify $(\mathcal{G}_e^*, \mathcal{C}_e^*, \Delta_e)$ with the Garside groupoid of cosets of $B_n^*(e)$ in B_n^*.

Using Section 5.3.3, we can construct a shoal for the groupoid of cosets $G_{B_n^*(e)}$, which in turn induces a shoal for the dual groupoid $(\mathcal{G}_e^*, \mathcal{C}_e^*, \Delta_e)$. Since the shoal of all standard parabolic subgroups of G is support-preserving by Corollary 8.4.2, we can apply the results of Section 5.3.3. The results are summarized in the following proposition:

Proposition 8.5.5 (Parabolic subgroups of $B_n^*(e)$).

(a) For $s \in \mathrm{NCP}(2,1,n)$, the map $s_e : i \mapsto s_e(i) = s_i$ is a parabolic Garside map in $(\mathcal{G}_e^*, \mathcal{C}_e^*, \Delta_e)$.

(b) The set $\mathcal{T}_e^* := \{(\mathcal{G}_e^*)_{s_e} \mid s \in \mathrm{NCP}(2,1,n)\}$ is a support preserving shoal for $(\mathcal{G}_e^*, \mathcal{C}_e^*, \Delta_e)$.

(c) The isomorphism $\mathcal{G}_e^*(u_0, u_0) \simeq B_n^*(e)$ induces a bijection between the $\mathcal{T}_e^*$-parabolic subgroups of $\mathcal{G}_e^*(u_0, u_0)$ and the parabolic subgroups of $B_n^*(e)$.

(d) The map $B_0 \mapsto B_0 \cap B_n^*(e)$, induces a bijection between the parabolic subgroups of B_n^* and the parabolic subgroups of $B_n^*(e)$.

Proof. (a) and (b) are immediate applications of Lemma 5.3.19 and Proposition 5.3.20 (Shoal for groupoid of cosets) since the parabolic Garside elements in G are all the simple elements by

Lemma 8.2.14. For point (c), Proposition 5.3.21 (Parabolic subgroups of a finite index subgroup) proves that the isomorphism $\mathcal{G}_e^*(u_0, u_0) \simeq B_n^*(e)$ induces a bijection between the $\mathcal{T}_e^*$-parabolic subgroups of $\mathcal{G}_e^*(u_0, u_0)$ and the intersections with $B_n^*(e)$ of the parabolic subgroups of B_n^*. We already know by Proposition 8.5.1 that these intersections are the parabolic subgroups of $B_n^*(e)$. Point (d) is another application of Proposition 5.3.21. □

We can actually understand the bijection given by Proposition 5.3.21 in topological terms in the following proposition (both points were proven in the proof Corollary 7.2.9 except the uniqueness part, which comes from Proposition 8.5.5).

Corollary 8.5.6. *Let p denote the covering map $X_n(de)/W(e) \twoheadrightarrow X_n(de)/W$.*

(a) Let $B_0 \subset B_n^*$ be a parabolic subgroup defined by a capillary path η in the pair $(X_n(r)/W, \mathbb{C}^n/W)$. The parabolic subgroup $B_0 \cap B_n^*(e)$ of $B_n^*(e)$ is defined using the unique lift of η under p. In particular this does not depend on the choice of η.

(b) Let $B_0' \subset B_n^*(e)$ be a parabolic subgroup defined by a capillary path γ in the pair $(X_n(r)/W(e), \mathbb{C}^n/W(e))$. The unique parabolic subgroup B_0 of $B_n^*(e)$ such that $B_0 \cap B_n^*(e) = B_0'$ is defined by $p \circ \gamma$. In particular this does not depend on the choice of γ.

Lastly, we construct a system of conjugacy representatives for the shoal $\mathcal{T}_e^*$. We know that the set $\{z_s \mid s \in \mathcal{S}\}$, where z_s is the smallest power of s which is central in $G(c)_s$, is a set of conjugacy representatives for the standard parabolic subgroups of (G, M, δ) by Proposition 8.4.6.

For $s \in \mathcal{S} \simeq \mathrm{NCP}(2, 1, n)$, let k_s is the smallest integer such that $(z_s)^{k_s} \in B_n^*(e)$. By Proposition 5.3.24 (system of conjugacy representatives), the set $\{(z_s)^{k_s} \mid s \in \mathcal{S}\}$ is a system of conjugacy representatives for $\mathcal{T}_e^*$. For H a parabolic subgroup in $\mathcal{G}_e^*$, we denote by Z_H the associated element in this system of conjugacy representatives. If H' is the unique parabolic subgroup of G such that $H' \cap B_n^*(e) = H$, then Z_H is the smallest power of $z_{H'}$ lying in H.

Lemma 8.5.7. *If $B_0' \subset B_n^*(e)$ is an irreducible parabolic subgroup, then the element $Z_{B_0'}$ is the positive generator of the center of B_0'.*

Proof. Let B_0 be the unique parabolic subgroup of B_n^* such that $B_0 \cap B_n^*(e) = B_0'$. By Proposition 8.5.1, the parabolic subgroup B_0 is irreducible. Since B_0 is the braid group of a parabolic subgroup of W, it is well-generated by Proposition 6.1.23.

Since $B_n^*(e)$ has finite index in B_n^*, we obtain that B_0' has finite index in B_0. Since B_0 is irreducible and well-generated, we obtain that $Z(B_0') \subset Z(B_0)$ by Theorem 8.4.3. Since $Z(B_0) = \langle z_{B_0} \rangle$ by Remark 8.4.7, $Z(B_0')$ is generated by the smallest power of z_{B_0} belonging to B_0', that is, $Z_{B_0'}$. Furthermore, $Z_{B_0'}$ is positive as a power of the positive element z_{B_0}. □

As a corollary (and by definition of a system of conjugacy representatives), we obtain the following characterization of conjugacy in irreducible parabolic subgroups of $B_n^*(e)$:

Corollary 8.5.8. *Let $B_1, B_2 \subset B_n^*(e)$ be two irreducible parabolic subgroups. If z_1, z_2 denote the respective positive generators of the centers of B_1, B_2, then an element $b \in B_n^*(e)$ conjugates B_1 to B_2 if and only if it conjugates z_1 to z_2.*

Remark 8.5.9. Applying Corollary 5.3.25 to the set of conjugacy representatives for $\mathcal{G}_e^*$ allows one to reprove one result of [GM22, Section 6.3], stating that the curve complexes of B_n^* and $B_n^*(e)$ are isomorphic graphs.

Chapter 9

Regular centralizers and parabolic subgroups

In this chapter we study regular braids and their centralizers in complex braid groups. We start in Section 9.1 by explaining how this theory provides a lift in complex braid groups of Springer theory of regular elements. The case of well-generated complex reflection groups was proven by Bessis, and the remaining cases were done in my first paper [Gar23a].

Then, we study Springer groupoids in general, and how they provide a good Garside-theoretic framework for studying regular centralizers in braid groups of well-generated complex reflection groups. The first part of this section is taken from my fourth preprint [Gar23b], while the second part about parabolic subgroups is new.

In Section 9.3 we prove a general result allowing us to completely describe the parabolic subgroups of the centralizer of a regular braid in terms of parabolic subgroups of the ambient group.

Contents

The first mention of a possible lift of Springer theory of regular elements to complex braid groups appears in [BM97, Section 3]. In this paper, Broué and Michel consider roots of the generator z_P of the center of the pure braid group attached to a real reflection group W (we will call such roots *regular braids*). They are able to show that d-regular braids are lifts in $B(W)$ of d-regular elements in W, and that d-regular braids exist in $B(W)$ if and only if d is a regular number for W. They conjecture that distinct d-regular braids are conjugate, and that similar results hold in complex braid groups.

Later, Bessis studies regular braids in complex braid groups in [Bes00], where he shows that the braid group of a regular centralizer in a complex reflection group W embeds in the centralizer of some regular braid in $B(W)$ (except in a handful of cases). Bessis, Digne, Michel show in [BDM02] that the embedding mentioned above is actually an isomorphism for $W = \widetilde{G}(1,1,n), G(r,1,n)$ and the groups which are isodiscriminantal to those.

These various results are generalized by Bessis to all well-generated groups in [Bes15]. He also shows that d-regular braids (should they exist) are all conjugate in this case. The proof of Bessis relies on the dual group attached to a well-generated complex reflection group, and on the construction of a Garside groupoid equivalent to the centralizer of a regular braid.

In this chapter, we start by defining regular braids, and we extend the result of Bessis to all complex reflection groups. Then, we study in depth the groupoids introduced by Bessis to study regular centralizers in well-generated braid groups. We call these groupoids *Springer groupoids*.

In the last chapter, we studied well-generated complex braid groups using dual groups. We were in particular able to give new proofs of [GM22, Theorem 1.1 and 1.2] for these groups. Using Springer groupoids, we give a description of parabolic subgroups in regular centralizers, first in well-generated groups, and then in all complex braid groups.

Throughout this chapter, we freely use the notation introduced in Chapter 6 and in Chapter 8.

9.1 Springer theory in complex braid groups

9.1.1 Regular braids

In this section, we fix a finite dimensional complex vector space V of dimension n, along with an irreducible complex reflection group $W \subset \mathrm{GL}(V)$. We also fix a positive integer d which is regular for W, along with a d-regular element $g \in W$. We also fix a basepoint $* \in X$ and we define $P(W) := \pi_1(X, *)$ and $B(W) := \pi_1(X/W, W.*)$.

By Section 6.1.3, the group $W_g := C_W(g)$ acts as a complex reflection group on the space $V_g := \mathrm{Ker}(g - \zeta_d \,\mathrm{Id})$. The hyperplane arrangement for W_g is $\{H \cap V_g \mid H \in \mathcal{A}\}$. We deduce that

$$X_g := V_g \setminus \bigcup \mathcal{A}_g = V_g \setminus \left(\left(\bigcup \mathcal{A}\right) \cap V_g\right) = V_g \cap X.$$

Following [BMR98, Notation 2.20], we consider the homotopy class z_B in $B(W)$ of the path $t \mapsto W.(e^{2i\pi t/|Z(W)|}.*)$ in X/W. We also consider the homotopy class z_P in $P(W)$ of the path $t \mapsto e^{2i\pi t}*$ in X. The element z_P is called the *full-twist* in $B(W)$.

Lemma 9.1.1. *[BMR98, Lemma 2.22] The image of z_B in W is the scalar multiplication by $e^{2i\pi/|Z(W)|}$ in V, it generates $Z(W)$. Furthermore, we have $z_B^{|Z(W)|} = z_P$ and $z_B \in Z(B(W))$.*

The key notion in this chapter is that of a regular braid. Just like Springer regular elements, regular braids will allow us to realize some braid groups as centralizers in possibly better behaved braid groups.

Definition 9.1.2 (Regular braid). An element $\rho \in B(W)$ is a *d-regular braid* if $\rho^d = z_P$.

This first section is dedicated to proving Theorem 9.1.7 below, which provides the required connection between regular braids and regular elements in the group W. First, we can construct regular braids starting from regular elements in W.

Proposition 9.1.3. *[Bes15, Theorem 1.9(3)] The embeddings $V_g \to V$ and $X_g \to X$ induce an isomorphism of topological pairs $p : (X_g/G, V_g/W_g) \simeq ((X/W)^{\mu_d}, (V/W)^{\mu_d})$, where the action of μ_d on V/W is the quotient of the natural action of μ_d on V.*

In particular, this proposition proves that, in order to study the braid groups $B(W_g)$ of W_g, it is sufficient to consider the fundamental group of the space $(X/W)^{\mu_d}$. Moreover, note that $(X/W)^{\mu_d}$ is nonempty if and only if d is regular for W. The if part is given by Proposition 9.1.3. For the only if part, if there is some $W.v \in (X/W)^{\mu_d}$, then we have by definition $\zeta_d.(W.v) = W.(\zeta_d v) = W.v$, in other words, there is some $g \in W$ such that $g.v = \zeta_d.v$, which is to say that g is ζ_d-regular.

Notation 9.1.4. If the orbit $W.*$ lies in $(X/W)^{\mu_d}$, then we set $B(W)^{\mu_d} := \pi_1((X/W)^{\mu_d}, W.*)$. Proposition 9.1.3 states that, if $g \in W$ is d-regular, then we have $B(W_g) \simeq B(W)^{\mu_d}$.

Let now $x \in X$ be a ζ_d-regular eigenvector for g. We define a path in X by

$$\widetilde{g} : t \mapsto e^{\frac{2i\pi t}{d}} x.$$

Since this path ends at $\zeta_d x = g.x$, it induces a well-defined element of $\pi_1(X/W, W.x)$. This element is a d-regular braid and a lift of g. Now if $y \in X$ is another basepoint, then a path from x to y in X induces an isomorphism $\pi_1(X/W, W.x) \simeq \pi_1(X/W, W.y)$, which sends the homotopy class $[\widetilde{g}]$ of $\widetilde{g}$ to an element of $\pi_1(X/W, W.y)$, which is also a d-regular braid and a lift of g. In general, a lift of g in $\pi_1(X/W, W.y)$ which is a d-th root of z_P will be called a *regular lift* of g (in $\pi_1(X/W, W.y)$).

Since W is irreducible, its center is cyclic and generated by the image z_W of z_B in W, which is a ζ_m-regular (it is just the scalar multiplication by ζ_m on V), with $m = |Z(W)|$. One sees that z_B is a regular lift of z_W, and a m-regular braid.

The existence of regular lifts of regular elements, which we just showed, implies in particular the following lemma:

Lemma 9.1.5. *There are d-regular braids in $B(W)$. Furthermore, any conjugate of a d-regular braid is again a d-regular braid.*

The second part comes from the fact that, as $z_P = z_B^{|Z(W)|}$ is central, being a root of z_P is invariant under conjugacy. The following lemma is folklore:

Lemma 9.1.6. *[Bro01, Theorem-Assumption 2.25] Let $x \in X_g$. The map $X_g/W_g \to (X/W)^{\mu_d} \hookrightarrow X/W$ induces a group morphism*

$$B(W_g) = \pi_1(X_g/W_g, W_g.x) \to \pi_1(X/W, W.x),$$

whose image lies inside $C_{\pi_1(X/W,W.x)}([\widetilde{g}])$, where $[\widetilde{g}]$ is the regular lift of g in $\pi_1(X/W, W.x)$ constructed above.

With the notation of the lemma, choosing a path from x to $*$ in X induces in turn an isomorphism $\pi_1(X/W, W.x) \simeq B(W)$, and thus a morphism $B(W_g) \to B(W)$. By construction, the image of this morphism lies in the centralizer in $B(W)$ of some regular braid.

The main result of this section is a characterization of regular braids and their connection with regular elements in the reflection group W. In particular, the morphism constructed in the above lemma will be proven to be injective, and we will compute its image.

Theorem 9.1.7 (Springer theory in braid groups). *Let W be an irreducible complex braid group, and let d be a positive integer.*

(a) The integer d is regular for W if and only if there are d-regular braids in $B(W)$.

(b) When d is regular, d-regular braids are all conjugate in $B(W)$, and a d-regular braid is mapped to a ζ_d-regular element in $B(W)$.

(c) Let d be a regular number, and let ρ be a d-regular braid. Let $g \in W$ be the image of ρ. Any morphism $B(W_g) \to B(W)$ given by Lemma 9.1.6 induces an isomorphism between $B(W_g)$ and $C_{B(W)}(\rho')$, where ρ' is a conjugate of ρ.

This theorem was first shown by Bessis in the case where W is well-generated [Bes15, Theorem 12.4] (although partial results were already known, see [Bes15, Remark 12.5]), as an answer to a question asked by Broué. The proof for the remaining irreducible groups was done in my first paper [Gar23a, Theorem 1.2]. We include the proof of the well-generated case here, as the underlying topological construction will be useful to us.

9.1.2 Proof of Theorem 9.1.7 for well-generated groups

In this section, we fix a finite dimensional complex vector space V of dimension n, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We also fix a system f of basic invariants for W such that $\Delta(W, f)$ has the form given in (8.1.1):

$$\Delta(W, f) = X_n^n + \alpha_2(X_1, \dots, X_{n-1})X_n^{n-2} + \cdots + \alpha_n(X_1, \dots, X_{n-1}),$$

where $\alpha_i \in \mathbb{C}[X_1, \dots, X_{n-1}]$ is weighted homogeneous of weighted degree ih. We consider the associated (extended) Lyashko-Looijenga morphism $\overline{\mathrm{LL}} : V/W \to E_n$ as in Chapter 8.

We fix h to be the highest degree of W, and d to be a regular number for W (note that $d = 1$ is a valid choice). We write

$$p := \frac{d}{d \wedge h} \text{ and } q := \frac{h}{d \wedge h},$$

where $d \wedge h$ is the gcd of d, h in $\mathbb{N}$.

The proof of Theorem 9.1.7 in this context, done by Bessis in [Bes15, Section 12], consists in relating the space $(X/W)^{\mu_d}$ with the category of (p, q)-regular elements associated to the dual group of type W.

Recall from Theorem 8.2.17 that we can define $B(W)$ as $\pi_1(X/W, \mathcal{U})$, where $\mathcal{U}$ is a simply connected subset of X/W. For any $x \in \mathcal{U}$, the circular tunnel $(x, \frac{2\pi}{h})$ defines the same element $\Delta \in B(W)$. The projection map $B(W) \to W$ induces a group isomorphism $B(W) \to G(W, c)$, where c is the image in W of Δ. Using this result, we identify $B(W)$ with the dual group $G(W, c)$ of type W. In particular, Δ designates both the aforementioned element of $B(W)$ and the Garside element of $G(W)$.

In [Bes15, Lemma 11.15], points in $(X/W)^{\mu_d}$ are characterized in terms of their Lyashko-Looijenga multisets and in terms of their cyclic label. Our definition (Definition 8.1.17) of cyclic label extended to V/W (and not only X/W) allows us to generalize this results into a characterization of points in $(V/W)^{\mu_d}$. This characterization is not mandatory in order to define Springer groupoids, but it will be important in Section 9.2.5 when we consider parabolic subgroups.

First, as the set E_n consists of multisets inside $\mathbb{C}$, it comes equipped with an action of $\mathbb{C}^*$. In particular we can define $(E_n)^{\mu_k}$ as the fixed points of E_n under the action of the k-th roots of unity. Since the map $\overline{\mathrm{LL}}$ is homogeneous of degree h by Lemma 8.1.4, we have $\overline{\mathrm{LL}}(\zeta_d x) = (\zeta_d)^h \overline{\mathrm{LL}}(x)$. As $(\zeta_d)^h$ is a primitive p-th root of unity, we obtain that $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$ if $x \in (V/W)^{\mu_d}$. Thus we are interested in describing points $x \in V/W$ such that $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$.

Let $x \in V/W$, both the cyclic label and the outer label of x are tuples of elements of $G(W,c)$. We can then consider the map τ defined in Section 4.4.1 acting on $\mathrm{clbl}(x)$ or on $\mathrm{olbl}(x)$.

Lemma 9.1.8. *Let $x \in V/W$ be nonzero. Assume that $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$. Then* $\mathrm{olbl}(x)$ *contains pk terms for some positive integer k, and* $\mathrm{olbl}(\zeta_{ph} x) = \tau^k(\mathrm{olbl}(x))$.

Proof. If $x \in X/W$, then this is simply [Bes15, Lemma 11.14]. We restrict our attention to the case where x lies on the discriminant hypersurface. Since $\zeta_p \overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(x)$ by assumption, the cyclic support of $\overline{\mathrm{LL}}(x)$ is a reunion of μ_p-orbits in $\mathbb{C}$. As each nonzero μ_p-orbit in $\mathbb{C}$ contains exactly p points, we obtain that $\mathrm{olbl}(x)$ contains pk terms, where k is the number of μ_p-orbits in the cyclic support of $\overline{\mathrm{LL}}(x) \setminus \{0\}$ (since $x \neq 0$, we have $k > 0$ by [Bes15, Lemma 5.6]). We write $\mathrm{olbl}(x) = (c_1, \ldots, c_{pk})$ and $c_{pk+1} := (c_1 \cdots c_{pk})^{-1} c$ so that $\mathrm{clbl}(x) = (c_1, \ldots, c_{pk}, c_{pk+1})$.

Let $\gamma : [0,1] \to V/W$ be a desingularization path for x. We denote by $\gamma_j : [0,1] \to \mathbb{C}$ the j-th component of the cyclic support of $\overline{\mathrm{LL}} \circ \gamma$, and let $(\theta_1(t), \ldots, \theta_{pk}(t), \theta_{pk+1}(t))$ be the cyclic argument of $\gamma(t)$. By [Bes15, Lemma 11.11], the cyclic label of $\zeta_{ph}\gamma(t)$ is given for t big enough by $(c_{k+1}, c_{k+2}, \ldots, c_{pk}, c_0, c_1^c, \ldots, c_k^c) = \tau^k(\mathrm{clbl}(\zeta_{ph}\gamma(t)))$ The cyclic argument of $\zeta_{ph}\gamma(t)$ is given by

$$\left(\theta_{k+1}(t) - \frac{2\pi}{p}, \ldots, \theta_{pk+1}(t) - \frac{2\pi}{p}, \theta_1(t) - \frac{2\pi}{p} + 2\pi, \ldots, \theta_k(t) - \frac{2\pi}{p} + 2\pi\right).$$

In particular, $\zeta_{ph}\gamma$ is not a desingularization path for $\zeta_{ph} x$, since it doesn't respect the condition on (the path $\zeta_{ph}\gamma_{pk+1}$ which terminates at $0 \in \overline{\mathrm{LL}}(x)$ does not give the last term of the cyclic support of $\zeta_{ph}\gamma$). We fix $r > 0$ such that, for $t < 1$ big enough, we have $|\gamma_{pk+1}(t)| < r < |\gamma_j(t)|$ for all $j \neq pk+1$. We can then rotate $\zeta_{ph}\gamma_{pk+1}(t)$ by an angle of $K(t)$ so that $\theta_k(t) - \frac{2\pi}{p} + 2\pi < \theta_{pk+1}(t) - \frac{2\pi}{p} + K(t) < 2\pi$. We obtain a new path $\gamma'_{pk+1}(t)$. The family $(\zeta_{ph}\gamma_1(t), \ldots, \zeta_{ph}\gamma_{pk}(t), \gamma'_{pk+1}(t))$ induces a desingularization path γ' for $\zeta_{ph} x$, which is homotopic to $\zeta_{ph}\gamma$.

However, by Lemma 8.1.20 (Hurwitz moves), the cyclic label of $\gamma'(t)$ is given (by definition), for t big enough, by

$$\mathrm{clbl}(\gamma'(t)) = (c_{k+1}, c_{k+2}, \ldots, c_{pk}, c_1^c, \ldots, c_k^c, (c_0)^{c^{-1} c_1 \cdots c_k c}).$$

Thus, $\mathrm{olbl}(x) = (c_{k+1}, c_{k+2}, \ldots, c_{pk}, c_1^c, \ldots, c_k^c)$ as claimed. ☐

Using this Lemma, we get a complete description of $(V/W)^{\mu_d}$ using the map $(\overline{\mathrm{LL}}, \mathrm{clbl})$.

Lemma 9.1.9. *For all nonzero $x \in V/W$, the following assertions are equivalent:*

(i) $x \in (V/W)^{\mu_d}$.

(ii) $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$ *and* $\tau^{kq}(\mathrm{olbl}(x)) = \mathrm{olbl}(x)$*, where* pk *is the cardinality of the cyclic support of* $\overline{\mathrm{LL}}(x) \setminus \{0\}$.

Proof. Assume (i). We have $\overline{\mathrm{LL}}(x) = (\zeta_d)^h \overline{\mathrm{LL}}(x)$, thus $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$ since $(\zeta_d)^h$ is a primitive p-th root of unity. By Lemma 9.1.8, we have $\mathrm{olbl}(\zeta_d x) = \mathrm{olbl}(\zeta_{d'h}^q x) = \tau^{kq}(\mathrm{olbl}(x))$, where k is the number of points of the cyclic support of $\overline{\mathrm{LL}}(x)$ with argument $\theta \in]0, 2\pi/p]$.

Conversely, assuming (ii), we conclude that x and $\zeta_d x$ satisfy $\overline{\mathrm{LL}}(x) = \overline{\mathrm{LL}}(\zeta_d x)$ and $\mathrm{olbl}(x) = \tau^{kq}(\mathrm{olbl}(x)) = \mathrm{olbl}(\zeta_d x)$ by Lemma 9.1.8. We then have $x = \zeta_d x$ by Corollary 8.1.19. $\square$

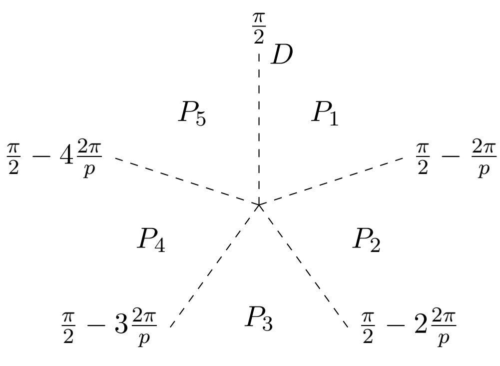
Corollary 9.1.10 (Path lifting in $(V/W)^{\mu_d}$). *Let* $\gamma : [0,1] \to (E_n)^{\mu_p}$ *be a path such that points are not unmerged as* t *increases (they can be merged). If* $x_0 \in (V/W)^{\mu_d}$ *is such that* $\overline{\mathrm{LL}}(x_0) = \gamma(0)$*, then there exists a unique lift* $\widetilde{\gamma} : [0,1] \to (V/W)^{\mu_d}$ *such that* $\widetilde{\gamma}(0) = x_0$.

Proof. The existence of a unique lift $\widetilde{\gamma}$ of γ in V/W starting at x_0 is given by Proposition 8.1.10 (path lifting). We only have to show that $\widetilde{\gamma}(t) \in (V/W)^{\mu_d}$ for all $t \in [0,1]$. Since $\gamma(t) \in (E_n)^{\mu_p}$ for all $t \in [0,1]$, the path $t \mapsto \zeta_d.\widetilde{\gamma}(t)$ is another lift of γ, which starts at $\zeta_d.\widetilde{\gamma}(0) = \zeta_d.x_0 = x_0$. By uniqueness of $\widetilde{\gamma}$, we obtain that $\widetilde{\gamma}(t) \in (V/W)^{\mu_d}$ for all $t \in [0,1]$. $\square$

We are now equipped to give the topological definition of the Springer groupoid. We consider the following union of half-lines:

$$D := \bigcup_{\zeta \in \mu_p} \zeta i \mathbb{R}_{\geqslant 0}.$$

The complement of D in $\mathbb{C}$ consists of p sectors $P_1, \dots, P_p$, which are labeled so that P_k contains all nonzero points of arguments $\theta \in [\frac{\pi}{2} - k\frac{2\pi}{p}, \frac{\pi}{2} - (k-1)\frac{2\pi}{p}[$, as in the example below (where $p = 5$):



We also consider the following subsets of V/W:

$$\mathcal{U}_p := \{x \in X/W \mid \overline{\mathrm{LL}}(x) \cap D = \varnothing\},$$
$$\mathcal{U}^{\mu_d} := \{x \in (X/W)^{\mu_d} \mid \overline{\mathrm{LL}}(x) \cap D = \varnothing\}.$$

Since $\overline{\mathrm{LL}}(x) \in (E_n)^{\mu_p}$ when $x \in (V/W)^{\mu_d}$, we have $\mathcal{U}^{\mu_d} := \mathcal{U}_p \cap (X/W)^{\mu_d} = \mathcal{U} \cap (X/W)^{\mu_d}$, where $\mathcal{U}$ is the (simply connected) basepoint for defining $B(W)$. For $x \in (V/W)^{\mu_d}$, the points of the cyclic support of $\overline{\mathrm{LL}}(x) \setminus \{0\}$ can be partitioned into p groups, according to which sector P_i they lie in for $i \in [\![1, p]\!]$.

Definition 9.1.11 (Cyclic content). Let $x \in V/W$. The *cyclic content* $\mathrm{cc}(x)$ of x is the sequence $(c_1, \dots, c_p)$, where c_i is the product (in the ordering given by the cyclic support of x) of the terms of $\mathrm{clbl}(x)$ associated to points in the sector P_i, for $i \in [\![1, p]\!]$.

This is a generalization of [Bes15, Definition 11.18], which covers the case where $x \in (X/W)^{\mu_d}$. Note that $\mathrm{cc}(x) = (1, \dots, 1)$ if and only if $x = 0$, again by [Bes15, Lemma 5.6]. A first consequence of Lemma 9.1.9 is that $\tau^q \,\mathrm{cc}(x) = \mathrm{cc}(x)$ if $x \in (V/W)^{\mu_d}$. In this case, since $\mathrm{cc}(x)$ has length p, Lemma 4.5.5 gives that $\mathrm{cc}(x)$ has the form $(s_1, \phi^{\lambda}(s_1), \cdots, \phi^{(p-1)\lambda}(s_1))$, where λ, μ is a couple of positive integers such that $p\lambda - q\mu = 1$, and where ϕ is the Garside automorphism of $G(W, c)$. In particular $\mathrm{cc}(x)$ is entirely determined by its first term.

Since $(G(W, c), M(W, c), \Delta)$ is a Garside group, we can then consider the set of decompositions of Δ, as defined in Section 4.4.1 and Section 4.5.1. Furthermore, as the monoid $M(W, c)$ is an interval monoid, we can identify by Remark 4.4.4 the set $D_p(\Delta)$ (resp. $D_p^q(\Delta)$) with the set $D_p(c)$ (resp. $D_p^q(c)$) of length-additive decompositions of c (resp. of length-additive decompositions of c which are invariant under the action of c^q).

By [Bes15, Lemma 11.10], for any $x \in X/W$, the products of the terms of $\mathrm{clbl}(x)$ is equal to Δ. Thus $\mathrm{cc}(x) \in D_p(\Delta)$ for $x \in \mathcal{U}_p$, and $\mathrm{cc}(x) \in D_p^q(\Delta)$ for $x \in \mathcal{U}^{\mu_d}$. We can actually show that the cyclic content completely characterizes the path connected components of $\mathcal{U}_p$ and $\mathcal{U}^{\mu_d}$.

Proposition 9.1.12. *[Bes15, Lemma 11.22]*

(a) The cyclic content $\mathrm{cc} : \mathcal{U}_p \to D_p(\Delta)$ *induces a bijection between the path connected components of* $\mathcal{U}_p$ *and* $D_p(\Delta)$

(b) The cyclic content $\mathrm{cc} : \mathcal{U}^{\mu_d} \to D_p^q(\Delta)$ *induces a bijection between the path connected components of* $\mathcal{U}^{\mu_d}$ *and* $D_p^q(\Delta)$.

(c) The path connected components of $\mathcal{U}_p$*, and of* $\mathcal{U}^{\mu_d}$*, are contractible.*

Since the connected components of $\mathcal{U}_p$ and $\mathcal{U}^{\mu_d}$ are contractible, we can use them as groupoid basepoints in the sense of [Bes15, Definition A.4].

Definition 9.1.13 (Springer groupoid). [Bes15, Definition 11.23]
The *Springer groupoid* associated to W and d is the groupoid

$$B_p^q(W) = \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d}).$$

We also set $B_p(W) := \pi_1(X/W, \mathcal{U}_p)$. The functoriality of π_1 gives natural functors $B_p^q(W) \to B_p(W) \to B(W)$.

By definition of $B_p(W)$, the functor $B_p(W) \to B(W)$ is an equivalence of groupoids. By definition of a fundamental groupoid with basepoint having several connected components, the elements of the Springer groupoid are homotopy classes of paths from some point of $\mathcal{U}^{\mu_d}$ to some other one, up to a homotopy leaving the endpoints inside $\mathcal{U}^{\mu_d}$. Let $u \in \pi_0(\mathcal{U}^{\mu_d}) \simeq D_p^q(\Delta)$. By definition of $B_p^q(W)$, the group $B_p^q(W)(u, u)$ is canonically identified with every group $\pi_1((X/W)^{\mu_d}, x)$ with $x \in u$.

We now establish an isomorphism between the topological groupoid $B_p^q(W)$ and the groupoid $G(W, c)_p^q$ of (p, q)-periodic elements in the dual group $G(W, c)$. Let $s := (a, b) \in D_{2p}(\Delta) \simeq D_{2p}(c)$ represent a simple morphism in the divided category $M(W, c)_p$. By Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$), there is a unique element $x_s \in \mathcal{U}_p$ such that $\mathrm{clbl}(x) = s$ and $\overline{\mathrm{LL}}(x_s)$ consists of the points $e^{i\pi(\frac{1}{2}-\frac{2j+1}{2p})}$ such that the j-th term of s is nontrivial. Following [Bes15, Definition 11.20], we call x_s the *standard image* of s.

The circular tunnel $(x_s, \frac{\pi}{ph})$ then defines an element of $B_p(W)$ which we denote by b_s. If $s \in D_{2p}^{2q}(\Delta)$, then the circular tunnel $(x_s, \frac{\pi}{ph})$ also represents an element of $B_p^q(W)$, which we denote by b'_s. By [Bes15, Lemma 11.26], the maps $s \mapsto b_s$ and $s \mapsto b'_s$ extend to groupoid morphisms $\psi : G(W,c)_p \to B_p(W)$ and $\psi' : G(W,c)_p^q \to B_p^q(W)$. We then have a commutative diagram of functors

$$\begin{tikzcd}
M(W,c)_p^q \arrow[r, hook] \arrow[d] & M(W,c)_p \arrow[r, two heads, "\pi"] \arrow[d, hook] & M(W,c) \arrow[d, hook] \\
G(W,c)_p^q \arrow[r, hook] \arrow[d, "\psi'"] & G(W,c)_p \arrow[r, "\pi", "\sim"'] \arrow[d, "\psi"] & G(W,c) \arrow[d, "\simeq"] \\
B_p^q(W) \arrow[r] & B_p(W) \arrow[r, "\sim"] & B(W)
\end{tikzcd}$$

Where $\pi : G(W,c)_p \to G(W,c)$ denotes the collapse functor of Definition 4.4.15, and where arrows marked with "$\sim$" denote equivalences of groupoids.

Theorem 9.1.14. *[Bes15, Theorem 11.28]*
Let W be an irreducible well-generated complex reflection group with highest degree h and let d be a regular number for W. We use the notation of Theorem 8.2.11. We write $p := \frac{d}{d \wedge h}$ and $q := \frac{h}{d \wedge h}$. The functor $\psi : G(W,c)_p \to B_p(W)$ defined above is an isomorphism of groupoids, which restricts to an isomorphism of groupoids $\psi' : G(W,c)_p^q \to B_p^q(W)$.

Remark 9.1.15. Using this theorem, we will allow ourselves to identify the Springer groupoid with the associated groupoid of periodic elements.

Theorem 9.1.14 proves in particular that the fundamental group of $(X/W)^{\mu_d}$ (i.e. the braid group associated with the centralizer of a regular element in W) is a weak Garside group. This was the last case we needed in order to prove the following result:

Corollary 9.1.16. *Let W be a complex reflection group. The braid group $B(W)$ of W is a weak Garside group.*

Proof. Since the direct product of weak Garside groups is again a weak Garside group (take the direct product of the associated Garside groupoids), it is sufficient to prove the result when W is irreducible. If W is well-generated, then $B(W)$ is isomorphic to the dual group of type W, which is a Garside group. If $W = G(de,e,n)$ for $d,e,n \geqslant 2$, then W is a finite index subgroup of the well-generated group $G(de,1,n)$, and thus $B(W)$ is a weak Garside group by Corollary 4.2.4. Since we are working up to isodiscriminantality, this leaves the groups G_{12}, G_{13}, G_{22} and G_{31}. The groups G_{22} and G_{31} are both 4-regular centralizers in the well-generated groups G_{30} and G_{37}, respectively, and thus $B(G_{22})$ and $B(G_{31})$ are weak Garside group by Theorem 9.1.14. Lastly, the groups $B(G_{12})$ and $B(G_{13})$ are both circular groups (see Section B.1.5) and in particular they are Garside groups. $\square$

Lemma 9.1.17. *Let u be an object of $B_p^q(W)$. The group isomorphisms*

$$B(W_g) \simeq B_p^q(W)(u,u) \simeq G(W,c)_p^q(u,u)$$

maps the full-twist in $B(W_g)$ to $\Delta^{ph}(u)$.

Proof. The morphism $\Delta_p(u)$ in $M(W,c)_p^q$ corresponds to a rotation of angle $\frac{2\pi}{ph}$, while the full-twist corresponds to a rotation of angle 2π. $\square$

Using the isomorphism of Theorem 9.1.14, it is fairly easy to prove Theorem 9.1.7 in the case where W is well-generated.

Theorem 9.1.18. *[Bes15, Theorem 12.4] Theorem 9.1.7 holds when W is a well-generated irreducible complex reflection group.*

Proof. (a) Let $d > 0$ be an integer. By Lemma 9.1.5, we only have to show that, if there are d-regular braids in $B(W)$, then d is regular for W. By construction, the element $\Delta \in G(W,c) \simeq B(W)$ is represented by a circular tunnel $(x, \frac{2\pi}{h})$. From this we deduce that Δ^h represents the full-twist $z_p \in B(W)$. A d-regular braid in $B(W)$ is thus the same thing as a (d,h)-regular element in $G(W,c) \simeq B(W)$. By Theorem 3.4.4 (conjugacy of periodic elements), $G(W,c)$ contains (d,h)-periodic elements if and only if it contains (p,q)-periodic elements. By Corollary 4.5.4, this is equivalent to having $D_p^q(c) \neq \varnothing$. By Lemma 9.1.12, this is equivalent to $\mathcal{U}^{\mu_d} \neq \varnothing$ and to $(X/W)^{\mu_d} \neq \varnothing$. And we know that $(X/W)^{\mu_d} \neq \varnothing$ if and only if d is regular for W (see the discussion after Proposition 9.1.3).

(b) By Theorem 9.1.14, we have a bijection between the connected components of $B_p^q(W)$ and the connected components of the groupoid $G(W,c)_p^q$ of (p,q)-periodic elements in $G(W,c)$. Since $(X/W)^{\mu_d}$ is path connected, the Springer groupoid $B_p^q(W)$ is connected. But, by Corollary 4.5.4, the connected components of $G(W,c)_p^q$ are in bijection with the conjugacy classes of (p,q)-periodic elements (i.e. of d-regular braids) in $B(W)$. Since all d-regular braids are conjugates, their images in W are conjugate. And since regular elements in W have regular lifts, the images of d-regular braids are all ζ_d-regular elements in W.

(c) Let $u \in \pi_0(\mathcal{U}^{\mu_d}) \simeq D_p^q(\Delta)$, and let $x \in u \subset \mathcal{U}^{\mu_d}$. The morphism $\Delta_p(u) \in M(W,c)_p^q$ is sent by ψ' to the homotopy class of a circular tunnel of angle $\frac{2\pi}{ph}$, and thus $\psi'(\Delta_p^{ph}(u)) = z_p \in B(W)$. Furthermore, the group $\pi_1((X/W)^{\mu_d}, x)$ is canonically isomorphic to $B_p^q(W)(u,u)$. The image of this group in $B(W)$ is identified in $G(W,c)$ with the image of $G(W,c)_p^q(u,u)$ under the collapse functor π. By Theorem 4.5.2 (periodic elements and groupoid of periodic elements), this subgroup of $G(W,c)$ is the centralizer of $\pi(\Delta_p^q(u))$, which is a (p,q)-regular element in $G(W,c)$, and thus a d-regular braid in $B(W)$. $\square$

9.1.3 Proof of Theorem 9.1.7 for groups in the infinite series

This section consists in showing that Theorem 9.1.7 holds for irreducible groups belonging to the infinite series.

In this section, we fix three integers $d, e, n \geqslant 2$, and we write $r := de$. We consider the groups $W := G(r,1,n)$ and $W(e) := G(de,e,n)$.

Recall that we have an inclusion $G(de,e,n) \subset G(r,1,n)$ as a normal subgroup of index e. This inclusion induces an inclusion $B_n^*(e) \simeq B(G(de,e,n)) \subset B(G(r,1,n)) \simeq B_n^*$ as we saw in Section 6.2.4.

The complex reflection group $G(r,1,n)$ is well-generated, its degrees (resp. codegrees) $r, 2r, \cdots, nr$ (resp. $0, r, \cdots, (n-1)r$). In particular, the highest degree is $h = nr$, and we have by that the integers k which are regular for $G(r,1,n)$ are exactly the divisors of rn. We saw in

Section 8.3.2 that a Coxeter element in $G(r,1,n)$ is given by

$$c := c(r,1,n) = \begin{pmatrix} 0 & & & & \zeta_r \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

We can then consider the dual braid monoid $M(W,c)$ associated to c, and we have $G(W,c) \simeq B_n^*$. We know that the Garside element Δ of $M(W,c)$ is a $rn-$regular braid by Section 9.1.2. Since $\Delta^{rn/k}$ is always a k-regular braid when k divides rn, we obtain by Theorem 9.1.7 applied to $G(r,1,n)$ that every regular braid in B_n^* is conjugate to some power of Δ. Actually, using the tools of Section 4.5.2, we can show a stronger statement.

Lemma 9.1.19. *The periodic elements in the Garside group $G(W,c) \simeq B_n^*$ are exactly the conjugates of powers of Δ.*

Proof. The fact that conjugates of powers of Δ are periodic is obvious since conjugates of powers of periodic elements are always periodic elements. Conversely, we know by Corollary 4.5.15 that all periodic elements of $G(c)$ are conjugates of powers of rootless periodic elements. It is then sufficient to show that Δ is the unique rootless periodic element in B_n^*.

We know that $z_P = z_{B_n^*}^r$, since r is the gcd of the degrees of $G(r,1,n)$, and thus $z_{B_n^*} = \Delta^n$ is the smallest central power of Δ. In particular, the order of the Garside automorphism ϕ on the dual braid monoid associated to $G(r,1,n)$ is n.

Let now $\rho \in B_n^*$ be a rootless (p,q)-periodic element, for $p,q > 0$ two coprime integers. By Proposition 4.5.13 and Remark 4.5.14, we have that $p \leqslant n$ and that q divides the order n of the Garside automorphism, say with $qk = rn$.

The element ρ is also (pk, rn)-periodic, and a pk-regular braid as $\Delta^{rn} = z_P$. We then have $pk = \frac{p}{q} rn | rn$, and thus q divides p, which implies $q = 1$. We then have that $k = rn$, and ρ is a prn-regular braid. Again this implies that $prn | rn$ and $p = 1$. And thus ρ is $(1,1)$-periodic, and equal to Δ by definition. $\square$

Recall that we have a morphism $\mathrm{wd} : B_n^* \to \mathbb{Z}$ such that $B_n^*(e)$ is the kernel of the reduction wd_e of wd modulo e. Since $c = r_{\zeta_r,1} s_{1,1,2} \cdots s_{1,n-1,n}$, we have $\mathrm{wd}(\Delta) = 1$, and $\lambda := \Delta^e \in B_n^*(e)$. Now, since the hyperplane arrangements for $G(de,e,n)$ and $G(r,1,n)$ in $\mathbb{C}^n$ are the same (since $d \geqslant 2$), the element z_P is the same for $B_n^*(e)$ and B_n^*. So there is no conflict of notation between $z_{P(de,e,n)}$ and $z_{P(r,1,n)}$, and we have $z_P = \lambda^{dn}$.

The following proposition has been proven in [CLL15, Theorem 4.14], using the tools they introduce there. We provide a more direct proof here.

Proposition 9.1.20. *Every element in $B_n^*(e)$ which admits a central power is conjugate in $B_n^*(e)$ to some power of λ.*

Proof. Thanks to [DMM11, Theorem 1.4], since $B_n^*(e)$ is a finite index subgroup of B_n^*, we have $Z(B_n^*(e)) \subset Z(B_n^*)$. So if $\rho \in B_n^*(e)$ admits a central power in $B_n^*(e)$, then it also does in B_n^*. Since the center of B_n^* is generated by $z_{B_n^*} = \Delta^n$, we deduce that ρ is periodic in B_n^*, the last proposition then implies that ρ is conjugate in B_n^* to some power Δ^r of Δ.

Since $B_n^*(e)$ is normal, we must have $\Delta^r \in B_n^*(e)$, that is $\mathrm{wd}(\Delta^r) = r \equiv 0[e]$, so $r = pe$ for some integer p and we have

$$\exists g \in B_n^* \mid \rho^g = \Delta^{pe} = \lambda^p.$$

Thus ρ is conjugate to some power of λ, but the conjugating element g needs not be in $B_n^*(e)$. But Δ is an element of winding number 1 that centralizes λ, so assuming $i = \mathrm{wd}(g)$, we get

$$\rho^{g\varepsilon^{-i}} = (\lambda^{\Delta^{-i}})^p = \lambda^p,$$

and $g\varepsilon^{-i} \in B_n^*(e)$ is an element of $B_n^*(e)$ that conjugates ρ to λ^p. $\square$

This result is actually stronger than property (b) applied to $B_n^*(e)$, as it applies to all elements having a central power, and not just to roots of the full-twist.

Proposition 9.1.21. *Theorem 9.1.7 holds when W is a member of the infinite series.*

Proof. Since Theorem 9.1.7 is already known for well-generated groups, it is sufficient to consider the case of $G(de, e, n)$ with $d, e, n \geqslant 2$.

Let $\rho \in B_n^*(e)$ be such that $\rho^k = z_P$. By the last proposition ρ is conjugate to some λ^r and we have $\lambda^{rk} = z_P = \lambda^{dn}$. Since $B_n^*(e)$ has no torsion, we have $rk = dn$ and k divides dn, which gives point (a).

For point (b), any two k-th roots of z_P are conjugate to the same power of λ, and so they are conjugate. As powers of λ are mapped in $G(de, e, n)$ to some conjugate of a regular element, it is also the case of any k-th root of z_P.

For the last point, we only have to show the result for powers of λ. We consider $g := c^{ep}$ the image of λ^p in $W(e)$. We write $W(e)_g := C_{W(e)}(g)$ and $W_g := C_W(g)$. We have

$$C_{B_n^*(e)}(\lambda^p) = C_{B_n^*}(\lambda^p) \cap B_n^*(e) = C_{B_n^*}(\lambda^p) \cap \ker(\mathrm{wd}_e) = \ker(\mathrm{wd}_{e|C_{B_n^*}(\lambda^p)}),$$

$$W(e)_g = W_g \cap W(e) = W_g \cap \ker(\overline{\mathrm{wd}_e}) = \ker(\overline{\mathrm{wd}}_{e|W_g}),$$

where $\overline{\mathrm{wd}_e}$ is the morphism $W \to \mathbb{Z}/e\mathbb{Z}$ obtained by quotienting wd_e. We already know that the natural morphism $B(W_g) \to C_{B_n^*}(\lambda^p)$ is an isomorphism. Furthermore, since Δ commutes with λ and has winding number 1, the morphisms $\overline{\mathrm{wd}}_{e|C_W(g)}$ and $\mathrm{wd}_{e|C_{B_n^*}(\lambda^p)}$ are surjective, and we have short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & B(W(e)_g) \arrow[r] \arrow[ddd] & B(W_g) \arrow[rr, "\mathrm{wd}_e"] \arrow[ddd, "\simeq"'] \arrow[dr] & & \mathbb{Z}/e\mathbb{Z} \arrow[r] \arrow[ddd, equal] & 1 \\
& & & W_g \arrow[ur, "\overline{\mathrm{wd}_e}"'] \arrow[d] & & \\
& & & W \arrow[dr, "\overline{\mathrm{wd}_e}"] & & \\
1 \arrow[r] & C_{B_n^*(e)}(\lambda^p) \arrow[r] & C_{B_n^*}(\lambda^p) \arrow[rr, "\mathrm{wd}_e"'] \arrow[ur] & & \mathbb{Z}/e\mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

We deduce from this that the morphism $B(W(e)_g) \to C_{B_n^*(e)}(\lambda^p)$ is an isomorphism. This proves property (c) for $B_n^*(e)$. $\square$

The argument for property (c) in this proof was communicated to us by Ivan Marin.

9.1.4 Proof of Theorem 9.1.7 for regular centralizers

This section consists in showing that Theorem 9.1.7 holds for centralizers of regular elements in well-generated groups. More precisely we show that if Theorem 9.1.7 is true for a given group W, then it is also true for centralizers of regular elements in W (this idea originates from [Bes15, Remark 12.6]).

In this section, we fix an irreducible complex reflection group W for which Theorem 9.1.7 is true, along with an integer r and a r-regular braid $\delta \in B(W)$. We define g as the image of δ in W, and $W_g := C(W_g)$. We also fix an integer d, and we set $k = d \wedge r$, $d = d'k$ and $r = r'k$.

Since Theorem 9.1.7 holds for W, we know that $B(W_g)$ identifies with $C_{B(W)}(\delta)$

Lemma 9.1.22. *Let $\rho \in B(W_g)$ be a d-th root of z_P. The value of $\rho^v\delta^u$ does not depend on the choice of (u, v) such that $d'u + r'v = 1$, we denote it by $q(\rho)$. It is an element of $B(W)$ such that*

(a) $q(\rho)^{d\vee r} = z_P$, where $d \vee r$ is the lcm of d and r.

(b) $q(\rho)^{d'} = \delta$

(c) $q(\rho)^{r'} = \rho$

Proof. Let (u, v) be such that $d'u + r'v = 1$, we have

$$(\rho^v\delta^u)^{d\vee r} = \rho^{dr'v}\delta^{d'ru} = z_P^{r'v+d'u} = z_P.$$

Let now (u', v') be another pair such that $d'u' + r'v' = 1$. It is known that $v' = v + d'q$, $u' = u - r'q$ for some integer q. We obtain

$$(\rho^{v'}\delta^{u'}) = \rho^v\rho^{d'q}\delta^u\delta^{-r'q} = (\rho^v\delta^u)(\rho^{d'}\delta^{-r'})^q.$$

And since $(\rho^{v'}\delta^{u'})^{d\vee r} = z_P$, we have $(\rho^{d'}\delta^{-r'})^{q(d\vee r)} = 1$. Since $B(W)$ is torsion-free, we have $\rho^{v'}\delta^{u'} = \rho^v\delta^u$ as we claimed.

Then, since d' and r are also coprime, we can consider (a, b) such that $d'a + rb = 1$, we then get

$$q(\rho)^{d'} = (\rho^{kb}\delta^a)^{d'} = \rho^{db}\delta^{ad'} = z_P^b\delta^{1-rb} = \delta.$$

Likewise, by taking x, y such that $dx + r'y = 1$, we get $q(\rho)^{r'} = \rho$. ☐

Using this key lemma, we can easily prove Theorem 9.1.7 for $B(W_g)$.

(a) If ρ is a d-th root of z_P in $B(W_g)$, then $q(\rho)$ is a $d \vee r$-th root of z_P in $B(W_g) \subset B(W)$. So $d \vee r$ is regular for W by point (A) for W: it divides as many degrees as codegrees of W. But by definition of the lcm, the (co)degrees of W_g divided by r are exactly the (co)degrees of W divided by d and r (i.e. divided by $d \vee r$). So d is regular for W_g.

(b) If ρ and ρ' are two d-th roots of z_P in $B(W_g)$, then $q(\rho)$ and $q(\rho')$ are conjugate by some $g \in B(W)$. But then

- $g\delta g^{-1} = g(q(\rho)^{d'})g^{-1} = q(\rho')^{d'} = \delta$, so $g \in B(W_g) = C_{B(W)}(\delta)$
- $g\rho g^{-1} = g(q(\rho)^{r'})g^{-1} = q(\rho')^{r'} = \rho'$

and ρ and ρ' are conjugate by $g \in B(W_g)$.

(c) Let ρ be a d-th root of z_P in $B(W_g)$ and w its image in W_g. Again thanks to Lemma 9.1.22, we have

$$C_{B(W_g)}(\rho) = B(W_g) \cap C_{B(W)}(\rho) = C_{B(W)}(\rho) \cap C_{B(W)}(\delta) = C_{B(W)}(q(\rho)).$$

If $\widetilde{w}$ is a regular lift of w in W_g, then the same reasoning gives $C_{B(W_g)}(\widetilde{w}) = C_{B(W)}(q(\widetilde{w}))$ and $C_{W_g}(\pi(q(\widetilde{w}))) = C_W(\pi(q(\widetilde{w})))$. The morphism $B(C_{W_g}(w)) \to C_{B(W_g)}(\widetilde{w})$ is the same morphism as $B(C_W(\pi(q(\widetilde{w})))) \to C_{B(W)}(q(\widetilde{w}))$, which is known to be an isomorphism, this concludes the proof.

It is known that G_{31} and G_{22} respectively admits regular embeddings into G_{37} and G_{30}. Since both G_{37} and G_{30} are well-generated groups for which Theorem 9.1.7 is already known to hold, we get that Theorem 9.1.7 is also true for G_{22} and G_{31}.

9.1.5 Proof of Theorem 9.1.7 for groups of rank 2

This section consists in showing that Theorem 9.1.7 holds for the remaining irreducible complex reflection groups of rank 2, that is $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}$. We will prove the theorem for $G_7, G_{11}, G_{15}, G_{19}$ by one method, and for G_{12}, G_{13} by another method.

In this section, we fix a two-dimensional complex vector space V, along with an irreducible complex reflection group $W \subset \mathrm{GL}(V)$.

Since we are in rank 2, we can give easy general reductions. Let k be a regular number for W. By Theorem 6.1.27 (characterization of regular numbers), we have either

- k divides all degrees and all codegrees of W, the centralizer of a k-regular element g is then equal to W, and we have $V_g = V, X_g = X$. The morphism of Lemma 9.1.6 is then obviously an isomorphism.
- k divides exactly one degree d and it doesn't divide the nonzero codegree (this is possible only if d does not divide the nonzero codegree). In this case, the integer d is regular for W, and a k-regular element g is a power of a d-regular element g'. We then have $V_g = V_{g'} \simeq \mathbb{C}$, $X_g = X_{g'} \simeq \mathbb{C}^*$.
  The braid group $B(W_g)$ is then cyclic and generated by some regular lift of g' in $B(W_{g'}) = B(W_g)$. The image of the morphism of Lemma 9.1.6 is then a cyclic subgroup of $B(W)$ generated by some regular lift of g'. In order to show property (c) of Theorem 9.1.7 in this case, it is the sufficient to show that, for any k-regular braid ρ, the centralizer of ρ in $B(W)$ is equal to the centralizer in $B(W)$ of some d-regular braid ρ' such that $\rho \in \langle \rho' \rangle$ (such a root always exists by construction of regular lifts).

Degrees, codegrees and isodiscriminantality

No exceptional badly-generated group other than G_{31} and G_{22} admits a regular embedding inside a well-generated group. However, the preceding sections can still be used for more than these two groups. For instance, G_7 has the same degrees and codegrees as the group $G(12,2,2)$, which is a member of the infinite series. By Section 9.1.3, we know that our theorem holds for $G(12,2,2)$. Here we show that this is sufficient to show that it also holds for G_7.

Proposition 9.1.23. *Let W, W' be two irreducible complex reflection groups having the same degrees and codegrees. If property (a) (resp. property (b)) in Theorem 9.1.7 holds for W, then it also holds for W'.*

Proof. The main tool for this proof is the classification of all the pairs irreducible groups having the same degrees and codegrees. This classification is easy by direct inspection of the degrees and codegrees of each irreducible group, and summarized by the following lemma:

Lemma 9.1.24. *The only pairs of irreducible complex reflection groups having same degrees and codegrees are*

$$\begin{array}{cccc} G_5 \leftrightarrow G(6,1,2), & G_{10} \leftrightarrow G(12,1,2), & G_{18} \leftrightarrow G(30,1,2), & G_7 \leftrightarrow G(12,2,2), \\ G_{11} \leftrightarrow G(24,2,2), & G_{15} \leftrightarrow G(24,4,2), & G_{19} \leftrightarrow G(60,2,2), & G_{26} \leftrightarrow G(6,1,3). \end{array}$$

We could also include $G(2,2,3) \leftrightarrow G(1,1,4)$, $G(3,3,2) \leftrightarrow G(1,1,3)$ *and* $G(2,1,2) \leftrightarrow G(4,4,2)$*, but these pairs are moreover isomorphic as complex reflection groups.*

Each of these pairs are pairs of isodiscriminantal groups having a center of the same order (this last point is obvious as the cardinality of the center is the gcd of the degrees).

The isomorphism $\varphi : B(W) \to B(W')$ induced by isodiscriminantality sends $z_{B(W)}$ to $z_{B(W')}$. As the centers of W and W' have the same size, it also sends $z_{P(W)}$ to $z_{P(W')}$. So the existence of d-regular braids in $B(W)$ implies the existence of d-regular braids in $B(W)$. As the groups W and W' have the same degrees and codegrees, we get that properties (a) and (b) for W imply properties (a) and (b) for W'. $\square$

It remains to prove property (c). We keep the notation of the proof of Lemma 9.1.24. Let k be a regular number for W which divides a unique degree d. If ρ is an k-regular braid, and ρ' is a d-regular braid which is a root of ρ. We already know by assumption that $C_{B(W)}(\varphi^{-1}(\rho)) = C_{B(W)}(\varphi^{-1}(\rho')) = \langle \varphi^{-1}(\rho') \rangle$, and the isomorphism φ gives the desired result. This settles the cases of G_7, G_{11}, G_{19} and G_{15}, since $G(12,2,2), G(24,2,2), G(60,2,2)$ and $G(24,4,2)$ belong to the infinite series.

The two remaining exceptional groups

By now, the only remaining groups are G_{12} and G_{13}. Contrary to the "direct" approach given in [Gar23a, Section 7], we are going to study $B(G_{12})$ and $B(G_{13})$ using circular groups (see Appendix B).

First, we know by [Ban76, Theorem 1 and 2] that $B(G_{12})$ is isomorphic to the circular group $G(3,4) = \langle s,t,u \mid stus = tust = ustu \rangle$. By Corollary B.1.11, the center of $G(3,4)$ is generated by $\Delta^3 = (stu)^4$, and thus the isomorphism $B(G_{12}) \simeq G(4,3)$ sends $z_P = z_B^2$ to Δ^6. A d-regular braid in $B(G_{12})$ is then a $(d,6)$-periodic element in $G(3,4)$.

Let ρ be a d-regular braid. By Proposition B.1.13, we can up to conjugacy assume that ρ is a power of either Δ, which is $(1,1)$-periodic, or of stu, which is $(4,3)$-periodic.

- If $\rho = \Delta^k$ is a power of Δ, then $\rho^d = \Delta^{dk} = \Delta^6$ implies that d divides 6.
- If $\rho = (stu)^k$ is a power of stu, then $\rho^d = (stu)^{dk} = \Delta^6 = (stu)^8$. By comparing the length, we obtain that d divides 8.

In either case, we have that d is a regular number of G_{12}, which proves property (a) of Theorem 9.1.7 (again, the converse statement of property (a) is known by Lemma 9.1.5). Property (b) is an immediate consequence of Proposition B.1.12.

Now, let d be a regular number which divides exactly one degree of G_{12}, and does not divide the nonzero codegree 10. We either have

- $d = 3, 6$, which only divides 6. By setting $k := 6/d$, a d-regular braid is conjugate to Δ^k with $k = 1, 2$. By Lemma B.1.9, we have $C_{G(3,4)}(\Delta^k) = \langle \Delta \rangle$.
- $d = 4, 8$, which only divides 8. By setting $k := 8/d$, a d-regular braid is conjugate to $(stu)^k$ with $k = 1, 2$. By Lemma B.1.10, we have $C_{G(3,4)}((stu)^k) = \langle stu \rangle$.

And property (c) follows.

Now, we know by [Ban76, Theorem 1 and 2] that $B(G_{13})$ is isomorphic to the circular group $G(2,6) = \langle s, t \mid ststst = tststs \rangle$. By Corollary B.1.11, the center of $G(2,6)$ is generated by $\Delta = ststst$, and thus the isomorphism $B(G_{13}) \simeq G(2,6)$ sends $z_P = z_B^4$ to Δ^4. A d-regular braid in $B(G_{13})$ is then a $(d,4)$-periodic element in $G(2,6)$.

Let ρ be a d-regular braid. By Proposition B.1.13, we can up to conjugacy assume that ρ is a power $(st)^k$ of st, which is $(3,1)$-periodic. We then have $(st)^{dk} = (st)^{12}$, and thus d divides 12. We then have that d is a regular number of G_{13}, which proves property (a) of Theorem 9.1.7 (again, the converse statement of property (a) is known by Lemma 9.1.5). Property (b) is an immediate consequence of Proposition B.1.12.

Now, let d be a regular number which divides exactly one degree of G_{13}, and does not divide the nonzero codegree 16. Since the degrees of G_{13} are 8 (which divides 16) and 12, we are looking at the integers $d = 3, 6, 12$. By setting $k = 12/d$, a d-regular braid is conjugate to $(st)^k$ with $k = 1, 2, 4$. By Lemma B.1.9, we have $C_{G(2,6)}((st)^k) = \langle st \rangle$, and property (c) follows.

9.1.6 A Kerékjártó type by-product

In [Bes15, Remark 12.5], Bessis suggests that our main theorem can be thought of as an analogue of the Kerékjártó Theorem, stating that every periodic homeomorphism of the disk is conjugate to a rotation. In this last section we establish a rephrasing of this analogy.

In this section, we fix an irreducible complex reflection group W.

Definition 9.1.25 (Periodic braid). An element $b \in B(W)$ is *periodic* if there is an integer $n > 0$ such that $b^n \in Z(B(W))$.

Recall that the *regular Springer set* (see [Shv96, Definition 3]) of W is the set of regular numbers for W which are maximal with respect to divisibility. The elements of this set will be called the *fundamental regular numbers*. We say that a d-regular braid is *fundamental* if d is in the regular Springer set of W.

Proposition 9.1.26. *The periodic elements in $B(W)$ are exactly the powers of fundamental regular braids. In particular, the image of a periodic element in W is a regular element.*

If $B(W)$ is a Garside group with Garside element Δ such that z_B is a power of Δ, then the center $Z(B(W))$ is infinite cyclic and an element of $B(W)$ is periodic if and only if it is periodic in the Garside sense. This also justifies the connection between Proposition 9.1.26 and the Kerékjártó Theorem.

Proof. This statement is already known to hold for the infinite series by Lemma 9.1.19 and Proposition 9.1.20. Indeed Δ (resp. λ) is the only fundamental root of z_P in B_n^* (resp. $B_n^*(e)$) up to conjugacy. Let us first prove proposition 9.1.26 in the case where $B(W)$ is a Garside group, and $z_B = \Delta^n$. Since z_B generates the center of $B(W)$, we have that n is the order of the Garside automorphism of $B(W)$ associated to Δ. By Corollary 4.5.15, we know that every periodic element in $B(W)$ is conjugate to a rootless periodic element. Let then ρ be a rootless (p, q)-periodic element, with p, q coprime. By Proposition 4.5.13, we have that q divides n. In particular, ρ is a root of z_B and of z_P, and it is a regular braid. Since the only regular braids which are rootless in $B(W)$ are the fundamental regular braids, we have the result in this case. This settles the case of almost every irreducible group, the exception being G_{31}. Let δ be a 4-th root of z_P in $B(G_{37})$. We know that $B(G_{31})$ is isomorphic to $C_{B(G_{37})}(\delta)$. We have $Z(C_{B(G_{37})}(\delta)) = \langle\delta\rangle$ (see [Bes15, Corollary 12.7]), that is the isomorphism $C_{B(G_{37})}(\delta) \simeq B(G_{31})$ sends δ to z_B. The case of $B(G_{31})$ is then a consequence of Lemma 9.1.22. □

The description of periodic elements we provide in the case of an irreducible complex braid group allows us to give a generalization of [Shv96, Theorem B]: As all fundamental regular numbers are divisible by the cardinality $|Z(W)|$ of the center of W, we can define the *modified Springer set* as the set of elements of the regular Springer set divided by $|Z(W)|$.

Proposition 9.1.27 (Shvartsman). *The order of any torsion element in $B(W)/Z(B(W))$ divides one of the elements of the modified Springer set of W. Conversely, for each divisor t of an element of the modified Springer set, there is an element $\gamma \in B(W)/Z(B(W))$ having order t.*

Proof. Let γ be an element of finite order. There is some periodic element $x \in B(W)$ having image γ in the quotient $B(W)/Z(B(W))$. By Proposition 9.1.26, x is equal to some ρ^p where ρ is a fundamental regular braid and p is a positive integer. Since ρ is a fundamental regular braid, there is an element d of the regular Springer set such that $\rho^d = z_P$. The integer $d' = \frac{d}{|Z(W)|}$ is an element of the modified Springer set such that $\rho^{d'} = z_B$. Let r be the image of ρ in $B(W)/Z(B(W))$. We have $\gamma = r^p$, so the order of γ is $\frac{p \vee d'}{p} = \frac{d'}{p \wedge d'}$, where $p \vee d'$ is the lcm of p and d', and $p \wedge d'$ is their gcd. So the order of γ divides d'.

Conversely, let d be an element of the regular Springer set, let $d' = \frac{d}{|Z(W)|}$ be the associated element of the modified Springer set. Let ρ be a d-th root of z_P, it induces an element of order d' in $B(W)/Z(B(W))$, so considering powers of ρ gives the desired results. □

In particular, we also obtain a convenient test of periodicity in complex braid groups.

Corollary 9.1.28 (Characterization of periodic braids). *An element $b \in B(W)$ is periodic if and only if $b^k \in Z(B(W))$ for k belonging to the modified Springer set of W.*

Proof. An element of $B(W)$ is periodic if and only if it is a torsion element in $B(W)/Z(B(W))$, the result is then an obvious consequence of Proposition 9.1.27. □

As we said, for the usual braid group B_n, associated to the complex reflection group $G(1, 1, n)$, the modified Springer set is $\{n - 1, n\}$, and we find that a (classical) braid b is periodic if and only if either b^{n-1} or b^n is central.

9.2 Springer groupoids

As stated earlier, Theorem 9.1.14 gives a way to study the braid group of the centralizer of a regular element in a well-generated complex reflection group using Garside theory. This section is dedicated to the study of the Springer groupoids attached to such a regular centralizer, and to the results we deduce for the associated braid groups.

Throughout this section, we use freely the notation of Chapter 4 and Chapter 5.

9.2.1 Elementary properties

In this section, we fix an irreducible well-generated complex reflection group W of rank n. We define h as the highest degree of W, and we fix d a regular number for W (the number $d = 1$ is a valid choice, which simply gives the dual group of type W). We set again

$$p := \frac{d}{d \wedge h} \text{ and } q := \frac{h}{d \wedge h}.$$

We write $(G(W), M(W), \Delta)$ for the dual group of type W. We denote by $\mathcal{S}$ its set of simple elements, and by ϕ its Garside automorphism. We denote by $(\mathcal{G}, \mathcal{C}, \Delta_p)$ the groupoid $G(W)_p^q$ of (p, q)-periodic elements attached to $(G(W), M(W), \Delta)$. We call $\mathcal{G}$ the *Springer groupoid* and we call $\mathcal{C}$ the *Springer category*. If need be, we will also denote by $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid associated to $(G(W), M(W), \Delta)$.

Remark 9.2.1. Notice that, up to isomorphism of Garside groupoids, the Springer groupoid depends only on W and on the integer d. We will then allow ourselves to talk about the Springer groupoid attached to a well-generated complex reflection group and to a regular integer for this group.

By definition, $(\mathcal{G}, \mathcal{C}, \Delta_p)$ is the groupoid of fixed points under the automorphism ϕ_p^q of $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ (notice the slight conflict of notation with Section 4.4: $\mathcal{G}_p$ does not denote the divided groupoid of the groupoid $\mathcal{G}$ in this section).

Recall from Definition 4.5.7 (germ of periodic elements) that $\mathcal{C}$ depends on the sets $D_p^q(\Delta)$, $D_{2p}^{2q}(\Delta)$ and $D_{3p}^{3q}(\Delta)$. Thanks to Remark 4.4.4 and Proposition 4.5.6, we have

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) := D_p^q(c) &\simeq \left\{ u \in \mathcal{S}^{\phi^q} \;\middle|\; \begin{cases} p.\ell_R(u) = \ell_R(c) = n \\ u(u^{c^\lambda}) \cdots (u^{c^{(p-1)\lambda}}) = c \end{cases} \right\} \\ \mathcal{S}_p^q := D_{2p}^{2q}(c) &= \left\{ (a,b) \in (\mathcal{S}^{\phi^q})^2 \mid ab \in \mathrm{Ob}(\mathcal{C}) \right\} \\ R_p^q := D_{3p}^{3q}(c) &= \left\{ (x,y,z) \in (\mathcal{S}^{\phi^q})^3 \mid xyz \in \mathrm{Ob}(\mathcal{C}) \right\}, \end{aligned}$$

where λ, μ are positive integers such that $p\lambda - q\mu = 1$, and where ℓ_R is the reflection length on W.

Under these definitions, a simple morphism (a, b) in the graph of simples $\mathcal{S}_p^q$ goes from ab to ba^{c^λ} in $\mathcal{C}$, and an element (x, y, z) of $D_{3p}^{3q}(c)$ induces the relation $(x, yz)(y, zx^{c^\lambda}) = (xy, z)$ in R_p^q. The following lemma is an obvious consequence of the definition of $\mathrm{Ob}(\mathcal{C})$.

Lemma 9.2.2. *Let $u \in \mathrm{Ob}(\mathcal{C})$, the morphism $\Delta_p(u) = (u, 1)$ has length $\ell_R(u) = \frac{n}{p}$. This length doesn't depend on u and every simple morphism in $\mathcal{C}$ has length at most $\frac{n}{p}$.*

Let $u \in D_p^q(c)$ be an object of $\mathcal{C}$. The collapse functor $\pi_0 : \mathcal{C} \to M(W)$ sends $\Delta_p^q(u)$ to some (p, q)-periodic element in $M(W)$ by Theorem 4.5.2 (periodic elements and groupoid of periodic elements). Actually, using Theorem 3.4.4 (conjugacy of periodic elements), we obtain explicit formulas for regular braids in $B(W)$, provided that one knows how to compute elements of the sets $D_m^n(c)$.

Applying Corollary 4.5.10 (atoms in category of periodic elements) to the Springer category yields

Lemma 9.2.3 (Atoms in the Springer category). *The atoms of $\mathcal{C}$ are exactly the simples $s = (a,b) \in \mathcal{S}_p^q(c)$ such that a is an atom in $\mathcal{S}^{\phi^q}$. In particular if ϕ^q is trivial, a simple morphism of $\mathcal{C}$ is an atom if and only if it has length 1.*

Proposition 9.2.4 (No pairs of parallel simples). *If $p > 1$, then there are no pairs of parallel simples in $\mathcal{C}_p$, that is, a simple morphism is uniquely determined by its source and target. In particular, for $u \in \mathrm{Ob}(\mathcal{C}_p)$, we have $\mathcal{S}_p(u,u) = \{1_u\}$ and $\mathcal{S}_p(u,\phi_p(u)) = \{\Delta_p(u)\}$.*

Proof. Let $s := (a,b)$ be a simple morphism in $\mathcal{C}_p$ (recall that a, b are then p-tuples in $\mathcal{S} \subset M(c)$), and let u, v denote the source and target of s, respectively. We have $u = ab$ and $v = b\tau(a)$. We claim that $b = u \wedge v$. The result is obvious when $a = (1, \ldots, 1)$, so we assume that a is nontrivial. We have $ab = ba^b$, so b is an obvious left-divisor of ab and $b\tau(a)$. If $b \neq u \wedge v$, then there is a nontrivial common divisor d of a^b and $\tau(a)$. Let $i \in [\![0, p-1]\!]$ be such that d_i is nontrivial. Since $p > 1$, we have $\Delta = a_0b_0a_1b_1 \cdots a_{p-1}b_{p-1}$ and $a_ib_i\tau(a)_i$ is simple. Since all elements if $\mathcal{S} = I(c)_T$ are balanced, we obtain that d_i^2 divides $a_ib_i\tau(a)_i$, thus d_i^2 is trivial by Lemma 8.2.12.

Now, let $(a', b') \in \mathcal{S}_p$ have the same source and target as (a,b). We have $u = ab = a'b'$ and $v = b\tau(a) = b'\tau(a')$. We have $b \preccurlyeq b'$ and $b' \preccurlyeq b$, so $b = b'$ and $(a,b) = (a',b')$. □

Since simple elements of $\mathcal{C}$ are in particular simple elements of $\mathcal{C}_p$, we have the following immediate corollary:

Corollary 9.2.5 (No pairs of parallel simples). *If $p > 1$, then there are no pairs of parallel simples in $\mathcal{C}$. That is, a simple morphism is uniquely determined by its source and target. In particular, for $u \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{S}_p^q(u,u) = \{1_u\}$ and $\mathcal{S}_p^q(u,\phi_p(u)) = \{\Delta_p(u)\}$.*

In the case $p = 1$. There is only one object in $\mathcal{C}$, which corresponds to $\Delta \in D_1(\Delta)$. The above corollary is then false in this case.

Lemma 9.2.6 (Lifting words expressing simples). *Let $s := (a,b)$ be a simple morphism in $\mathcal{C}$, and let $a_1 \cdots a_r$ be a word in $\mathcal{S}^{\phi^q}$ expressing a in $M(W)$. There is a unique path $s_1 \cdots s_r$ in $\mathcal{S}_p^q$ expressing s in $\mathcal{C}$ and such that $\pi_0(s_i) = a_i$ for all $i \in [\![1, r]\!]$*

Proof. Let $u := ab$ be the source of s. We proceed by $\succcurlyeq$-induction on s. If s is an atom, then a is an atom of $\mathcal{S}^{\phi^q}$ by Lemma 9.2.3. The only word in $\mathcal{S}^{\phi^q}$ expressing a is then a itself and the result is trivial. Now for the general case, we have $a_1x = a$ with $x = a_2 \cdots a_r$. By Corollary 4.5.11 (characterization of divisibility), $s_1 := (a_1, xb)$ is the only atom in $\mathcal{C}$ with source u and such that $\pi_p(s_1) = a_1$. By induction hypothesis, there is a unique path $s_2 \cdots s_r$ expressing $(x, ba_1^{c^\lambda})$ in $\mathcal{C}$ and such that $\pi_p(s_i) = a_i$ for $i \in [\![2, r]\!]$. The path $s_1s_2 \cdots s_r$ is then the unique path expressing s and such that $\pi_p(s_i) = a_i$ for $i \in [\![1, n]\!]$. □

Let $s := (a,b) \in \mathcal{S}_p$ be a simple morphism in $\mathcal{C}_p$. Its source is $ab = ba^b$ and its target is $b\tau(a) = \tau(a)^{b^{-1}}b$. By Corollary 4.5.11, we deduce the existence of the following simple morphisms in $\mathcal{C}$:

$$s^\flat := (\tau^{-1}(a^b), b) \text{ and } s^\# := (\tau(a)^{b^{-1}}, b).$$

Lemma 9.2.7. *Let $s := (a,b)$ be a simple morphism in $\mathcal{C}_p$.*

(a) The target of $s^\flat$ is the source of s, and the target of s is the source of $s^\#$.

(b) *We have* $(s^\flat)^\# = (s^\#)^\flat = s$.

(c) *The paths* $s^\flat s$ *and* $ss^\#$ *are both in greedy normal form in* $\mathcal{C}$.

(d) *We have* $\phi_p(s^\#) = (\phi_p(s))^\#$ *and* $\phi_p(s^\flat) = (\phi_p(s))^\flat$. *In particular, for* $s \in \mathcal{S}_p^q$, *we have* $s^\flat, s^\# \in \mathcal{S}_p^q$.

Proof. (a) The target of $s^\flat$ is $ba^b = ab$ and the source of $s^\#$ is $\tau(a)^{b^{-1}}b = b\tau(a)$.
(b) We have $s^{\flat\#} = (\tau^{-1}(a^b), b)^\# = (\tau(\tau^{-1}(a^b))^{b^{-1}}, b) = (a, b) = s$. The same reasoning proves that $(s^\#)^\flat = s$.
(c) Since $s = (a, b)$ is a simple morphism, $ab = u$ is a p-tuple of simple element in $M(c)$. In particular, $a \wedge b = 1$ by Proposition 8.2.13. We get that the path $s^\flat s$ is greedy by Corollary 4.4.14 (greediness in divided category). The path $ss^\# = (s^\#)^\flat s^\#$ is also greedy by the same argument.
(d) We have

$$(\phi_p(s))^\flat = (\tau(a), \tau(b))^\flat = (\tau^{-1}(\tau(a)^{\tau(b)}), \tau(b)) = (a^b, \tau(b)) = \phi_p(s^\flat),$$

and

$$(\phi_p(s))^\# = (\phi_p(s^{\#\flat}))^\# = ((\phi_p(s^\#))^\flat)^\# = \phi_p(s^\#).$$

□

Let $s \in \mathcal{S}_p^q$ be a simple morphism in $\mathcal{C}$. By using the description of s as a pair of elements (a, b) rather than a pair of p-tuples (Proposition 4.5.6), the source of s is source is $ab = b\phi^\lambda(a^{bc^{-\lambda}})$ and its target is $b\phi^\lambda(a) = a^{c^\lambda b^{-1}}b$. The morphisms $s^\flat$ and $s^\#$ are then given by

$$s^\flat := (a^{bc^{-\lambda}}, b) \text{ and } s^\# := (a^{c^\lambda b^{-1}}, b).$$

The transformation $s \mapsto s^\#$ is a bijection of the finite set S_p: it then has finite order. As the source of $s^\#$ is the target of s, there is then a smallest integer n such that $s^{(n\#)} = s$.

Definition 9.2.8 (Simple loop). Let $s : u \to v$ be a simple morphism in $\mathcal{C}_p$. The *simple loop* (of the object u) associated to s is the morphism

$$\lambda(s) := ss^\# s^{\#\#} \cdots s^{\flat\flat} s^\flat \in \mathcal{C}_p(u, u).$$

If s is an atom of $\mathcal{C}$, then we say that $\lambda(s)$ is an *atomic loop* (in $\mathcal{C}$, of the object u).

Lemma 9.2.9. *Let* s *be a simple morphism in* $\mathcal{C}$. *The simple loop* $\lambda(s)$ *is rigid, in particular it lies in its own super-summit set.*

Proof. By Lemma 9.2.7 (c), the greedy normal form of $\lambda(s)$ is given by

$$\lambda(s) := ss^\# s^{\#\#} \cdots s^{\flat\flat} s^\flat.$$

Since $s^\flat s$ is greedy, we get that $\lambda(s)$ is rigid (i.e. the sliding $\mathrm{sl}(\lambda(s))$ is equal to $\lambda(s)$). We then have $\mathrm{cyc}(\lambda(s)) = \lambda(s^\#)$ and $\mathrm{dec}(\lambda(s)) = \lambda(s^\flat)$. In particular we see that cycling and decycling a simple loop doesn't change its inf or its sup. By Corollary 3.2.7, we deduce that $\lambda(s)$ belongs to its own super-summit set. □

Remark 9.2.10. Let u be an object of $M(c)_p$. We have

$$\Delta_p(u)^{\#} = (u,1)^{\#} = (u^{c^\lambda},1) = \phi_p(\Delta_p(u)) = \Delta_p(\phi_p(u)),$$

so $\lambda(\Delta_p(u)) = \Delta_p^k(u)$ where k is the smallest integer such that $\phi_p^k(u) = u$. By definition of $\mathcal{C}$, we have that k divides q: there is some k' such that $kk' = q$. The element $(\Delta_p)^q(u)$ is then equal to $\lambda(\Delta_p(u))^{k'}$. By Theorem 4.5.2 (periodic elements and groupoid of periodic elements), the group $\mathcal{B}(u,u)$ then identifies with the centralizer in $G(c)$ of $\pi_p(\lambda(\Delta_p(u))^{k'})$.

9.2.2 Springer groupoids with one objects are dual groups

In this section, we show that, when Springer groupoids only have one object, they coincide with the dual group of the associated regular centralizer (which is then well-generated). Let W be a well-generated complex reflection group of highest degree h, and let d be a regular number for W. By Corollary 9.2.5 (no pairs of parallel simples), the Springer groupoid attached to W and d has one object if and only if $p = d/d \wedge h = 1$, i.e. if d divides h.

Exceptionally in this section, the interval in a complex reflection group W attached to the set of all its reflections and to a Coxeter element c will be denoted by $I(c)_W$.

In this section, we fix an irreducible well-generated complex reflection group of highest degree h, along with an integer d dividing h. We fix $q := h/d$. Let also $c \in W$ be a Coxeter element, and let $(G(W,c), M(W,c), \Delta)$ be the associated dual group. The Garside automorphism of $G(W,c)$ is denoted by ϕ, and its set of simple elements is identified with $I(c)_W$.

In this situation, the Springer groupoid associated to (W,d) is the groupoid of $(1,q)$-periodic elements of the dual group $G(W)$. Since $G(W)$ is a group, the Springer groupoid is simply the Garside group $(G(W)^{\phi^q}, M(W)^{\phi^q}, \Delta)$. Its set of simple elements corresponds to the set $(I(c)_T)^{c^q}$ of simple elements which commute with c^q in W.

We know that the centralizer $W_{c^q} := C_W(c^q)$ acts as a complex reflection group on the eigenspace $V_{c^q} := \mathrm{Ker}(c^q - \zeta_h^q)$. Furthermore, it is easy to show that W_{c^q} acting on V_{c^q} is a well-generated complex reflection group: Let d be the order of ζ_h^q in $\mathbb{C}^*$, we know that d is a regular integer for W, and by Theorem 6.1.26 (Springer theory in complex reflection groups), the degrees (resp. codegrees) of W_{c^q} are exactly the degrees (resp. codegrees) of W which are divisible by d. Let $d_1 \leqslant \ldots \leqslant d_n$ be the degrees of W, and let $d_1^* \geqslant \ldots \geqslant d_n^*$ be its codegrees. Since d divides $h = d_i + d_i^*$ for all $i \in [\![1,n]\!]$, we have that d divides d_i if and only if it divides d_i^*. Thus, if the degrees of W_{c^q} are $d_{i_1}, \ldots, d_{i_k}$, then the codegrees of W_{c^q} are $d_{i_1}^*, \ldots, d_{i_k}^*$. In particular for $j \in [\![1,k]\!]$, we have $d_{i_j} + d_{i_j}^* = h = d_{i_k}$ and W_{c^q} is well-generated.

We also know that $c \in W_{c^q}$ is still a Coxeter element for W_{c^q}, and thus we can consider the dual group $(G(W_{c^q}, c), M(W_{c^q}, c), \Delta')$. We then have two candidate Garside structures for the braid group $B(W_{c^q})$: on the one hand its associated dual group, and on the other hand the Garside group $(G(W,c)^{\phi^q}, M(W,c)^{\phi^q}, \Delta)$. We are going to show that this two Garside structures are in fact isomorphic.

Theorem 9.2.11. *Let W be a well-generated irreducible complex reflection group with Coxeter element c, and let q be a positive integer. Let also $T := \mathrm{Ref}(W)$ and $T_q := \mathrm{Ref}(W_{c^q})$. Then the posets $((I(c)_W)^{c^q}, \preccurlyeq_T)$ and $(I(c)_{W_{c^q}}, \preccurlyeq_{T_q})$ are equal. That is, we have $(I(c)_W)^{c^q} = I(c)_{W_{c^q}}$ as subsets of W_{c^q} and, for $s,t \in I(c)_{W_{c^q}}$, we have $s \preccurlyeq t$ in W if and only if $s \preccurlyeq t$ in W_{c^q}.*

Corollary 9.2.12. *The monoids $M(W,c)^{\phi^q}$ and $M(W_{c^q}, c)$ are isomorphic.*

The corollary comes from the fact that the defining presentation of an interval monoid only depends on the poset structure of the associated interval $(I, \preccurlyeq)$ and on the group structure: the defining relations are all relations of the form $s(s^{-1}t) = t$ for $s, t \in I$ with $s \preccurlyeq t$.

Remark 9.2.13. If W is a complexified real reflection groups, and if (W, S) is a Coxeter system for W. We know that the Artin-Tits monoid associated to (W, S) is a Garside monoid (with enveloping group isomorphic to $B(W)$). It is also the interval monoid associated with the data (W, S, w_0) (where w_0 is the longest element of W). If $Z(W)$ is trivial, then the action of w_0 induces an automorphism of W which stabilizes S globally (a *diagram automorphism*). The centralizer of w_0 in W is known to again be a Coxeter group, whose Artin group is identified with the centralizer of Δ in $B(W)$ (see [Cri00]). Theorem 9.2.11 is an analogue of this situation for the dual braid monoid.

The proof of Theorem 9.2.11 is ultimately going to rely on a case-by-case analysis, but we can do some easy reductions. First we assumed in the beginning of the section that the integer q divides h. This assumption does not appear in Theorem 9.2.11. The following lemma shows that this assumption is not needed.

Lemma 9.2.14. *We keep the notation of Theorem 9.2.11. If Theorem 9.2.11 holds when q divides the highest degree of W, then it holds for all values of q.*

Proof. Let h be the highest degree of W. Since h is the order of c, we have that the order d of c^q divides h. We set $q' := h/d$. The two elements c^q and $c^{q'}$ both have order d in the cyclic group $\langle c\rangle$ of order h. There are two integers a and b with $c^{aq} = c^{q'}$ and $c^{q'b} = c^q$. We then have $W_{c^q} = W_{c^{q'}}$, $(I(c)_W)^{c^q} = (I(c)_W)^{c^{q'}}$ and $M(W, c)^{\phi^q} = M(W, c)^{\phi^{q'}}$. Thus Theorem 9.2.11 holds for q if and only if it holds for q', which divides h. □

The element c^q is $\zeta_d = \zeta_h^q$-regular. If d divides all the degrees of W, then c^q is central in W and we have $W_{c^q} = W$, Theorem 9.2.11 is obvious in this case. If h is the only degree of W divisible by d, then W_{c^q} is a complex reflection group of rank 1: it is cyclic and equal to $\langle c\rangle$. We have $I(c)_{W_{c^q}} = \{\mathrm{Id}, c\}$. On the other hand, any $s \in (I(c)_W)^{c^q}$ lies in $W_{c^q} = \langle c\rangle$. We then have $I_c^q = \{\mathrm{Id}, c\}$ by [Bes15, Lemma 12.2]. The poset structure is induced by $\mathrm{Id} \preccurlyeq c$ in both cases and Theorem 9.2.11 holds.

Note that these two extreme cases are sufficient to prove Theorem 9.2.11 when W has rank 2. We now distinguish whether or not W belongs in the infinite series. The case where W is a well-generated irreducible exceptional group of rank > 2 is handled directly by computer.

If W is a well-generated irreducible group of rank $n \geqslant 3$ belonging to the infinite series then we have either

- $W \simeq \widetilde{G}(1, 1, n+1)$, the set of reflections of this group is equal to that of $G(1, 1, n+1)$, and their sets of Coxeter elements are the same (they are the $n+1$-cycles in $\mathfrak{S}_{n+1}$). Thus the interval monoids given by W and $G(1, 1, n+1)$ are the same and we can restrict our attention to the latter group. In this case, Theorem 9.2.11 was already proven in Section 8.3.2 (see Proposition 8.3.18).
- $W \simeq G(r, 1, n)$ for $r \geqslant 2$.
- $W \simeq G(e, e, n)$ for $e \geqslant 2$.

We then prove Theorem 9.2.11 for the two families of remaining cases.

The case $W = G(r,1,n)$ for $r \geqslant 2$

The highest degree of W is rn. We saw in Section 8.3.2 that we can endow W with the Coxeter element

$$c(r,1,n) = \begin{pmatrix} 0 & & & & \zeta_r \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

We also saw that $G(r,1,n)$ can be realized as the centralizer in $W' := G(1,1,rn)$ of $c(1,1,rn)^r$, and this identification sends $c(r,1,n)$ to $c(1,1,rn)$. Let q be a divisor of rn. We have

$$W_{c(r,1,n)^q} \simeq (W_{c(1,1,rn)^r})_{c(1,1,rn)^q} = W'_{c(1,1,rn)^{q \vee r}}.$$

This identification induces the following isomorphisms of posets:

$$(I(c(r,1,n))_W)^{c(r,1,n)^q} \simeq (I(c(1,1,rn))_{W'_{c(1,1,rn)^r}})^{c(1,1,rn)^q},$$

$$I(c(r,1,n))_{W_{c(r,1,n)^q}} \simeq I(c(1,1,rn))_{W'_{c(1,1,rn)^{q\vee r}}}.$$

By Proposition 8.3.18, we have equalities of posets

$$\begin{aligned}(I(c(1,1,rn))_{W'_{c(1,1,rn)^r}})^{c(1,1,rn)^q} &= ((I(c(1,1,rn))_{W'})^{c(1,1,rn)^r})^{c(1,1,rn)^q} \\ &= I(c(1,1,rn))_{W'}^{c(1,1,rn)^{r\vee q}} \\ &= I(c(1,1,rn))_{W'_{c(1,1,rn)^{q\vee r}}},\end{aligned}$$

which gives the desired results.

The case $W = G(e,e,n)$ for $e \geqslant 2$, $n \geqslant 3$

The highest degree of W is $e(n-1)$. We start by considering the group $W' := G(e,1,n-1)$. As we saw in Section 8.3.3, the character $\chi : W' \to \mathbb{C}^*$ giving the product of the nonzero entries allows us to define an embedding $W' \to W$, which sends the Coxeter element $c(e,1,n-1)$ to a Coxeter element $c(e,e,n)$ of W. Let q be a divisor of $e(n-1)$. We have

$$c(e,e,n)^q = \begin{pmatrix} \zeta_e^{-q} & 0 \\ 0 & c(e,1,n-1)^q \end{pmatrix}.$$

We can compute directly the centralizer of this element in W:

Proposition 9.2.15. *Let q be a positive integer dividing $e(n-1)$.*

(a) If $\frac{e(n-1)}{e\wedge n}$ divides q, then $W = W_{c(e,e,n)^q}$. In particular Theorem 9.2.11 holds.

(b) If $\frac{e(n-1)}{e\wedge n}$ does not divide q, then i induces an isomorphism between $W'_{c(e,1,n-1)^q}$ and $W_{c(e,e,n)^q}$.

Proof. (a) Assume that $\frac{e(n-1)}{e\wedge n} j = q$ for some integer j. Since $dq = e(n-1)$, we obtain that $d = \frac{e\wedge n}{j}$ divides $e \wedge n$, which is the gcd of the degrees of W (cf. [BMR98, Table 2]). The degrees

of $W_{c(e,e,n)}$ are then the same as that of W.
(b) We denote d the integer such that $dq = e(n-1)$. Let $w \in W$. We write w as a block matrix.

$$w = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

with $X \in \mathcal{M}_1(\mathbb{C}), Y \in \mathcal{M}_{1,n-1}(\mathbb{C}), Z \in \mathcal{M}_{n-1,1}(\mathbb{C})$ and $T \in \mathcal{M}_{n-1,n-1}(\mathbb{C})$. We find

$$w \in W_{c(e,e,n)^q} \Leftrightarrow \begin{cases} c(e,1,n-1)^q Z = \zeta_e^{-q} Z, \\ Y c(e,1,n-1)^q = \zeta_e^{-q} Y, \\ T c(e,1,n-1)^q = c(e,1,n-1)^q T. \end{cases}$$

as w is a monomial matrix, Z and Y have at most one nonzero coefficient. Suppose that $Z \neq 0$, then $c(e,1,n-1)^q$ has a diagonal coefficient equal to ζ_d^{-q}. The underlying permutation of $c(e,1,n-1)$ is an $(n-1)$-cycle. Thus $c(e,1,n-1)^q$ has a nonzero diagonal coefficient if and only if $(n-1)$ divides q.

Let j be an integer such that $j(n-1) = q$. Since $dq = e(n-1)$, we have $dj = e$. We also have $c(e,1,n-1)^q = (c(e,1,n-1)^{n-1})^j = \zeta_e^j \operatorname{Id}$. Thus $c(e,1,n-1)^q Z = \zeta_e^{-q} Z$ if and only if $\zeta_e^j = \zeta_e^{-q}$, that is, $j = -q \bmod e$. Since $j(n-1) = q$, we obtain that there is some integer k with $jn = kjd$. Thus $kd = n$ and d divides n. We have already shown that d divides e, we obtain that d divides $e \wedge n$ and that $\frac{e(n-1)}{e \wedge n}$ divides q.

The same reasoning shows that $Y \neq 0$ implies that $\frac{e(n-1)}{e\wedge n}$ divides q. We obtain $Y = 0$, $Z = 0$, and $w \in W_{c(e,e,n)^q}$ if and only if $w = i(T)$ with $T \in W'_{c(e,1,n-1)^q}$. $\square$

From now on, we suppose that $\frac{e(n-1)}{e\wedge n}$ does not divide q. The isomorphism between $W_{c(e,e,n)^q}$ and $W'_{c(e,1,n-1)^q}$ induced by i induces in turn an isomorphism of posets

$$I(c(e,1,n-1))_{W'_{c(e,1,n-1)^q}} \simeq I(c(e,e,n))_{W_{c(e,e,n)^q}}.$$

On the other hand, we know from Proposition 8.3.18 that the two posets

$$(I(c(e,1,n-1))_W)^{c(e,1,n-1)^q} \text{ and } I(c(e,1,n-1))_{W'_{c(e,1,n-1)^q}}$$

are equal. To show that Theorem 9.2.11 holds in this case, it only remains to show that the morphism i induces an isomorphism of posets between $(I(c(e,1,n-1))_{W'})^{c(e,1,n-1)^q}$ and $(I(c(e,e,n))_W)^{c(e,e,n)^q}$. By [BC06, Remark after Lemma 1.22], $c(e,e,n)$ induces a free action (of a cyclic group of order $\frac{e(n-1)}{e\wedge n}$) on the set $I(c(e,e,n))_W \setminus i(I(c(e,1,n-1))_{W'})$. Since $\frac{e(n-1)}{e\wedge n}$ does not divide q by assumption, we get

$$(I(c(e,e,n))_W)^{c(e,e,n)^q} = i(I(c(e,1,n-1))_{W'})^{c(e,e,n)^q} = i\left((I(c(e,1,n-1))_{W'})^{c(e,1,n-1)^q}\right),$$

which finishes the proof.

9.2.3 Presentation by Hurwitz relations

We saw in Proposition 8.2.20 that dual groups admit a presentation where the generators are the atoms, and where the relations are Hurwitz relations. This presentation is smaller (both in terms of numbers of generators and in terms of number of relators) than the presentation induced by the germ of simple elements. In this section, we provide a similar result for Springer groupoids.

In this section, we fix an irreducible well-generated complex reflection group W. We define h as the highest degree of W, and we fix d a regular number for W. We set again $p := \frac{d}{d \wedge h}$ and $q := \frac{h}{d \wedge h}$. We write $(G(W), M(W), \Delta)$ for the dual group of type W. We denote by $\mathcal{S}$ its set of simple elements, and by ϕ its Garside automorphism.
We also consider the Springer groupoid $(\mathcal{G}, \mathcal{C}, \Delta_p)$ attached to W and d, and we denote by $\mathcal{A}$ the set of atoms of $\mathcal{C}$.

Recall from Lemma 9.2.3 (atoms in Springer category) that $\mathcal{A}$ is made of the simples (a, b) such that a admits no proper ϕ^q-invariant left-divisor in $M(W)$. Recall also that the Springer groupoid comes equipped with a collapse functor $\pi_0 : \mathcal{G} \to G(W)$. For $u \in \mathrm{Ob}(\mathcal{G})$, this functor identifies $\mathcal{G}(u, u)$ with the centralizer in $G(W)$ of $\pi_0(\Delta_p^q(u))$.

Lemma 9.2.16. *Consider a square of atoms in $\mathcal{C}$:*

$$\begin{tikzcd} u \arrow[r, "s"] \arrow[d, "\sigma"'] & v \arrow[d, "t"] \\ v' \arrow[r, "\tau"'] & w \end{tikzcd}$$

The square is commutative if and only if $\pi_0(st) = \pi_0(\sigma\tau)$. In this case we either have $s = \sigma$ and $t = \tau$, or $\pi_0(\sigma\tau) = \pi_0(st) \in \mathcal{S}$ and $st = \sigma\tau$ is a simple morphism.

Proof. The fact that $st = \sigma\tau$ if and only if $\pi_0(st) = \pi_0(\sigma\tau)$ comes from Corollary 4.5.11 (characterization of divisibility). If we have $s \neq \sigma$, then $st = \sigma\tau$ is a common right-multiple of s and σ. By atomicity of t and τ, we obtain that it is the right-lcm of s and σ. Since the lcm of simple morphisms is simple, we obtain that $st \in \mathcal{S}_p^q$ and $\pi_0(st) \in \mathcal{S}$ is simple. □

Definition 9.2.17 (Hurwitz relations). We call *Hurwitz relations* on $\mathcal{C}$ the relations of the form $st = \tau\sigma$, where s, t, τ, σ are atoms in $\mathcal{C}$, and $\pi_p(\sigma\tau) = \pi_p(st) \in \mathcal{S}$.

Theorem 9.2.18 (Presentation of Springer Groupoids). *Let W be an irreducible well-generated complex reflection group, and let d be a regular number for W. Let also $(\mathcal{G}, \mathcal{C}, \Delta)$ be the associated Springer groupoid, and let $\mathcal{A}$ be its set of atoms. Let also $\mathcal{H}$ be the set of Hurwitz relations in $\mathcal{C}$.*
We have $\mathcal{C} = \langle \mathcal{A} \mid \mathcal{H} \rangle^+$ and $\mathcal{G} = \langle \mathcal{A} \mid \mathcal{H} \rangle$. In other words, Springer groupoids are presented by their atoms, endowed with the Hurwitz relations.

Proof. If $p = 1$, this is already known from Corollary 9.2.12, since dual braid monoids are presented by their atoms, endowed with Hurwitz relations (Proposition-Definition 8.2.20). Suppose now that $p > 1$. We know that $\mathcal{C}$ is generated by its atoms, and that the defining relations of $\mathcal{C}$ imply the Hurwitz relations. It remains to show that the defining relations of $\mathcal{C}$ are implied by the Hurwitz relations.

Let $st = u$ be a defining relation of $\mathcal{C}$. We consider three paths of atoms in $\mathcal{C}$

$$s_1 \cdots s_r,\ t_1 \cdots t_k,\ u_1 \cdots u_m,$$

expressing s, t and u, respectively. We set $a_i = \pi_p(s_i)$ for $i \in [\![1, r]\!]$, $b_i := \pi_p(t_i)$ for $i \in [\![1, k]\!]$ and $\alpha_i := \pi_p(u_i)$ for $i \in [\![1, m]\!]$. In $M(W)^{\phi^q}$, the two words

$$a_1 \ldots a_r b_1 \cdots b_k \text{ and } \alpha_1 \cdots \alpha_m$$

express the same element $\pi_p(u)$. Since $M(W)^{\phi^q}$ is presented by the Hurwitz relations (this is the case $p = 1$), there is a sequence of words $\mu_1, \ldots, \mu_n$ in the atoms of $M(W)^{\phi^q}$ such that

- $\mu_1 = a_1 \cdots a_r b_1 \cdots b_r$
- $\mu_n = \alpha_1 \cdots \alpha_m$
- For $i \in [\![1, n-1]\!]$, μ_i and μ_{i+1} are related by a Hurwitz relation in $M(W)^{\phi^q}$.

In particular, each of the μ_i expresses the element $\pi_p(u)$ in $M(W)^{\phi^q}$. By Lemma 9.2.6 (lifting words expressing simples), each μ_i admits a unique lift p_i in $\mathcal{C}$, and each of the p_i expresses u in $\mathcal{C}$. By Lemma 9.2.16, the paths p_i and p_{i+1} are related by a Hurwitz relation in $\mathcal{C}$ for $i \in [\![1, n-1]\!]$. In particular the equality $st = u$ is implied by the Hurwitz relations in $\mathcal{C}$. □

The above presentation can be seen as an analogue of Remark 8.2.21 for categories of periodic elements of dual braid monoids. This new presentation will be useful for computational purposes in Section 10.1.

9.2.4 A shoal for Springer groupoids

Our goal is now to construct a support-preserving shoal for Springer groupoids, and to show that this shoal does give the topological parabolic subgroups of the associated complex braid group. We begin by constructing a shoal for divided groupoids of dual groups attached to well-generated complex braid groups, and we then use the results of Section 5.3.2.

Throughout this section, we fix an irreducible well-generated complex reflection group W. We denote by $(G(W), M(W), \Delta)$ the dual group of type W. We denote by $\mathcal{S}$ its set of simple elements, and by ϕ its Garside automorphism.

A shoal for the divided groupoid

In this section, we fix a positive integer p, and we denote by $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid of the group $(G(W), M(W), \Delta)$.

We construct a shoal of standard parabolic subcategories of $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$. We will then show that the parabolic subgroups for this shoal are equal to the parabolic subgroups of $G(W)$, and that this shoal is a support-preserving shoal for $\mathcal{G}_p$.

Notation 9.2.19. For $s \in \mathcal{S}$, and $i \in [\![0, p-1]\!]$, we denote by $s[i]$ the tuple with all entries equal to 1, except th i-th one, equal to s. For $\beta \in M(W)^p$, we denote by $\prod \beta$ the product $\beta_0 \cdots \beta_{p-1} \in M(W)$.

We start by defining a certain family of parabolic Garside maps in $\mathcal{G}_p$. For $\beta \in M(W)^p$ such that $\prod \beta \in \mathcal{S}$, we define a partial map δ_β from $\mathrm{Ob}(\mathcal{C}_p)$ to $\mathcal{C}_p$ as follows.

- If $u \not\succcurlyeq \beta$ as p-tuples, then $\delta_\beta(u)$ is not defined.
- If $u = \alpha\beta$ for some p-tuple α, then $\delta_\beta(u) := (\alpha, \beta) \in \mathcal{S}_p(u, \beta\tau(\alpha))$.

The fact that $\prod \beta$ is simple ensures that $\delta_\beta(u)$ is defined at least for some object u of $\mathcal{G}_p$ (for instance $u = (\beta_0, \ldots, \beta_{p-2}, \beta_{p-1}\overline{\beta_0 \cdots \beta_{p-1}})$).

Lemma 9.2.20 (Parabolic Garside map). *Let $\beta \in M(W)^p$ be such that $\prod \beta \in \mathcal{S}$. The map δ_β is a parabolic Garside map in $\mathcal{G}_p$. We denote by $((\mathcal{G}_p)_\beta, (\mathcal{C}_p)_\beta, \delta_\beta)$ the associated parabolic subgroupoid of $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$. Its set of simple morphisms is given by $(\mathcal{S}_p)_\beta := \{(a, b) \in \mathcal{S}_p \mid \beta \preccurlyeq b\}$.*

Proof. By definition, $\delta_\beta(u)$ is always a simple morphism when it is defined. Then let, u be such that $\delta_\beta(u)$ is defined, and let v be the target of $\delta_\beta(u)$. By Proposition 4.4.8 (characterization of divisibility in divided germ), we have

$$\forall(a,b) \in \mathcal{S}_p(u,-),\ (a,b) \preccurlyeq \delta_\beta(u) \Leftrightarrow b \succcurlyeq \beta \text{ and } \forall(a,b) \in \mathcal{S}_p(-,v),\ \delta_\beta(u) \succcurlyeq (a,b) \Leftrightarrow \beta \preccurlyeq b.$$

And thus $\mathrm{Div}(\delta_\beta) = \{(a,b) \in \mathcal{S}_p \mid b \succcurlyeq \beta\}$ and $\mathrm{Div}_R(\delta_\beta) = \{(a,b) \in \mathcal{S}_p \mid \beta \preccurlyeq b\}$. Those two sets are equal since every simple element of $M(W)$ is balanced, and thus δ_β is a balanced map.

Not, let $u, v, w \in \mathrm{Ob}(\mathcal{G}_p)$, and let $s := (a,b) \in \mathcal{S}_p(u,v)$, $t := (d,e) \in \mathcal{S}_p(v,w)$ such that $s, t \in \mathrm{Div}(\delta_\beta)$. By Lemma 4.4.6, the product st is a simple morphism if and only if $dx = b$ for some p-tuple x, and we have $st = (ad, x)$. By assumption, we have $\beta \preccurlyeq b, e$. We saw in the proof of Proposition 9.2.4 (no pairs of parallel simples) that b (resp. e, x) is the gcd of the tuples u, v (resp. v, w, u, w). Since $b \succcurlyeq x$, we have $x \preccurlyeq b \preccurlyeq v$, and thus

$$x = u \wedge w = u \wedge v \wedge w = (u \wedge v) \wedge (v \wedge w) = b \wedge e.$$

By definition of a left-gcd, we obtain that $\beta \preccurlyeq x$, and thus $st \in \mathrm{Div}(\delta_\beta)$, which is then a parabolic Garside map. □

Proposition 9.2.21 (Shoal for divided groupoids of a dual group). *The set*

$$\mathcal{T} = \left\{(\mathcal{G}_p)_\beta \mid \beta \in M(W)^p,\ \prod \beta \in \mathcal{S}\right\} \cup \{\{1_u\}_{u \in X} \mid X \subset \mathrm{Ob}(\mathcal{G}_p)\}$$

is a shoal for the divided groupoid $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$.

Proof. First, $\mathcal{T}$ contains $\{1_u\}_{u \in \mathrm{Ob}(\mathcal{G}_p)}$ by definition, and $\mathcal{G}_p = (\mathcal{G}_p)_\beta$ with $\beta = (1, 1, \ldots, 1)$. Then, for $\beta \in M(W)^p$ such that $\prod \beta \in \mathcal{S}$, we have $\phi_p(\delta_\beta) = \delta_{\tau(\beta)}$, and thus $\mathcal{T}$ is stable under ϕ_p.

Lastly, we show that the intersection of two elements of $\mathcal{T}$, if nonempty, lies in $\mathcal{T}$. If one of the factors has the form $\{1_u\}_{u \in X}$ for $X \subset \mathrm{Ob}(\mathcal{G}_p)$, then the result is obvious. Let then $(\mathcal{G}_p)_\beta, (\mathcal{G}_p)_{\beta'} \in \mathcal{T}$ and such that $\mathrm{Ob}((\mathcal{G}_p)_\beta) \cap \mathrm{Ob}((\mathcal{G}_p)_{\beta'}) \neq \varnothing$. For u in this intersection, we have $\beta, \beta' \preccurlyeq u$, and thus $\beta \vee \beta' \preccurlyeq u$ (in particular, $\prod(\beta \vee \beta') \in \mathcal{S}$). Furthermore, we have $\delta_\beta(u) \wedge \delta_\beta(u) = \delta_{\beta \vee \beta'}(u)$ by Corollary 4.4.11 (join and meet in divided germ). Since $\delta_{\beta \vee \beta'}$ is a parabolic Garside map, it is balanced, and equal to $\delta_\beta \wedge_R \delta_\beta = \delta_{\beta \vee_L \beta'}$ since simple elements in $M(W)$ are balanced. By Proposition 5.1.27 (intersection of standard parabolic subgroupoids), we obtain that $(\mathcal{G}_p)_\beta \cap (\mathcal{G}_p)_{\beta'} = (\mathcal{G}_p)_{(\beta \vee \beta')} \in \mathcal{T}$ and the result is shown. □

At this stage, it is unclear what is the connection between the shoal $\mathcal{T}$ and the shoal of all standards parabolic subgroups of $(G(W), M(W), \Delta)$. In order to establish that these shoals give the same parabolic subgroups, we need a few intermediate results. First, we have natural isomorphisms occurring between $\mathcal{T}$-standard parabolic subcategories.

Proposition 9.2.22 (Isomorphism of standard parabolic subgroupoids).
Let $(\mathcal{G}_p)_\beta \in \mathcal{T}$, and let $s \preccurlyeq \beta$. For every object $u \in (\mathcal{C}_p)_\beta$, we have $s \preccurlyeq u$. We then define a functor $\psi : (\mathcal{C}_p)_\beta \to \mathcal{G}_p$ as follows

- *For $u \in \mathrm{Ob}((\mathcal{C}_p)_\beta)$, $\psi(u)$ is the target of $\sigma_u := (s, s^{-1}u) \in \mathcal{S}_p(u,-)$.*
- *For $f \in (\mathcal{C}_p)_\beta(u,v)$, we define $\psi(u) := \sigma_u^{-1} f \sigma_v \in \mathcal{G}_p(\psi(u), \psi(v))$.*

The functor ψ is an isomorphism between $(\mathcal{C}_p)_\beta$ and $(\mathcal{C}_p)_{\beta'}$, with $\beta' = s^{-1}\beta\tau(s)$.

Proof. First, an object u of $\mathcal{C}_p$ is an object of $(\mathcal{C}_p)_\beta$ if and only if $\beta \preccurlyeq u$. We then have $s \preccurlyeq \beta \preccurlyeq u$, and thus σ_u is well-defined.

Now, we show that ψ defines a functor from $(\mathcal{C}_p)_\beta$ to $(\mathcal{C}_p)_{\beta'}$. Let $u = \alpha\beta \in \mathrm{Ob}((\mathcal{C}_p)_\beta)$. By definition of σ_u, we have $\psi(u) = s^{-1}u\tau(s) = \alpha^s\beta' \in \mathrm{Ob}((\mathcal{C}_p)_{\beta'})$. Let now $(a,b) \in (\mathcal{S}_p)_\beta(u,v)$ for some $u, v \in \mathrm{Ob}((\mathcal{C}_p)_\beta)$. We show that $\psi(a,b) \in (\mathcal{S}_p)_{\beta'}(\psi(u),\psi(v))$. Since $s \preccurlyeq \beta \preccurlyeq b$, $as = sa^s$ lies in $(\mathcal{S})^p$, and we have

$$\begin{aligned}\psi(a,b) = \sigma_u^{-1}(a,b)\sigma_v &= (s, s^{-1}u)^{-1}(a,b)(s,s^{-1}v)\\ &= (s,s^{-1}u)^{-1}(as, s^{-1}b)\\ &= (s,s^{-1}u)^{-1}(sa^s, s^{-1}b)\\ &= (a^s, s^{-1}b\tau(s)).\end{aligned}$$

This is a simple element, which lies in $(\mathcal{C}_p)_{\beta'}$ since $\beta' \preccurlyeq s^{-1}b\tau(s)$. We also obtain that ψ preserves positivity, thus it induces a functor from $(\mathcal{C}_p)_\beta$ to $(\mathcal{C}_p)_{\beta'}$.

In order to show that ψ is an isomorphism of categories, we first show that it induces a bijection between $\mathrm{Ob}((\mathcal{C}_p)_\beta)$ and $\mathrm{Ob}((\mathcal{C}_p)_{\beta'})$. The injectivity is obvious by definition of ψ and by cancellativity. Now, let $w \in \mathrm{Ob}((\mathcal{C}_p)_{\beta'})$, we have

$$\beta' \preccurlyeq w \Leftrightarrow s^{-1}\beta\tau(s) \preccurlyeq w \Leftrightarrow \beta\tau(s) \preccurlyeq sw.$$

In particular, $sw \succcurlyeq \tau(s)$ and $w' := sw\tau(s)^{-1} \in \mathrm{Ob}(\mathcal{C}_p)$ is divided by β. As we have $\psi(w') = w$, we showed that ψ is bijective on objects.

Now, ψ induces an injective map from $(\mathcal{S}_p)_\beta$ to $(\mathcal{S}_p)_{\beta'}$ by the first part of the proof. It only remains to show that this map is also surjective. Let (a,b) be a simple element of $(\mathcal{C}_p)_{\beta'}$. Since $b \succcurlyeq s^{-1}\beta\tau(s)$, and since $b\tau(a) \in \mathcal{S}^p$, we have that $sa \in \mathcal{S}^p$. The element $({}^s a, sb\tau(s)^{-1}) \in (\mathcal{S}_p)_\beta$ is a preimage of (a,b) by ψ. □

A miscellaneous consequence of this proposition is that $\mathcal{T}$-standard parabolic subgroupoids of $\mathcal{G}_p$ are connected as groupoids.

Corollary 9.2.23. *Let $(\mathcal{G}_p)_\beta \in \mathcal{T}$. The standard parabolic subgroupoid $(\mathcal{G}_p)_\beta$ is connected.*

Proof. For $i \in [\![0, p-1]\!]$, we define $\gamma_i := (1, \ldots, \beta_0 \cdots \beta_i, \beta_{i+1}, \ldots, \beta_{p-1})$. For $i \in [\![1, p-1]\!]$, we define $s_i := (1, \ldots, 1, \beta_0 \cdots \beta_{i-1}, 1, \ldots, 1)$. For $i \in [\![0, p-2]\!]$, we have $s_{i+1} \preccurlyeq \gamma_{i+1}$ and $s_{i+1}^{-1}\gamma_{i+1}\tau(s_{i+1}) = \gamma_i$. We then have that $(\mathcal{G}_p)_{\gamma_i} \simeq (\mathcal{G}_p)_{\gamma_{i+1}}$ by Proposition 9.2.22. Since $\gamma_0 = \beta$, we obtain that $(\mathcal{G}_p)_\beta$ is isomorphic to $(\mathcal{G}_p)_{\gamma_{p-1}}$. It is then sufficient to show that $(\mathcal{G}_p)_{\gamma_{p-1}}$ is connected.

We show that, for $s \in \mathcal{S}$, the groupoid $(\mathcal{G}_p)_{s[p-1]}$ is connected, where $s[p-1] = (1, 1, \cdots, 1, s)$. By definition, we have $\iota(\Delta) = \Delta[p-1] \in \mathrm{Ob}(\mathcal{G}_p)$. Of course, it is sufficient to show that, for each $u \in \mathrm{Ob}((\mathcal{G}_p)_{s[p-1]})$, there is a morphism $\Delta[p-1] \to u$ in $(\mathcal{G}_p)_{s[p-1]}$. We show that the morphism $\mathrm{Dil}(u) : \Delta[p-1] \to u$ belongs to $(\mathcal{C}_p)_{s[p-1]}$. Recall that, for $u \in \mathrm{Ob}(\widetilde{\mathcal{C}})$ (and with the above notation), we have

$$\mathrm{Dil}(u) = \prod_{i=0}^{p-2} \left((u_0 \cdots u_i)[i+1], (1, \cdots, 1, u_{i+1}, \cdots, u_{p-1})\right).$$

For $u \in \mathrm{Ob}((\mathcal{G}_p)_{s[p-1]})$, we have $s \preccurlyeq u_{p-1}$ and $s[p-1] \preccurlyeq (1, \cdots, 1, u_{i+1}, \cdots, u_{p-1})$ for all $i \in [\![0, p-2]\!]$. Hence $\mathrm{Dil}(u) \in (\mathcal{C}_p)_{s[p-1]}$, which shows the claim. □

A more important consequence of Proposition 9.2.22 is the following proposition, which proves that the collapse functor $\pi : \mathcal{G}_p \to G(W)$ sends $\mathcal{T}$-standard parabolic subgroups to standard parabolic subgroups of $G(W)$.

Proposition 9.2.24 (Identification of standard parabolic subgroups).
Let $(\mathcal{G}_p)_\beta \in \mathcal{T}$, and let $u \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$. The isomorphism $\mathcal{G}_p(u,u) \simeq G(W)$ induced by the collapse functor identifies $(\mathcal{G}_p)_\beta(u,u)$ with the standard parabolic subgroup $G(W)_{(\prod \beta)^*}$, where $(\prod \beta)^*$ is the right-complement of $\prod \beta$ to Δ in $M(W)$.

Proof. First, let $s := (a,b) \in \mathrm{Div}(\delta_\beta)$. By definition, there is some tuple x such that $\beta x = b$. The collapse functor sends s to a_0, whose left-complement in Δ (in $M(W)$) is $b_0 a_1 b_1 \cdots b_{p-1} = x_0\beta_0 a_1 x_1 \beta_1 \cdots x_{p-1}\beta_{p-1}$. Since this is a simple element of $M(W)$, and since every simple element of $M(W)$ is balanced, one easily show that this simple can be written as $s \prod \beta$. We then have $as \prod \beta = \Delta$, and thus $a \preccurlyeq (\prod \beta)^*$. We then have $\pi((\mathcal{G}_p)_\beta(u,u)) \subset G(W)_{(\prod \beta)^*}$.

For the converse, we start with a particular case. Let $s \in \mathcal{S}$. We have $\Delta[p-1] \in \mathrm{Ob}((\mathcal{G}_p)_{s[p-1]})$. Let $t \in \mathcal{S}$ be a simple morphism dividing s^*. We can consider the morphism $\iota(t) : \Delta[p-1] \to \Delta[p-1]$ in $(\mathcal{C}_p)$ (see the proof of Theorem 4.4.21 (equivalence of groupoids)) we have $\iota(t) \in (\mathcal{C}_p)_{s[p-1]}$ since $ts \in \mathcal{S}$, and $\pi\iota(t) = t$. Thus every simple of $(G(W)_{s^*}, M(W)_{s^*}, s^*)$ lies in the image of $\pi : (\mathcal{G}_p)_{s[p-1]}(\Delta[p-1], \Delta[p-1]) \to G(W)$, and this image is then equal to $G(W)_{s^*}$.

Now, for the general case. Let $u, v \in \mathrm{Ob}(\mathcal{G}_p)$, and let $f \in \mathcal{G}_p(u,v)$. We denote by c_f the isomorphism $\mathcal{G}_p(u,u) \simeq \mathcal{G}_p(v,v)$ induced by f. We have a commutative square of groups

$$\begin{array}{ccc} \mathcal{G}_p(u,u) & \xrightarrow{c_f} & \mathcal{G}_p(v,v) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G(W) & \xrightarrow[\gamma_{\pi(f)}]{} & G(W) \end{array}$$

Let $\beta \in M(W)^p$ be such that $\prod \beta \in \mathcal{S}$. For $i \in [\![0, p-1]\!]$, we define, as in the proof of Corollary 9.2.23, $\gamma_i := (1, \ldots, \beta_0 \cdots \beta_i, \beta_{i+1}, \ldots, \beta_{p-1})$. For $i \in [\![1, p-1]\!]$, we define $s_i := (1, \ldots, 1, \beta_0 \cdots \beta_{i-1}, 1, \ldots, 1)$. For $i \in [\![0, p-2]\!]$, we have $s_{i+1} \preccurlyeq \gamma_{i+1}$ and $s_{i+1}^{-1}\gamma_{i+1}\tau(s_{i+1}) = \gamma_i$. We then have an isomorphism $\psi_{i+1} : (\mathcal{G}_p)_{\gamma_{i+1}} \simeq (\mathcal{G}_p)_{\gamma_i}$ induced by s_{i+1} by Proposition 9.2.22. Since $\gamma_0 = \beta$, we obtain that $(\mathcal{G}_p)_\beta$ is isomorphic to $(\mathcal{G}_p)_{\gamma_{p-1}}$.

Let $i \in [\![0, p-2]\!]$, and let $u \in (\mathcal{G}_p)_{\gamma_{i+1}}$. By construction, the isomorphism $(\mathcal{G}_p)_{\gamma_{i+1}}(u,u) \simeq (\mathcal{G}_p)_{\gamma_i}(\psi_{i+1}(u), \psi_{i+1}(u))$ induced by ψ_{i+1} is $c_{(\sigma_{i+1})_u}$, where $(\sigma_{i+1})_u = (s_{i+1}, s_{i+1}^{-1}u)$. Since the first term of s_{i+1} is trivial for $i \geqslant 0$, we obtain a commutative square of groups

$$\begin{array}{ccc} \mathcal{G}_p(u,u) & \xrightarrow{c_{(\sigma_{i+1})_u}} & \mathcal{G}_p(\psi_{i+1}(u), \psi_{i+1}(u)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G(W) & = & G(W) \end{array}$$

And thus, the image of $(\mathcal{G}_p)_{\gamma_{i+1}}(u,u)$ in $G(W)$ is equal to that of $(\mathcal{G}_p)_{\gamma_i}(\psi(u), \psi(u))$.

Let now $s := \prod \beta$, we have $\gamma_{p-1} = s[p-1]$, and thus $(\mathcal{G}_p)_{\gamma_{p-1}} = (\mathcal{G}_p)_{s[p-1]}$. By the above argument (along with an immediate induction), the isomorphism $\psi : (\mathcal{G}_p)_{s[p-1]} \to (\mathcal{G}_p)_\beta$ is such that $(\mathcal{G}_p)_{s[p-1]}(\Delta[p-1], \Delta[p-1])$ and $(\mathcal{G}_p)_\beta(\psi(\Delta[p-1]), \psi(\Delta[p-1]))$ have the same image in $G(W)$ under π. We know that the image of the former group is $G(W)_{s^*}$, as this was the first particular case we treated, thus, the image of the latter group is also $G(W)_{s^*}$.

Finally, if $f \in (\mathcal{G}_p)_\beta(u,v)$ is an arbitrary morphism, and if we know that $\pi((\mathcal{G}_p)_\beta(u,u)) = G(W)_{s^*}$, then we have

$$\pi((\mathcal{G}_p)_\beta(v,v)) = (\pi((\mathcal{G}_p)_\beta(u,u)))^{\pi(f)} = (G(W)_{s^*})^{\pi(f)} = G(W)_{s^*}.$$

since we already know that $\pi(f) \in G(W)_{s^*}$, this finishes the proof. □

We also want to show that $\mathcal{T}$ is a support-preserving shoal. In order to do so, we will need to study minimal positive conjugators in $\mathcal{G}_p$. Fortunately, they are quite easy to describe using minimal positive conjugators in $G(W)$.

Let $u \in \mathrm{Ob}(\mathcal{G}_p)$, and let $x \in \mathcal{C}_p(u,u)$. By Corollary 4.4.9 (atoms of divided category), the atoms of $\mathcal{S}_p(u,-)$ all have the form $a = (\alpha[i], \alpha[i]^{-1}u)$, where $\alpha \in \mathcal{S}$ is an atom.

Let $\pi_i : \mathcal{G}_p \to G(W)$ be the i-th collapse functor. By Proposition 4.4.18 (collapse functors and divisibility), the k-th converging prefix of $\rho_a(x)$ is $(c_k[i], c_k[i]^{-1}u)$, where c_k is the k-th converging prefix of $\rho_\alpha(\pi_i(x))$. We then have that $\rho_a(x) = (\rho_\alpha(\pi_i(x))[i], \rho_\alpha(\pi_i(x))[i]^{-1}u)$. In particular, $\rho_a(x)$ is a minimal positive conjugator of x if and only if $\rho_\alpha(\pi_i(x))$ is a minimal positive conjugator of $\pi_i(x)$.

Using this description of minimal positive conjugators, we obtain the following result:

Theorem 9.2.25 (Support-preserving shoal for divided groupoid of dual group). *Let W be an irreducible well-generated complex reflection group, and let p be a positive integer. Let also $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ be the p-divided groupoid of the dual group of type W. The set*

$$\mathcal{T} = \left\{(\mathcal{G}_p)_\beta \mid \beta \in M(W)^p,\ \prod \beta \in \mathcal{S}\right\} \cup \{\{1_u\}_{u\in X} \mid X \subset \mathrm{Ob}(\mathcal{G}_p)\}$$

is a support-preserving shoal for $\mathcal{G}_p$.

Proof. Let $x \in \mathcal{C}_p$ be an endomorphism, written as a product of simple morphisms $s_1 \cdots s_r$ with $s_j = (a_j, b_j) \in \mathcal{S}_p$ for $j \in [\![1,r]\!]$ (we denote by u_j the source of s_j). Since a $\mathcal{T}$-standard parabolic subcategory $(\mathcal{C}_p)_\beta$ contains x if and only if $\beta \preccurlyeq b_j$ for all $j \in [\![1,r]\!]$, the $\mathcal{T}$-standard categorical parabolic closure of x is given by $(\mathcal{C}_p)_\beta$ where $\beta = b_1 \wedge \cdots \wedge b_r$.

Let $a = (\alpha[i], \alpha[i]^{-1}u)$ be an atom of $\mathcal{S}_p(u,-)$ such that $\rho_a(x)$ is a minimal positive conjugator of x. We have to show that $\mathrm{SPC}(x)^{\rho_a(x)} = \mathrm{SPC}(x^{\rho_a(x)})$. Up to applying ϕ_p (which preserve the $\mathcal{T}$-standard parabolic closure), we can assume that $i = 0$. As we stated above, we have $\rho_a(x) = (\rho[0], \rho[0]^{-1}u)$ where $\rho := \rho_\alpha(\pi(x))$ is a minimal positive conjugator for $\pi(x)$.

By construction, we have $\pi(x) = a_{1,0}a_{2,0}\cdots a_{r,0}$ written as a product of simple morphisms. We then know that $\mathrm{SPC}(\pi(x)) = G(W)_t$ with $t = a_{1,0} \vee \cdots \vee a_{r,0}$ (the simple t is a parabolic Garside element of $M(W)$, as is every element of $\mathcal{S}$). By Theorem 8.4.1 (minimal positive conjugators in dual braid monoids), we either have $\rho \in M(W)_t$ of $t\rho \in \mathcal{S}$

- If $t\rho \in \mathcal{S}$, we claim that $\rho \preccurlyeq \beta_0$. First, for all $j \in [\![1,r]\!]$, we have $a_{i,0}\rho \in \mathcal{S}$ since $a_{i,0}$ right-divides t. Since $\rho \preccurlyeq u_{1,0}$ (as $\rho_a(x) \in \mathcal{S}_p(u,-)$ is well-defined), we have

$$\rho \preccurlyeq u_{1,0} \wedge \overline{a_{1,0}} = a_{1,0}b_{1,0} \wedge b_{1,0}u_{1,1}\cdots u_{1,p-1} = b_{1,0}(a_{1,0}^{b_{1,0}} \wedge u_{1,1}\cdots u_{1,p-1}).$$

  However, since the product $a_{1,0}^{b_{1,0}}u_{1,1}\cdots u_{1,p-1} = \overline{b_{1,0}}$ is simple. We obtain by Proposition 8.2.13 that this last gcd is trivial, and $\rho \preccurlyeq b_{1,0}$. In particular, $\rho \preccurlyeq u_{2,0} = b_{1,0}a_{1,1}$. Since $\rho \preccurlyeq \overline{a_{2,0}}$, we can iterate this reasoning to deduce that $\rho \preccurlyeq b_{j,0}$ for all $j \in [\![1,r]\!]$. We then

have $\rho \preccurlyeq \beta_0$, which is the gcd of all the $\beta_{j,0}$ for $j \in [\![1, r]\!]$.
Now, since $\rho \preccurlyeq \beta_0$, we have $\rho[0] \preccurlyeq \beta$, and we can apply Proposition 9.2.22, and we have $((\mathcal{G}_p)_\beta(u,u))^{\rho_a(x)} = (\mathcal{G}_p)_{\beta'}(v,v)$, where $\beta' = \rho[0]^{-1}\beta\tau(\rho[0])$, and v is the target of $\rho_a(x)$. Furthermore, $x^{\rho_a(x)}$ is given by $(a_1, b_1')\cdots(a_r, b_r')$, where $b_j' = \rho[0]^{-1}b_j\tau(\rho[0])$. It is clear that the $\mathcal{T}$-standard parabolic closure of $x^{\rho_a(x)}$ is $(\mathcal{G}_p)_{\beta'}(v,v)$, as wanted.

- If $\rho \in M(W)_t$, then $\rho_a(x)$ lies in the standard categorical parabolic closure of x. Thus $\mathrm{SPC}(x^{\rho_a(x)}) \subset \mathrm{SPC}(x)^{\rho_a(x)} = (\mathcal{G}_p)_\beta(v,v)$. In particular, if $\mathrm{SPC}(x^{\rho_a(x)}) = (\mathcal{G}_p)_{\beta''}(v,v)$, then the length of $\delta_{\beta''}$ is inferior to that of δ_β. We can then imitate the end of the proof of Corollary 8.4.2 to obtain the result.

□

Proposition 9.2.26 (Identification of parabolic subgroups). *Let u be an object of $\mathcal{G}_p$. The isomorphism $\mathcal{G}_p(u,u) \simeq G(W)$ induced by collapse functor induces a bijection between the $\mathcal{T}$-parabolic subgroups of $\mathcal{G}_p(u,u)$ and the parabolic subgroups of $G(W)$. In particular, the set $\mathcal{P}_\mathcal{T}(\mathcal{G}_p(u,u))$ is stable under intersection.*

Proof. We already know by Proposition 9.2.24 that the isomorphism $\mathcal{G}_p(u,u) \to G(W)$ sends $\mathcal{T}$-standard parabolic subgroups of $\mathcal{G}_p(u,u)$ to standard parabolic subgroups of $G(W)$, from this we easily deduce that it sends $\mathcal{T}$-parabolic subgroups of $\mathcal{G}_p(u,u)$ to parabolic subgroups of $G(W)$. Conversely, let $H \subset G(W)$ be a parabolic subgroup. There is some $g \in G(W)$ such that $H^g = G(W)_s$ is a standard parabolic subgroup. Taking $\beta = s[p-1]$ as in the proof of Proposition 9.2.24, we obtain that the isomorphism $\pi : \mathcal{G}_p(\Delta[p-1], \Delta[p-1]) \to G(W)$ identifies $P := (\mathcal{G}_p)_\beta(\Delta[p-1], \Delta[p-1])$ with $G(W)_s$. The subgroup H is then identified with $P^{\iota(g^{-1})}$ (where ι is the inverse equivalence of π constructed in the proof of Theorem 4.4.21 (equivalence of groupoids)), in particular, $P^{\iota(g^{-1})}$ is a $\mathcal{T}$-parabolic subgroup. Conjugating this subgroup by $\mathrm{Dil}(u)$ in $\mathcal{G}_p$ yields a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}_p(u,u)$, whose image in $G(W)$ is H.

The stability of $\mathcal{P}_\mathcal{T}(\mathcal{G}(u,u))$ under intersection is an immediate consequence of the first part, since we already know that the parabolic subgroups of $G(W)$ are stable under intersection by Theorem 8.4.5. □

The last step before we can move on to Springer groupoids is to give a system of conjugacy representatives for the shoal $\mathcal{T}$.

For $(\mathcal{G}_p)_\beta \in \mathcal{T}$, we consider $z_\beta := \delta_\beta^k$ the smallest power of δ_β which is central in $\mathcal{G}$. A priori, for a given $u \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$, the smallest integer k_u such that $\delta_\beta^{k_u}(u)$ is central in $(\mathcal{G}_p)_\beta(u,u)$ is smaller than the smallest integer k such that δ_β^k is central in $(\mathcal{G}_p)_\beta$ as a whole. The following lemma ensures that this is not the case.

Lemma 9.2.27. *Let $(\mathcal{G}_p)_\beta \in \mathcal{T}$ be a standard parabolic subgroupoid of $\mathcal{G}_p$, and let φ_β denote its Garside automorphism. If $u \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$ and $k > 0$ are such that $\delta_\beta^k(u)$ is central in $(\mathcal{G}_p)_\beta(u,u)$, then φ_β^k is trivial*

Proof. Let $u \in \mathrm{Ob}((\mathcal{G}_p)_\beta))$, and let $k > 0$ such that $\delta_\beta^k(u)$ is central in $(\mathcal{G}_p)_\beta(u,u)$. Let $s \in \mathcal{S}_p$. By construction of $s^\#$ and $s^\flat$, we have $s \in (\mathcal{S}_p)_\beta$ if and only if $s^\# \in (\mathcal{S}_p)_\beta$ if and only if $s^\flat \in (\mathcal{S}_p)_\beta$. Thus, if $s \in \mathcal{S}_p$, we can consider the simple loop $\lambda(s)$ associated to s, and this loop lies in

$(\mathcal{C}_p)_\beta(u,u)$. By assumption, we have $\lambda(s) = \lambda(s)^{\delta_\beta^k(u)} = \varphi_\beta^k(s)\varphi_\beta^k(s^\#)\cdots\varphi_\beta^k(s^\flat)$. Since the greedy normal form of $\lambda(s)$ is $ss^\#\cdots s^\flat$, we obtain in particular that $s = \varphi_\beta^k(s)$. We then have

$$\delta_\beta^k(u)s = s\delta_\beta^k(v),$$

and $\delta_\beta^k(v) = (\delta_\beta^k(u))^s \in Z((\mathcal{G}_p)_\beta(v,v))$. By an immediate induction, we obtain that $\delta_\beta^k(w) \in Z((\mathcal{G}_p)_\beta(w,w))$ for all w such that $\mathcal{G}_p(u,w)$ is nonempty, and that $\varphi_\beta^k(f) = f$ for all $f \in \mathcal{G}_p(u,w)$. By Corollary 9.2.23, we obtain that this is true for all $w \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$. □

Proposition 9.2.28 (System of conjugacy representatives).
The set $\{z_\beta \mid \beta \in M(W)^p,\ \prod\beta \in \mathcal{S}\}$, where z_β is the smallest central power of δ_β in $(\mathcal{G}_p)_\beta$, is a set of conjugacy representatives for $\mathcal{T}$.

Proof. Let $(\mathcal{G}_p)_\beta(u,u)$ be a $\mathcal{T}$-standard parabolic subgroups in $\mathcal{T}$. First, we have by definition that $z_\beta(u) = \delta_\beta^k(u)$ for some $k>0$. In particular, $(\mathcal{G}_p)_\beta(u,u) = \mathrm{SPC}(z_\beta(u))$ by construction.

Then, by definition of the divided germ, we can define a length function on simple morphisms in $\mathcal{G}_p$ by setting $\ell(a,b) = \sum_{i=0}^{p-1}\ell_T(a_i)$ (where ℓ_T is the natural length function of $M(W)$). This is also equal to $n - \sum_{i=0}^{p-1}\ell_T(b_i)$ since (a,b) is a decomposition of $\Delta \in M(W)$. In particular, for every $u \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$, the length $\ell(\delta_\beta(u))$ is the same, we write it ℓ_β. If k_β denotes the smallest integer such that $\delta_\beta^{k_\beta}$ is central in $(\mathcal{G}_p)_\beta$, then we have $\ell(z_\beta(u)) = k_\beta\ell_\beta$ for every object u in $(\mathcal{G}_p)_\beta$.

Let now $u,v \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$, and let $(\mathcal{G}_p)_\beta(u,u)$, $(\mathcal{G}_p)_{\beta'}(v,v)$ be two standard parabolic subgroups. For $f \in \mathcal{G}_p(u,v)$, support-preservingness of $\mathcal{T}$ implies that, if $(z_\beta(u))^f = z_\beta(v)$, then $((\mathcal{G}_p)_\beta(u,u))^f = (\mathcal{G}_p)_{\beta'}(v,v)$.

Conversely, assume that $f \in \mathcal{G}_p(u,v)$ is such that $((\mathcal{G}_p)_\beta(u,u))^f = (\mathcal{G}_p)_{\beta'}(v,v)$. Let $x := (z_\beta(u))^f \in Z((\mathcal{G}_p)_{\beta'}(v,v))$. Since x is conjugate in $\mathcal{G}_p$ to the positive endomorphism $z_\beta(u)$, there is a conjugating element $g \in (\mathcal{G}_p)_{\beta'}$ (conjugating element for iterated swaps) such that $x^g \in (\mathcal{C}_p)_\beta$. Since $z_\beta(u)$ is a power of the simple loop $\lambda(\delta_\beta(u))$, which is rigid by Lemma 9.2.9, it is rigid and in particular $z_\beta(u) \in \mathrm{SSS}(z_\beta(u))$. We then have

$$0 = \inf(z_\beta(u)) \geqslant \inf(x^g) \geqslant 0$$

(where inf denotes the infimum in $(\mathcal{G}_p, \mathcal{C}_p, \Delta)$), and thus $\inf(x^g) = 0$. Applying iterated decyclings to x^g gives a conjugating element h, which lies in $(\mathcal{C}_p)_\beta$ by construction of decycling, and such that $\sup(x^{gh}) = \sup(z_\beta(u)) = k_\beta$ by Corollary 3.2.6. Since the supremum in $\mathcal{C}_p$ or in $(\mathcal{C}_p)_{\beta'}$ are equal, we have $x^{gh} \preccurlyeq \delta_{\beta'}^{k_\beta}(v')$, where v' is the source of x^{gh}. In particular, we get

$$k_\beta\ell_\beta = \ell(z_\beta(u)) = \ell(x^{gh}) \preccurlyeq \ell(\delta_{\beta'}^{k_\beta}(v')) = k_\beta\ell'_\beta,$$

and thus $\ell_\beta \preccurlyeq \ell_{\beta'}$. Since f^{-1} conjugates $(\mathcal{G}_p)_{\beta'}(v,v)$ to $(\mathcal{G}_p)_\beta(u,u)$, we can apply the same reasoning to obtain $\ell_{\beta'} \preccurlyeq \ell_\beta$. We then have $\ell_\beta = \ell_{\beta'}$, and thus $x^{gh} = \delta_{\beta'}^{k_\beta}(v')$ since one divides the other and they have the same length. Since x^{gh} is central in $(\mathcal{G}_p)_{\beta'}(v',v')$ by construction, we have $k_{\beta'} \leqslant k_\beta$. By symmetry, we obtain $k_\beta \leqslant k_{\beta'}$ and $k_\beta = k_{\beta'}$. Thus, we have $x^{gh} = z_{\beta'}(v')$ and $x = (z_{\beta'}(v'))^{h^{-1}g^{-1}} = z_{\beta'}(v)$ since $gh \in (\mathcal{G}_p)_{\beta'}$.

□

A shoal for the Springer groupoid

Let h denote the highest degree of W. In this section, we fix a regular number d for W, and we set again $p := \frac{d}{d\wedge h}$ and $q := \frac{h}{d\wedge h}$. We denote by $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid of $(G(W), M(W), \Delta)$, and by $(\mathcal{G}, \mathcal{C}, \Delta_p)$ the Springer groupoid attached to W, d.

In the last section, we constructed a support-preserving shoal $\mathcal{T}$ for $\mathcal{G}_p$. It is then easy to build a shoal for the Springer groupoid $\mathcal{G}$ using the tools of Section 5.3.2. Indeed, $\mathcal{G}$ is defined as the groupoid of fixed points in $\mathcal{G}_p$, under the automorphism ϕ_p^q.

Let $(\mathcal{G}_p)_\beta \in \mathcal{T}$. By construction, we have $\phi_p((\mathcal{G}_p)_\beta) = (\mathcal{G}_p)_{\tau(\beta)}$, and thus $\phi_p^q((\mathcal{G}_p)_\beta) = (\mathcal{G}_p)_\beta$ if and only if $\tau^q(\beta) = \beta$. Note that, if λ, μ are positive integers such that $p\lambda - q\mu = 1$, then Lemma 4.5.5 gives us that

$$\tau^q(\beta) = \beta \Leftrightarrow \phi^q(\beta_0) = \beta_0 \text{ and } \forall i \in [\![1, p-1]\!],\ \beta_i = \phi^{i\lambda}(\beta_0).$$

Recall from Theorem 4.5.2 (periodic elements and groupoid of periodic elements) that, for $u \in \mathrm{Ob}(\mathcal{G})$, the collapse functor $\pi_0 : \mathcal{G} \to G(W)$ induces an isomorphism between $\mathcal{G}(u,u)$ and the centralizer in $G(W)$ of $\rho(u) := \pi_0(\Delta_p^q(u))$, which is a d-regular braid in $B(W) \simeq G(W)$.

Proposition 9.2.29 (Shoal for Springer groupoid).

(a) The set $\mathcal{T}' := \{\mathcal{G}_\beta := (\mathcal{G}_p)_\beta \mid (\mathcal{G}_p)_\beta \in \mathcal{T}, \tau^q(\beta) = \beta\}$ is a support-preserving shoal for $\mathcal{G}$.

(b) The set $\{z_{\beta|\,\mathrm{Ob}(\mathcal{G})} \mid \tau^q(\beta) = \beta\}$ is a system of conjugacy representative for $\mathcal{T}'$.

(c) Let $u \in \mathrm{Ob}(Gg)$. The isomorphism $\mathcal{G}(u,u) \simeq C_{B(W)}(\rho(u))$ induced by the collapse functor identifies the elements of $\mathcal{P}_{\mathcal{T}'}(\mathcal{G}(u,u))$ with groups of the form $H \cap C_{G(W)}(\rho(u))$, where H is a parabolic subgroup of $G(W)$ which is normalized by $\rho(u)$.

(d) Let $u \in \mathrm{Ob}(\mathcal{G})$. If H, H' are two parabolic subgroups of $G(W)$ which are normalized by $\rho(u)$, then $H = H'$ if and only if $H \cap C_{G(W)})(\rho(u)) = H' \cap C_{G(W)}(\rho(u))$.

Proof. By definition of a shoal, $\mathcal{T}$ is preserved by ϕ_p. Point (a) is then an immediate application of Proposition 5.3.11 (shoal for groupoid of fixed points). By Lemma 5.1.25, the system $\{z_\beta \mid \beta \in M(W)^p,\ \prod \beta \in \mathcal{S}\}$ is preserved by ϕ_p^q. Point (b) is thus an application of Proposition 5.3.13 (system of conjugacy representatives). Points (c) and (d) are obtained by combining Proposition 5.3.16 (parabolic subgroups in groupoid of fixed points) with Proposition 9.2.26 (identification of parabolic subgroups). □

It remains to show (see Section 9.2.5) that the parabolic subgroups obtained for a regular centralizer using this shoal coincide with the "topological" parabolic subgroups. This is the goal of the next section.

9.2.5 Local braid groupoids and parabolic subgroups

In this section, we fix a finite dimensional complex vector space V of dimension n, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We define h as the highest degree of W, and we fix d a regular number for W. We set again $p := \frac{d}{d\wedge h}$ and $q := \frac{h}{d\wedge h}$. We write $(G(W), M(W), \Delta)$ for the dual group of type W. We denote by $\mathcal{S}$ its set of simple elements, and by ϕ its Garside automorphism.
We denote by $(\mathcal{G}, \mathcal{C}, \Delta_p)$ the Springer groupoid attached to W, d. We will also denote by $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid associated to $(G(W), M(W), \Delta)$.

Let $g \in W$ be a d-regular element. By Proposition 9.1.3, we have a homeomorphism of topological pairs $(X_g/W_g, V_g/W_g) \simeq ((X/W)^{\mu_d}, (V/W)^{\mu_d})$. Since the definitions of capillary paths, local fundamental group and parabolic subgroup are purely topological, this isomorphism induces a bijection between the parabolic subgroups of $B(W_g)$ and the parabolic subgroups defined for the topological pair $((X/W)^{\mu_d}, (V/W)^{\mu_d})$.

In Chapter 8, the parabolic subgroups of W are understood in terms of the Garside group structure on $B(W)$. In the case of $(X/W)^{\mu_d}$, the Garside structure only exists at the level of the Springer groupoid. Thus we wish to adapt the notion of local fundamental group to a notion of local fundamental groupoid, more suitable for understanding the relation with the Springer groupoid. We will also define a notion of local fundamental groupoid for the divided groupoid.

Confining neighborhoods

We begin by defining a particular neighborhood basis of elements of V/W and of $(V/W)^{\mu_d}$.

Let $\kappa := \{\!\{z_1, \ldots, z_n\}\!\} \in E_n$. By definition of the topology of E_n, a neighborhood of κ in E_n is the image of a neighborhood of $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ under the projection map $\mathbb{C}^n \to E_n$. In particular, choosing a neighborhood U_{z_i} in $\mathbb{C}$ of each of the z_i induces a neighborhood U of κ in E_n (the image of $\prod_{i=1}^n U_{z_i}$ under the projection map $\mathbb{C}^n \to E_n$). For $z \in \mathbb{C}$ and $r > 0$, we denote by $\mathbb{D}(z, r)$ the open disk with center z and radius r.

Definition 9.2.30 (Confining neighborhood). Let $\kappa = \{\!\{z_1, \ldots, z_n\}\!\} \in E_n$ be such that $\kappa \cap D \subset \{0\}$. A *confining neighborhood* of κ in E_n is the image in E_n of a set of the form $\prod_{i=1}^n \mathbb{D}(z_i, r_i) \subset \mathbb{C}^n$ such that

- $r_i = r_j$ if $z_i = z_j$.
- $\mathbb{D}(z_i, r_i) \cap \mathbb{D}(z_j, r_j) = \varnothing$ if $z_i \neq z_j$.
- For $z_i \in \kappa \setminus \{0\}$, we have $\mathbb{D}(z_i, r_i) \cap D = \varnothing$.
- There is some $r > 0$ such that, for all $i \in [\![1, n]\!]$, we either have
    - $z_i = 0$, in which case $r_i < r$ and $\mathbb{D}(z_i, r_i) \subset \mathbb{D}(0, r)$,
    - $z_i \neq 0$, in which case $r + r_i < |z_i|$ and $\mathbb{D}(z_i, r_i) \cap \mathbb{D}(0, r) = \varnothing$.

If U is a confining neighborhood of κ, the sets $\mathbb{D}(z_i, r_i)$ with $z_i \neq 0$ are called *outer disks*, and the set $\mathbb{D}(z_i, r_i)$ with $z_i = 0$ (which is unique by the first condition) is called the *central disk*. Let $z \in V/W$ (resp. $(V/W)^{\mu_d}$) be a point such that $\overline{\mathrm{LL}}(z) \cap D \subset \{0\}$. By extension, we say that a neighborhood of z in V/W (resp. in $(V/W)^{\mu_d}$) is *confining* if it is path connected and if its image under $\overline{\mathrm{LL}}$ is included in a confining neighborhood of $\overline{\mathrm{LL}}(z)$.

Note that, by the second condition, the notion of outer and central disk of a confining neighborhood depends only on the confining neighborhood. The second condition is what justifies the name confining, the different disks isolates the points of the cyclic support of κ from one another, so that no singularization can happen inside of a confining neighborhood. The fourth condition is a strengthening of the second condition in the case where one of the points z_i is 0. This condition will allow us to apply Lemma 8.1.20 (Hurwitz moves). Indeed, since points in the central disk cannot move farther than points in outer disks, they cannot change their cyclic label by Lemma 8.1.20. This will be important in the proof of Lemma 9.2.32.

Let $\kappa \in E_n$ be such that $\kappa \cap D \subset \{0\}$. It is easily seen that confining neighborhoods of κ form a neighborhood basis of κ in E_n. Let $z \in V/W$ (resp $z \in (V/W)^{\mu_d}$) be such that

$\overline{\mathrm{LL}}(z) \cap D \subset \{0\}$, and let U be a confining neighborhood of $\overline{\mathrm{LL}}(z)$. By Proposition 8.1.10 (path lifting), the path connected component U' of z in $\overline{\mathrm{LL}}^{-1}(U)$ (resp. in $\overline{\mathrm{LL}}^{-1}(U) \cap (V/W)^{\mu_d}$) is a confining neighborhood of z in V/W (resp. in $(V/W)^{\mu_d}$) such that $\overline{\mathrm{LL}}(U') = U$ (resp. $\overline{\mathrm{LL}}(U') = U \cap E_n^{\mu_p}$). From this we see that confining neighborhoods of z form a neighborhood basis of z in V/W (resp. in $(V/W)^{\mu_d}$). Furthermore, if U is a confining neighborhood of z, then $U \cap \mathcal{U}_p$ (resp. $U \cap \mathcal{U}^{\mu_d}$) is nonempty since $\mathcal{U}_p$ is dense in V/W (resp. $\mathcal{U}^{\mu_d}$ is dense in $(V/W)^{\mu_d}$).

Let $z \in V/W$ be such that $\overline{\mathrm{LL}}(z) \cap D \subset \{0\}$, and let U be a confining neighborhood of z in V/W. For $x \in U$, we distinguish between the *outer points* of $\overline{\mathrm{LL}}(x)$, which lie in an outer disk of U, and the *central points* of $\overline{\mathrm{LL}}(x)$, which lie in the central disk of U.

We see that the definition of confining neighborhood does not make sense for arbitrary points of V/W. However, in our study of parabolic subgroups up to conjugacy, we can restrict our attention to a family of points for which confining neighborhoods are defined. Our first tool to do so is the following proposition, which states that a path which preserves the multiplicity in the discriminant stratification must remain in the same stratum of the discriminant stratification. Note that a priori, we need to distinguish the discriminant stratification of V/W and that of $(V/W)^{\mu_d}$ (which is the same as that of $\simeq V_g/W_g$, where $g \in W$ is d-regular by Theorem 6.1.32 (reflection arrangement of regular centralizer)).

Proposition 9.2.31. *Let $x \in V/W$, and let $\gamma : [0,1] \to V/W$ be a path starting at x.*

(a) If the multiplicity of $\gamma(t)$ in the discriminant stratification stays the same for all $t \in [0,1]$, then $\gamma(1)$ and x lie on the same stratum of the discriminant stratification of V/W.

(b) Furthermore, if $x \in (V/W)^{\mu_d}$, and if $\gamma(t) \in (V/W)^{\mu_d}$ for all $t \in [0,1]$, then $\gamma(1)$ and x lie on the same stratum of the discriminant stratification of $(V/W)^{\mu_d}$.

Proof. By Lemma 8.1.5, the multiplicity of x in the discriminant stratification is equal to the multiplicity of 0 in the multiset $\overline{\mathrm{LL}}(x)$.

(a) Let $p : V \to V/W$ be the canonical projection map, and let $\mathcal{A}$ be the set of reflecting hyperplanes of W. Let also $L \in L(\mathcal{A})$ be the unique flat such that $x \in L^0$. The set $I_L := \{t \in [0,1] \mid \gamma(t) \in \pi(L)\}$ is closed since $\pi(L)$ is a closed subset of V/W. We claim that $I_L = [0,1]$. If this is not the case, then we can consider $[0,t_0]$ the connected component of 0 in I_L, with $t_0 < 1$.

Let $v \in V$ be such that $p(v) = \gamma(t_0)$. We can consider a neighborhood $\widetilde{U}$ of v in V which intersects only the reflecting hyperplanes of W which contain v. Since p is an open map, $U := p(\widetilde{U})$ is a neighborhood of $\gamma(t_0)$ in V/W. By construction, U intersects some $p(H)$ for $H \in \mathcal{A}$ if and only if $\gamma(t_0) \in p(H)$.

For $\varepsilon > 0$ small enough, we have $\gamma(t_0+\varepsilon) \in U$. If H is such that $\gamma(t_0+\varepsilon) \in p(H)$, then $p(H) \cap U \neq \varnothing$ and $\gamma(t_0) \in p(H)$. Thus we have

$$\{p(H) \mid \gamma(t_0+\varepsilon) \in p(H)\} \subset \{p(H) \mid \gamma(t_0) \in p(H)\}.$$

And thus $\gamma(t_0)$ belongs to some $p(L')$ with $L' \leqslant L$ in the intersection lattice $L(\mathcal{A})$. Since $L' < L$ by assumption, we obtain by Lemma 8.1.5 that the multiplicity of $\gamma(t_0+\varepsilon)$ in the discriminant stratification, which is equal to the codimension of L' in V, is strictly smaller than the multiplicity of $\gamma(t_0)$, which is equal to the codimension of L in V. Since this contradicts the assumption, we obtain that $t_0 = 1$.

We have that $\gamma(t) \in p(L)$ for all $t \in [0,1]$. Since the only stratum contained in $p(L)$ on which the multiplicity of the discriminant is equal to the multiplicity of 0 in $\overline{\mathrm{LL}}(x)$ is $p(L^0)$, we obtain the result.

(b) By the first point, the path γ remains in the stratum $p(L^0)$, where L^0 is the unique stratum of V containing v. By Corollary 6.2.7, the path γ then admits a unique lift $\widetilde{\gamma} : [0,1] \to L^0$ starting at v. Since $x \in (V/W)^{\mu_d}$ and $\gamma(t) \in (V/W)^{\mu_d}$ for all $t \in [0,1]$, we obtain that there is some d-regular element $g \in W$ such that $\widetilde{\gamma}(t) \in V_g$ for all $t \in [0,1]$. The path $\widetilde{\gamma}$ remains in the stratum $L^0 \cap V_g$ of the stratification of V_g, and thus γ remains in the same stratum of the discriminant stratification of $(V/W)^{\mu_d}$. □

In particular, in the situation of the above proposition, two capillary paths (either in V/W or in $(V/W)^{\mu_d}$) terminating at x and $\gamma(1)$ define conjugate parabolic subgroups. Another useful result is that, in a confining neighborhood, the product of the terms of the cyclic content corresponding to outer points is constant.

Lemma 9.2.32. *Let $z \in V/W$, and let U be a confining neighborhood of z in V/W. For $x \in U$ and $i \in [\![1,p]\!]$, the product of the terms of* $\mathrm{clbl}(x)$ *corresponding to the outer points in the sector P_i (in the ordering given by the cyclic support of x) is equal to the i-th term of* $\mathrm{cc}(z)$.

Proof. By definition, there is a path γ from x to z in U. We fix $i \in [\![1,p]\!]$. For $t \in [0,1]$, let $\beta(t)$ be the subsequence of $\mathrm{clbl}(\gamma(t))$ corresponding to outer points in the sector P_i. By definition of a confining neighborhood, the outer points of $\overline{\mathrm{LL}}(\gamma(t))$ always have strictly greater modulus than central points. By Lemma 8.1.20 (Hurwitz moves), moving points in the cyclic support of some point in V/W may only affect the terms of the cyclic label corresponding to points of the cyclic support with lower modulus. Thus, $\beta(t)$ can be obtained from $\beta(0)$ by a sequence of elementary moves including

- Replacing a term β_i by a subsequence $\beta_{i,1}, \ldots, \beta_{i,m}$ such that $\beta_{i,1} \cdots \beta_{i,m} = \beta_i$.
- Replacing a subsequence $\beta_i, \ldots, \beta_{i+m}$ by its product $\beta_i \cdots \beta_{i+m}$.
- Replacing a length 2 subsequence β_i, β_{i+1} by either $\beta_{i+1}, \beta_i^{\beta_{i+1}}$ or ${}^{\beta_i}\beta_{i+1}, \beta_i$.

Since all these elementary moves preserve the product of the sequence, we obtain that the product of $\beta(t)$ is equal to the product of $\beta(0)$. Since this is true for all $t \in [0,1]$, we obtain that the product of $\beta(0)$ is equal to the product of $\beta(1)$, i.e. to the i-th term of $\mathrm{cc}(z)$. □

Now we can define the particular set of points of V/W (resp. of $(V/W)^{\mu_d}$) which we will use to study parabolic subgroups up to conjugacy. Let again $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ denote the p-divided groupoid of the Garside group $(G(c), M(c), \Delta)$, and let $\mathcal{S}_p$ denote its simple morphisms. For $s := (a,b) \in \mathcal{S}_p$, we can consider the standard image x_s of s in X/W [Bes15, Definition 11.20].

- If a or b is trivial, then $\overline{\mathrm{LL}}(x_s)$ contains at most one point in each sector P_i for $i \in [\![1,p]\!]$, and $\mathrm{clbl}(x_s) = \mathrm{cc}(x_s)$. If $a = 1$, then we consider $\gamma_s : [0,1] \to V/W$ to be the constant path equal to x_s. If $b = 1$, then we consider $\gamma_s : [0,1] \to V/W$ to be the unique lift inside V/W of the path $[0,1] \to E_n$ which sends t to $(1-t)\overline{\mathrm{LL}}(x_s)$.
- If $a \neq 1$ and $b \neq 1$, then we consider the path $\gamma : [0,1] \to E_n$ which consists, in each sector P_i such that $a_i \neq 1$, in sliding the first point in P_i (which corresponds to a_i in the cyclic label) towards 0 for $i \in [\![1,p]\!]$, as in Lemma 8.1.21. This path admits a unique lift $\gamma_s : [0,1] \to V/W$ with $\gamma_s(0) = x_s$.

In each of these cases, we define z_s as the endpoint of the path γ_s. Note that, if $s \in \mathcal{S}_p^q$, then $x_s \in (X/W)^{\mu_d}$, $\gamma_s : [0,1] \to (V/W)^{\mu_d}$ and $z_s \in (V/W)^{\mu_d}$.

Lemma 9.2.33. *Let $s := (a,b) \in \mathcal{S}_p$ be a simple morphism in $\mathcal{C}_p$.*

- *If a is trivial, then $z_s = x_s$.*
- *If b is trivial, then $z_s = 0$.*
- *If $a \neq 1 \neq b$, then $\operatorname{olbl}(z_s)$ consists of the nontrivial terms of b.*

Furthermore, if $s' := (a',b') \in \mathcal{S}_p$ is such that $b = b'$, then $z_s = z_{s'}$.

Proof. The first point is immediate. The second point follows from [Bes15, Lemma 5.6]. The third point follows from Lemma 8.1.21. Now, if $s' := (a',b') \in \mathcal{S}_p$ is such that $b = b'$ then we either have

- a is trivial, then the product of the terms of b is Δ. The product of the terms of b' is then also equal to Δ, and thus a' is also trivial. Thus $z_s = z_{(1,b)} = z_{(1,b')} = z_{s'}$.
- b is trivial, in which case b is also trivial, and $z_s = 0 = z_{s'}$.
- a and b are both nontrivial, in which case z_s and $z_{s'}$ share the same outer label and $\overline{\mathrm{LL}}(z_s) = \overline{\mathrm{LL}}(z_{s'})$ by construction. Thus, $z_s = z_{s'}$ by Corollary 8.1.19.

$\square$

Using this Lemma, we can write z_b instead of z_s for $s = (a,b) \in \mathcal{S}_p$ from now on. The following proposition shows in particular that, in our study of parabolic subgroups up to conjugacy, we can restrict our attention to points of the form z_b for $s = (a,b) \in \mathcal{S}_p$.

Proposition 9.2.34. *Let $x \in V/W$, and let $b = \operatorname{cc}(x)$.*

(a) The point z_b lies on the same stratum of the discriminant stratification of V/W as x.

(b) If $x \in (V/W)^{\mu_d}$, then the point $z_b \in (V/W)^{\mu_d}$ lies on the same stratum of the discriminant stratification of $(V/W)^{\mu_d}$ as x.

Proof. First, if $x \in X/W$, then by construction, z_b is just the standard image of $(1,b)$, which also lies in X/W, that is, on the same stratum as x (the same thing goes if $x \in (X/W)^{\mu_d}$).

(a) Now, if $x \notin X/W$, then let U be a confining neighborhood of x in V/W. Since $\mathcal{U}_p$ is dense in V/W, we can consider some $y \in U \cap \mathcal{U}_p$. Consider the path in E_n starting from $\overline{\mathrm{LL}}(y)$ which consists in

- In each sector P_i for $i \in [\![1,p]\!]$, slide affinely together the central points, and rotate the resulting point counterclockwise next to the associated half-line.
- Slide affinely together the points in each outer disks towards its center.

This path admits a unique lift $\gamma_y : [0,1] \to V/W$ by Proposition 8.1.10 (path lifting). By Lemma 9.2.32, the product of the terms of $\operatorname{clbl}(\gamma(1))$ corresponding to the outer points in the sector P_i is equal to b_i. We now consider the path in E_n, starting from $\overline{\mathrm{LL}}(\gamma_y(1))$, and which consists in sliding affinely together the outer points of $\overline{\mathrm{LL}}(\gamma_y(1))$ in each sector. It admits a unique lift $\eta : [0,1] \to V/W$ starting from $\gamma_y(1)$ (it may leave the neighborhood U). By construction, the cyclic label of $\eta(1)$ has the form $(\alpha, b) \in \mathcal{S}_p$.

By construction, the point z_b is such that $\mathrm{olbl}(z_b) = b = \mathrm{olbl}(x)$. We can consider an path $\theta : [0,1] \to E_n$ from $\overline{\mathrm{LL}}(x)$ to $\overline{\mathrm{LL}}(z_b)$ (again, consisting in sliding affinely together points in P_i), whose unique lift in V/W is homotopically trivial. It gives that x and z_b lie on the same stratum of the discriminant stratification by Proposition 9.2.31.

(b) It suffices to notice that, by Corollary 9.1.10 (path lifting in $(V/W)^{\mu_d}$), all the paths constructed in the proof of point (a) lie in $(V/W)^{\mu_d}$ if $x \in (V/W)^{\mu_d}$ (and y is taken in $U \cap \mathcal{U}^{\mu_d}$). □

In particular, we obtain that two points of V/W (resp. of $(V/W)^{\mu_d}$) sharing the same cyclic content belong to the same stratum of the discriminant stratification.

Corollary 9.2.35. *Let $x, x' \in V/W$.*

(a) If $\mathrm{cc}(x) = \mathrm{cc}(x')$, then x and x' belong to the same stratum of the discriminant stratification of V/W.

(b) If $x, x' \in (V/W)^{\mu_d}$ and if $\mathrm{cc}(x) = \mathrm{cc}(x')$, then x and x' belong to the same stratum of the discriminant stratification of $(V/W)^{\mu_d}$.

Relative local braid groupoids

Let now $U \subset V/W$ be a confining neighborhood of some point of the form $z_b \in V/W$. We wish to consider $U \cap \mathcal{U}_p$ as a basepoint with several path connected components for $U \cap X/W$. To do this, we need to show that the connected components of $U \cap \mathcal{U}_p$ are simply connected. We actually show an analogue of [Bes15, Lemma 11.22] in this context. That is, path connected components of $U \cap \mathcal{U}_p$ are in bijection with the possible values of the cyclic content, and they are contractible.

Proposition 9.2.36. *Let $s := (a, b) \in \mathcal{S}_p$, and let U be a confining neighborhood of z_b in V/W.*

(a) Two points of $U \cap \mathcal{U}_p$ are in the same path-connected component if and only if they share the same cyclic content. Moreover, the path connected components of $U \cap \mathcal{U}_p$ are contractible.

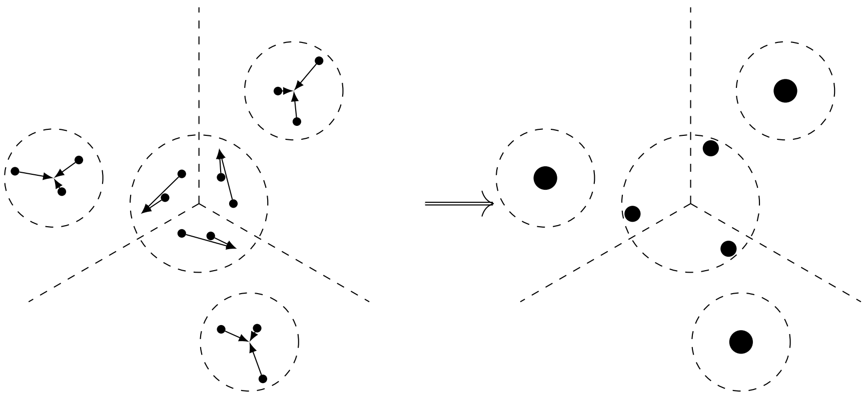
(b) If $s \in \mathcal{S}_p^q$, then two points of $U \cap \mathcal{U}^{\mu_d}$ are in the same path-connected component if and only if they share the same cyclic content. Moreover, the path connected components of $U \cap \mathcal{U}^{\mu_d}$ are contractible.

Proof. The only if part is clear: if x, y belong to the same path connected component of $U \cap \mathcal{U}_p$ (resp. $U \cap \mathcal{U}^{\mu_d}$), then in particular, x and y belong to the same path connected component of $\mathcal{U}_p$ (resp. of $\mathcal{U}^{\mu_d}$, and thus $\mathrm{cc}(x) = \mathrm{cc}(y)$ by Proposition 9.1.12.

(a) Now, let $x, y \in U \cap \mathcal{U}_p$ share the same cyclic content. First, we consider the paths in E_n which consist in

- In each sector P_i for $i \in [\![1, p]\!]$, slide affinely together the points in the central disk, and rotate the resulting point counterclockwise next to the associated half-line.
- Slide affinely together the points in each outer disks towards its center.

The example below (with $p = 3$) illustrates such a path



These paths admit unique lifts γ_x and γ_y in U starting at x and y, respectively. The paths γ_x and γ_y both lie in $\mathcal{U}_p$ by construction, thus they are homotopically trivial and they only impact the cyclic label. We claim that $\gamma_x(1) = \gamma_y(1)$. Since the cyclic supports of $\gamma_x(1)$ and $\gamma_y(1)$ are equal by construction, it only remains to show that $\mathrm{clbl}(\gamma_x(1)) = \mathrm{clbl}(\gamma_y(1))$.

If $z_b = 0$, then there are no outer disks in U, and we have $\mathrm{clbl}(\gamma_x(1)) = \mathrm{cc}(\gamma_x(1)) = \mathrm{cc}(x) = \mathrm{cc}(y) = \mathrm{cc}(\gamma_y(1)) = \mathrm{clbl}(\gamma_y(1))$. If $z_b \neq 0$, then there is at most one outer disk in U per sector P_i for $i \in [\![1, p]\!]$. By construction, we then have

$$\mathrm{clbl}(\gamma_x(1)) = (\alpha_0, \beta_0, \ldots, \alpha_{p-1}, \beta_{p-1}) \text{ and } \mathrm{clbl}(\gamma_y(1)) = (\alpha_0', \beta_0', \ldots, \alpha_{p-1}', \beta_{p-1}'),$$

where α_i (resp. α_i') corresponds to the point of the cyclic support of $\gamma_x(1)$ (resp. $\gamma_y(1)$) in the central disk in the sector P_{i+1}, and β_i (resp. β_i') corresponds to the point of the cyclic support of $\gamma_x(1)$ (resp. $\gamma_y(1)$) in the outer disk in the sector P_{i+1} (if $b_i = 1$, i.e. if z_b has no outer point in the sector P_i, then $\beta_i = \beta_i' = 1$ by convention). By Lemma 9.2.32, we have $\beta_i = \beta_i'$ for $i \in [\![0, p-1]\!]$. For $i \in [\![0, p-1]\!]$, the i-th term of $\mathrm{cc}(x)$ (resp. $\mathrm{cc}(y)$) is given by $\alpha_i\beta_i$ (resp. $\alpha_i'\beta_i'$), We obtain that $\alpha_i = \alpha_i'$ for $i \in [\![0, p-1]\!]$. By Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$), we have $\gamma_x(1) = \gamma_y(1)$, thus x and y lie in the same path connected component of $U \cap \mathcal{U}_p$.

Furthermore, the motions defined above define a deformation-retraction of any path-connected component of $U \cap \mathcal{U}_p$ onto a single point, which proves the contractibility.

Again, point (b) simply comes from the observation that the paths constructed in order to prove (a) are paths in $(V/W)^{\mu_d}$ when $x, y \in \mathcal{U}^{\mu_f}$ by Corollary 9.1.10 (path lifting in $(V/W)^{\mu_d}$). $\square$

We are also able to determine the image of the cyclic content when restricted to $U \cap \mathcal{U}_p$ and to $U \cap \mathcal{U}^{\mu_d}$.

Corollary 9.2.37. *Let $s := (a, b) \in \mathcal{S}_p$, and let U be a confining neighborhood of z_b in V/W.*

(a) The cyclic content restricts to a bijection $\pi_0(U \cap \mathcal{U}_p) \simeq \{u \in D_p(\Delta) \mid b \preccurlyeq u\}$.

(b) If $s \in \mathcal{S}_p^q$, then the cyclic content restricts to a bijection $\pi_0(U \cap \mathcal{U}^{\mu_d}) \simeq \{u \in D_p^q(\Delta) \mid b \preccurlyeq u\}$.

Proof. (a) We know by Proposition 9.2.36 that the cyclic content induces a bijection between $\pi_0(U \cap \mathcal{U}_p)$ and the set

$$\{u \in D_p(\Delta) \mid \exists x \in U \cap \mathcal{U}_p, \mathrm{cc}(x) = u\},$$

we just have to show that this latter set is equal to $\{u \in D_p(c) \mid b \preccurlyeq u\}$. First, let $x \in U \cap \mathcal{U}_p$. We have $b \preccurlyeq \mathrm{cc}(x)$ by Lemma 9.2.32. Conversely, let $u \in D_p(c)$ with $b \preccurlyeq u$. We write $u = ab$ and we set $\sigma := (a, b) \in D_p(\Delta)$. We consider the standard image x_σ of σ in X/W [Bes15, Definition

11.20]. Let $\gamma : [0,1] \to E_n$ be the path starting at $\overline{\mathrm{LL}}(x_\sigma)$ and consisting in shrinking the first point in each sector towards 0, as in Lemma 8.1.21. Let $\widetilde{\gamma}$ be the unique lift of γ in V/W obtained by Proposition 8.1.10 (path lifting). By Lemma 8.1.21, we have $\mathrm{clbl}(\widetilde{\gamma}(1)) = \mathrm{clbl}(z_b)$ and $\widetilde{\gamma}(1) = z_b$ by Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$). By definition, there is some point $\widetilde{\gamma}(t)$ which lies in U. Since $\mathrm{cc}(\gamma_\sigma(t)) = \mathrm{cc}(x_\sigma) = u$, we have the other inclusion.

(b) We apply the same reasoning: if $u \in D_p^q(\Delta)$ is such that $b \preccurlyeq u$, then, since $\tau^q(u) = u$ and $\tau^q(b) = b$, then the tuple a such that $ab = u$ is also such that $\tau^q(a) = a$. The simple σ then lies in $D_p^q(\Delta)$. The path γ is then a path in $(E_n)^{\mu_p}$, and $\widetilde{\gamma}$ is a path in $(V/W)^{\mu_d}$ by Corollary 9.1.10, which gives the result. □

Let U be a confining neighborhood of a standard point in V/W. By Proposition 9.2.36, the set $U \cap \mathcal{U}_p$ can be used as a basepoint (with several path connected components) for the space $U \cap X/W$.

Definition 9.2.38 (Local braid groupoid). Let $s := (a,b) \in \mathcal{S}_p$, and let U be a confining neighborhood of z_b. We define the *local braid groupoid* associated to U by

$$\mathcal{B}_U := \pi_1(U \cap X/W, U \cap \mathcal{U}_p).$$

If $s \in \mathcal{S}$, then we define the *relative local braid groupoid* associated to U by

$$\mathcal{B}_U^q := \pi_1(U \cap (X/W)^{\mu_d}, U \cap \mathcal{U}^{\mu_d}).$$

By Corollary 9.2.37, we can see $\mathrm{Ob}(\mathcal{B}_U)$ as a subset of $D_p(c) \simeq \pi_0(\mathcal{U}_p) = \mathrm{Ob}(B_p(W))$ (see Definition 9.1.13). The inclusion $U \cap (X/W)^{\mu_d} \hookrightarrow (X/W)^{\mu_d}$ then give a natural functor $\mathcal{B}_U \to B_p(W) \simeq \mathcal{G}_p$, which is injective on objects. As we will see later, it is also injective on morphisms, but at this stage we are not supposed to know this. Likewise, we have a functor $\mathcal{B}_U^q \to B_p^q(W) \simeq \mathcal{G}$ to the Springer groupoid. We then have a commutative square of functors

$$\begin{tikzcd}
\mathcal{B}_U^q \arrow[r] \arrow[d] & \mathcal{B}_U \arrow[d] \\
B_p^q(W) \arrow[r] & B_p(W)
\end{tikzcd}$$

At this point, it is unclear whether or not $\mathcal{B}_U$ and $\mathcal{B}_U^q$ depend on the choice of U, and whether or not the functor $\mathcal{B}_U \to B_p(W)$ (resp. $\mathcal{B}_U^q \to B_p^q(W)$) sends $\mathcal{B}_U(u,u)$ to a parabolic subgroup of $B_p(W)(u,u)$ (resp. of $B_p^q(W)(u,u)$). The next Section is devoted to the study of these problems.

Correspondence between standard parabolic subgroupoids and local braid groupoids

Let $u \in \pi_0(\mathcal{U}_p)$, since all the groups $\pi_1(X/W, x)$ with $x \in u$ are canonically identified with one another, we can talk about the parabolic subgroups of $B_p(W)(u,u)$ (we can also talk about the parabolic subgroups of $B_p^q(W)(u,u)$).

Using Proposition 9.2.34, we show that parabolic subgroups are realized up to conjugacy as fundamental groups of the intersection of X/W with confining neighborhoods of points of the form z_b for $s = (a,b) \in \mathcal{S}_p$.

Corollary 9.2.39. *(a) Let $B_0 \subset B_p(W)(u,u)$ be a parabolic subgroup. There is some $s := (a,b) \in \mathcal{S}_p$, along with a confining neighborhood U of z_b in V/W, and some $v \in \pi_0(U \cap \mathcal{U}_p)$ such that B_0 is conjugate in $B_p(W)$ to the image of $\mathcal{B}_U(v,v)$ in $B_p(W)(v,v)$.*

(b) Let $B_0 \subset B_p^q(W)(u,u)$ be a parabolic subgroup. There is some $s := (a,b) \in \mathcal{S}_p^q$, along with a confining neighborhood U of z_b in $(V/W)^{\mu_d}$, and some $v \in \pi_0(U \cap \mathcal{U}^{\mu_d})$ such that B_0 is conjugate in $B_p^q(W)$ to the image of $\mathcal{B}_U^q(v,v)$ in $B_p^q(W)(v,v)$.

Proof. (a) Let η be a capillary path such that B_0 is the image in $B_p(W)(u,u)$ of $\pi_1^{\mathrm{loc}}(X/W,\eta)$. Up to replacing B_0 by a conjugate subgroup, we can replace $\eta(1)$ by another point lying on the same stratum of the discriminant stratification. By Proposition 9.2.34, we can then assume that $\eta(1)$ has the form z_b for some $s = (a,b) \in \mathcal{S}$.

Let U be a neighborhood of z_b in V/W, suitable for defining $\pi_1^{\mathrm{loc}}(X/W,\eta)$. Up to taking a subneighborhood of U, we can assume that U is a confining neighborhood of z_b. Up to replacing η and z_b by $e^{i\varepsilon}\eta$ and $e^{i\varepsilon}z_b$ for $\varepsilon > 0$ small enough, we can assume that $\eta(t) \in U \cap \mathcal{U}^{\mu_d}$ for some $t < 1$ suitable for defining $\pi_1^{\mathrm{loc}}((X/W)^{\mu_d},\eta)$. If we denote by v the connected component of $\eta(t)$ in $U \cap \mathcal{U}_p$, we get that $\pi_1^{\mathrm{loc}}(X/W,\eta)$ is identified with $\mathcal{B}_U(v,v)$. The restriction of the capillary path η to $[0,t]$ gives a morphism f from u to v in $B_p(W)$. The conjugation by f in $B_p(W)$ sends the image of $\mathcal{B}_U(v,v)$ in $B_p(W)(v,v)$ to B_0 as claimed. The same reasoning works for point (b). $\square$

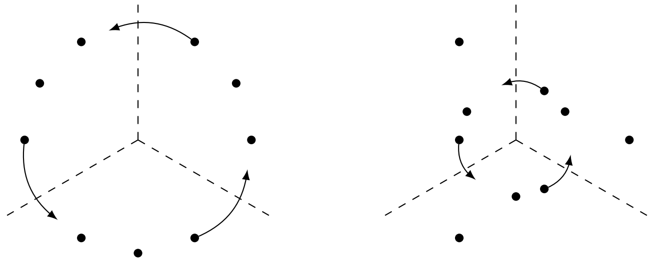
At this stage, we still need to show that the functor $\mathcal{B}_U \to B_p(W)$ associated to some confining neighborhood U in V/W always sends $\mathcal{B}_U(u,u)$ to a parabolic subgroup of $B_p(W)(u,u)$.

Let $s := (a,b) \in \mathcal{S}_p$, that we fix from now on. We plan to show that, for any confining neighborhood of z_b in V/W, the isomorphism $\mathcal{G}_p \to B_p(W)$ of Theorem 9.1.14 restricts to an isomorphism between $(\mathcal{G}_p)_b$ and $\mathcal{B}_U$. And similarly, if $s \in \mathcal{S}_p^q$, then the isomorphism $\mathcal{G} \simeq B_p^q(W)$ induces an isomorphism $\mathcal{G}_b \simeq \mathcal{B}_U^q$.

Let $\sigma = (d,e) \in (\mathcal{S}_p)_b$. By definition of $(\mathcal{S}_p)_b$, we can write $e = e'b$ and consider the standard image x_0 of the tuple (d,e',b) in V/W. Let γ be the path in E_n starting from $\overline{\mathrm{LL}}(x_0)$ and consisting in shrinking the points corresponding to d and to e' in the cyclic label towards 0 as in Lemma 8.1.21, and let $\widetilde{\gamma}$ be the unique lift of γ in V/W starting from x_0. By Lemma 8.1.21 and Proposition 8.1.18 (trivialization of $\overline{\mathrm{LL}}$), we have $\widetilde{\gamma}(1) = z_b$.

For $t \in [0,1]$, let $H_\sigma(t,-)$ be the unique lift in V/W of the path in E_n starting from $\overline{\mathrm{LL}}(\gamma(t))$ and consisting in rotating the points of the cyclic support of $\gamma(t)$ corresponding to d by an angle of $3\pi/4p$. We obtain a continuous map $H_\sigma : [0,1] \times [0,1] \to V/W$. By the Hurwitz rule, the path $H_\sigma(t,-)$ represents σ in $B_p(W)$ for all $t < 1$.

Here are two examples with $p = 3$, of paths of the form $H_\sigma(0,-)$ and $H_\sigma(1/2,-)$



Proposition-Definition 9.2.40. *Let U be a confining neighborhood of z_b in V/W.*

(a) For all $\sigma \in (\mathcal{S}_p)_b$, the path $H_\sigma(t,-)$ lies in U for $t < 1$ big enough, we denote by f_σ the morphism it induces in $\mathcal{B}_U$. The map $(\mathcal{S}_p)_b \to \mathcal{B}_U$, $\sigma \mapsto f_\sigma$ extends to a groupoid morphism

$\psi : (\mathcal{G}_p)_b \to \mathcal{B}_U$.

(b) If $s \in \mathcal{S}_p^q$, then for all $\sigma \in (\mathcal{S}_p^q)_b$, the path $H_\sigma(t, -)$ lies in $U \cap (X/W)^{\mu_d}$ for $t < 1$ big enough, and we denote by g_σ the morphism is induces in $\mathcal{B}_U^q$. The map $(\mathcal{S}_p^q)_b \to \mathcal{B}_U^q$, $\sigma \mapsto g_\sigma$ extents to a groupoid morphism $\psi' : \mathcal{G}_b \to \mathcal{B}_U^q$.

(c) We have commutative squares of functors

$$\begin{tikzcd} (\mathcal{G}_p)_b \arrow[r, hook] \arrow[d, "\psi"'] & \mathcal{G}_p \arrow[d, "\simeq"] \\ \mathcal{B}_U \arrow[r] & B_p(W) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}_b \arrow[r, hook] \arrow[d, "\psi'"'] & \mathcal{G} \arrow[d, "\simeq"] \\ \mathcal{B}_U^q \arrow[r] & B_p^q(W) \end{tikzcd}$$

Proof. (a) We keep the notation from above. Let $\theta \in [0, 1]$. By Lemma 8.1.21, we have $H_\sigma(1, \theta) = z_b$. Since U is a neighborhood of z_b, and since H_σ is continuous, there is a neighborhood K_θ of $(1, \theta)$ in $[0, 1] \times [0, 1]$ that is sent inside of U by H. The union $\bigcup K_\theta$ is an open cover of the compact set $\{(1, \theta) \mid \theta \in I\}$. By extracting a finite cover, we see that there is some $t_0 \in [0, 1[$ such that $H_\sigma(t, -)$ is a path in U for $t > t_0$.

Furthermore, as all the paths $H_\sigma(t, -)$ are homotopic for $t < 1$, we see that the morphism f_σ does not depend on the choice of $t < 1$ big enough.

The standard parabolic subgroupoid $((\mathcal{G}_p)_b, (\mathcal{C}_p)_b, \delta_b)$ is in particular a Garside category. By Proposition 2.2.2 (germ from Garside) and Theorem 2.2.4 (Garside from germ), we have a presentation of $(\mathcal{G}_p)_b$, where the generators are the elements of $(\mathcal{S}_p)_b$, and the relations are all the equalities of the form $xy = z$ which hold in $\mathcal{G}_p$, with $w, y, z \in (\mathcal{S}_p)_b$. Thus, replacing the circular tunnels of [Bes15, Definition 11.25] by paths of the form $H_\sigma(t, -)$ for $t < 1$ big enough, we can imitate the proof of [Bes15, Lemma 11.26] to obtain that the map $\sigma \mapsto f_\sigma$ is compatible with the defining relations of $(\mathcal{C}_p)_b$ and $(\mathcal{G}_p)_b$. Thus $\sigma \mapsto f_\sigma$ induces a functor $\psi : (\mathcal{G}_p)_b \to \mathcal{B}_U$.

(b) If $s, \sigma \in \mathcal{S}_p^q$, then the path γ defined above lies in $(E_n)^{\mu_p}$, and its lift $\widetilde{\gamma}$ actually is a path in $(V/W)^{\mu_d}$ by Corollary 9.1.10 (path lifting in $(V/W)^{\mu_d}$). Likewise, we obtain that H_σ is actually a continuous map $[0, 1] \times [0, 1] \to (V/W)^{\mu_d}$, and thus, it represents a morphism $g_\sigma \in \mathcal{B}_U^q$ for $t < 1$ big enough. We then reproduce the same reasoning as for point (a) in order to conclude.

(c) By the Hurwitz rule, the path $H_\sigma(t, -)$ represents σ in $B_p^q(W) \subset B_p(W)$ for all $t < 1$. The image of f_σ (resp. g_σ) under the natural functor $\mathcal{B}_U \to B_p(W)$ (resp. $\mathcal{B}_U^q \to B_p^q(W)$) is then equal to σ, and the given squares are commutative as claimed. □

Considering the embedding $\mathcal{G} \to \mathcal{G}_p$, and its restriction to $\mathcal{G}_b$, we obtain a commutative diagram of functors.

$$\begin{tikzcd} (\mathcal{G})_\beta \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right=60] & (\mathcal{G}_p)_\beta \arrow[d, hook] \arrow[dd, bend left=60] \\ B_p^q(W) \simeq \mathcal{G} \arrow[r, hook] & B_p(W) \simeq \mathcal{G}_p \\ \mathcal{B}_U^q \arrow[r] \arrow[u] & \mathcal{B}_U \arrow[u] \end{tikzcd}$$

From this we finally deduce that standard parabolic subgroupoids and local braid groupoids are identified by the isomorphism $\mathcal{G}_p \to B_p(W)$.

Theorem 9.2.41 (Isomorphism of groupoids). *Let W be an irreducible well-generated complex reflection group with highest degree h and let d be a regular number for W. We use the notation of Theorem 8.2.11. We write $p := \frac{d}{d \wedge h}$ and $q := \frac{h}{d \wedge h}$. We consider $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid attached to $(G(W,c), M(W,c), \Delta)$, and $(\mathcal{G}, \mathcal{C}, \Delta_p)$ the Springer groupoid attached to W, d.*
Let $s = (a,b) \in \mathcal{S}$, and let U be a confining neighborhood of z_b in $(V/W)^{\mu_d}$. The functor $\psi : (\mathcal{G}_p)_b \to \mathcal{B}_U$ of Proposition-Definition 9.2.40 is an isomorphism of groupoids. Furthermore, if $s \in \mathcal{S}_p^q$, then ψ restricts to an isomorphism $\psi' : \mathcal{G}_b \to \mathcal{B}_U^q$.

Proof. The commutative squares of functors of Proposition-Definition 9.2.40 prove in particular that ψ and ψ' are faithful. To prove that ψ and ψ' are full, we use the same generic position argument as in the proof of [Bes15, Theorem 11.28]: every morphism f in $\mathcal{B}_U$ (resp. in $\mathcal{B}_U^q$) can be represented by a path γ such that at any given $t \in [0,1]$, at most one point in $\overline{\mathrm{LL}}(\gamma(t))$ lies on the half-line $i\mathbb{R}_{\geqslant 0}$. This expresses f as a composition of paths homotopic to some f_σ with $\sigma \in (\mathcal{S}_p)_b$ (resp. to some g_σ with $\sigma \in (\mathcal{S}_p^q)_b$).

Now, by Proposition 9.2.36, we know that ψ and ψ' are bijective on objects, which terminates the proof. □

In particular, we obtain that the groupoids $\mathcal{B}_U$ and $\mathcal{B}_U^q$ do not depend on the confining neighborhood U, but only on the point z_b. Lastly, we show the main theorem of this section.

Theorem 9.2.42 (Parabolic subgroups up to conjugacy). *Let W be an irreducible well-generated complex reflection group with highest degree h and let d be a regular number for W. We use the notation of Theorem 8.2.11. We write $p := \frac{d}{d \wedge h}$ and $q := \frac{h}{d \wedge h}$. We consider $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ the p-divided groupoid attached to $(G(W,c), M(W,c), \Delta)$, and $(\mathcal{G}, \mathcal{C}, \Delta_p)$ the Springer groupoid attached to W, d.*

- *(a) Let $u \in \mathrm{Ob}(\mathcal{G}_p)$ such that $B(W) \simeq B_p(W)(u,u)$. A subgroup $B_0 \subset B(W) \simeq \mathcal{G}_p(u,u)$ is parabolic if and only if it is a $\mathcal{T}$-parabolic subgroup of $\mathcal{G}_p(u,u)$.*
- *(b) Let $u \in \mathrm{Ob}(\mathcal{G})$ such that $B(W)^{\mu_d} \simeq B_p^q(W)(u,u)$. A subgroup $B_0 \subset B(W)^{\mu_d} \simeq \mathcal{G}(u,u)$ is parabolic if and only if it is a $\mathcal{T}'$-parabolic subgroup of $\mathcal{G}(u,u)$.*

Proof. (a) First, let $(\mathcal{G}_p)_\beta$ be a standard parabolic subgroupoid of $\mathcal{G}_p$. Let also $f \in \mathcal{G}_p(u,v)$ be such that $v \in \mathrm{Ob}((\mathcal{G}_p)_\beta)$. By definition we have $v = \alpha\beta$, and we can set $s := (\alpha, \beta)$. Let x_s be the standard image of s in X/W. The natural path η from x_s to z_β consisting in shrinking points towards 0 is a capillary path. The image of $\pi_1^{\mathrm{loc}}(X/W, \eta)$ in $\pi_1(X/W, x_s) = B_p(W)(v,v)$ is a parabolic subgroup of the latter by definition.

Let U be a small enough confining neighborhood of z_β in V/W. By definition of a local fundamental group and by Lemma 7.3.1, we have identifications

$$\pi_1^{\mathrm{loc}}(X/W, \eta) \simeq \pi_1(U \cap X/W, \eta) = \mathcal{B}_U(v,v).$$

Thus, the image of $\mathcal{B}_U(v,v)$ in $\mathcal{G}_p(v,v)$ is a parabolic subgroup of $\mathcal{G}_p(v,v)$. By Theorem 9.2.41, this image is $(\mathcal{G}_p)_\beta(v,v)$, which is then a parabolic subgroup of $\mathcal{G}_p(v,v)$.

Let now δ be a path from u to v in V/W representing f in $\mathcal{G}_p$. The isomorphism between $\mathcal{G}_p(u,u)$ and $\mathcal{G}_p(v,v)$ induced by δ is a mere change of basepoint, and as such, it preserves the set of parabolic subgroups. Thus $B_0 := ((\mathcal{G}_p)_\beta(v,v))^f$ is a parabolic subgroup of $\mathcal{G}_p(u,u) = B_p(W)(u,u)$.

Conversely, let $B_0 \subset B_p(W)(u,u)$ be a parabolic subgroup. By Corollary 9.2.39, there is some $\alpha \in B_p(W)(u,v)$ such that $(B_0)^\alpha$ is the image of $\mathcal{B}_U(v,v)$ in $B_p(W)(v,v)$, where U is a confining neighborhood of some z_β in V/W. We have by Theorem 9.2.41 that the image of $\mathcal{B}_U(v,v)$ is $(\mathcal{G}_p)_b(v,v)$, which terminates the proof in this case.

(b) We apply the same reasoning as in point (a). First, let $\mathcal{G}_\beta$ be a standard parabolic subgroupoid of $\mathcal{G}$. Let also $f \in \mathcal{G}(u,v)$ be such that $v \in \mathrm{Ob}(\mathcal{G}_\beta)$. By definition we have $v = \alpha\beta$, and we can set $s := (\alpha,\beta)$. Let x_s be the standard image of s in $(X/W)^{\mu_d}$. The natural path η from x_s to z_β consisting in shrinking points towards 0 is a capillary path. The image of $\pi_1^{\mathrm{loc}}((X/W)^{\mu_d},\eta)$ in $\pi_1((X/W)^{\mu_d},x_s) = B_p^q(W)(v,v)$ is a parabolic subgroup of the latter by definition.

Let U be a small enough confining neighborhood of z_β in $(V/W)^{\mu_d}$. By definition of a local fundamental group, we have identifications

$$\pi_1^{\mathrm{loc}}((X/W)^{\mu_d},\eta) \simeq \pi_1(U \cap (X/W)^{\mu_d},\eta) = \mathcal{B}_U^q(v,v).$$

Thus, the image of $\mathcal{B}_U^q(v,v)$ in $\mathcal{G}(v,v)$ is a parabolic subgroup of $\mathcal{G}(v,v)$. By Theorem 9.2.41, this image is $\mathcal{G}_\beta(v,v)$, which is then a parabolic subgroup of $\mathcal{G}(v,v)$.

Let now δ be a path from u to v in $(V/W)^{\mu_d}$ representing f in $\mathcal{G}$. The isomorphism between $\mathcal{G}(u,u)$ and $\mathcal{G}(v,v)$ induced by δ is a mere change of basepoint, and as such, it preserves the set of parabolic subgroups. Thus $B_0 := (\mathcal{G}_\beta(v,v))^f$ is a parabolic subgroup of $\mathcal{G}(u,u) = B_p^q(W)(u,u)$.

Conversely, let $B_0 \subset B_p^q(W)(u,u)$ be a parabolic subgroup. By Corollary 9.2.39, there is some $\alpha \in B_p^q(W)(u,v)$ such that $(B_0)^\alpha$ is the image of $\mathcal{B}_U^q(v,v)$ in $B_p^q(W)(v,v)$, where U is a confining neighborhood of some z_β in $(V/W)^{\mu_d}$. We have by Theorem 9.2.41 that the image of $\mathcal{B}_U^q(v,v)$ is $\mathcal{G}_b(v,v)$, which terminates the proof. □

Remark 9.2.43. In this proof, we did not use Proposition 8.2.24. In particular, Theorem 9.2.42 applied to $d = 1$ gives a proof of Proposition 8.2.24.

9.2.6 Braided reflections and centers of finite index subgroups

In this short section, we examine the consequences of the above constructions on the center of Springer groupoids. In particular, we give a Garside-theoretic proof of [DMM11, Theorem 1.4] for regular centralizers in irreducible well-generated complex braid groups. Our method is to describe the conjugacy of distinguished braided reflections in Springer groupoids using atomic loops.

In this section, we fix a finite dimensional complex vector space V of dimension n, along with an irreducible well-generated complex reflection group $W \subset \mathrm{GL}(V)$. We define h as the highest degree of W, and we fix d a regular number for W. We set again $p := \frac{d}{d\wedge h}$ and $q := \frac{h}{d\wedge h}$. We denote by $(\mathcal{G},\mathcal{C},\Delta_p)$ the Springer groupoid attached to W,d.

We begin by computing the center of $\mathcal{G}$ and of $\mathcal{C}$. We recall some basic facts about centers of categories.

Definition 9.2.44 (Center of a category). Let $\mathcal{E}$ be a category. The *center* of $\mathcal{E}$ is the set $Z(\mathcal{E})$ of natural transformation from the functor $1_{\mathcal{E}}$ to itself.

Let $\mathcal{E}$ be a category. The data of an element z of $Z(\mathcal{E})$ is equivalent to the data, for every $u \in \mathrm{Ob}(\mathcal{E})$, of a morphism z_u such that

$$\forall f \in \mathcal{E}(u,v), z_u f = f z_v.$$

If $\mathcal{E} = M$ is a monoid, we recover the classical definition of the center. Note that $Z(\mathcal{E})$ is always a monoid. Moreover, the center of a groupoid is a group.

Lemma 9.2.45. *Let $\mathcal{K}$ be a connected groupoid, and let u be an object of $\mathcal{K}$. The map*

$$\begin{array}{rccc} r: & Z(\mathcal{K}) & \longrightarrow & Z(\mathcal{K}(u,u)) \\ & z & \longmapsto & z_u \end{array}$$

is an isomorphism of groups.

Proof. The map r is clearly a morphism of groups. We choose, for every $v \in \mathrm{Ob}(\mathcal{K})$, a morphism $m_v : u \to v$. Let $z_0 \in Z(\mathcal{K}(u,u))$ be a central element. We define an element z of $Z(\mathcal{K})$ by defining, for every $v \in \mathrm{Ob}(\mathcal{K})$:

$$z_v := m_v^{-1} z_0 m_v.$$

We have in particular $z_u = m_u^{-1} z_0 m_u = z_0$ since z_0 is central. Let $f : v \to w$ be a morphism in $\mathcal{K}$, we have

$$z_v f = m_v^{-1} z_0 m_v f = m_v^{-1} z_0 (m_v f m_w^{-1}) m_w = m_v^{-1} (m_v f m_w^{-1}) z_0 m_w = f z_w,$$

and z is indeed in $Z(\mathcal{K})$. The map $z_0 \mapsto z$ is the inverse of r. $\square$

Let us go back to the Springer groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$. It is known from [BMR98, Theorem 2.24] and [Bes15, Theorem 12.3 and Corollary 12.7] that the center of an irreducible complex braid group is cyclic. Since $\mathcal{G}$ is connected and equivalent to the braid group of a regular centralizer in W (which is irreducible by Theorem 6.1.30), the center of the Springer groupoid $\mathcal{G}$ is also cyclic since $\mathcal{G}$ is connected. Under a combinatorial assumption on the integer d, we get that the center of $\mathcal{C}$ (and $\mathcal{G}$) is actually generated by some power of the Garside map Δ_p.

Proposition 9.2.46. *Assume that the regular number d is the gcd of the degrees of the complex reflection group W which it divides, then both $Z(\mathcal{C})$ and $Z(\mathcal{G})$ are cyclic and generated by Δ_p^q.*

Proof. First, as Δ_p is a natural transformation from $1_{\mathcal{C}}$ to ϕ_p, and as $\phi_p^q = 1_{\mathcal{C}}$, we get that Δ_p^q lies in the center of $\mathcal{C}$.

We claim that $Z(\mathcal{G})$ is generated by $Z(\mathcal{C})$ and Δ_p^{-q}. Let $z \in Z(\mathcal{G})$, and let $u \in \mathrm{Ob}(\mathcal{G})$. We set $n_u := \inf(z_u)$. Since $\mathcal{G}$ admits a finite number of objects ($D_p^q(c)$ is finite by definition), there is some kq for $k \in \mathbb{Z}_{\geqslant 0}$ such that $kq + n_u > 0$ for all object u. We then have that $\Delta_p^{kq} z \in Z(\mathcal{C})$ and $z = \Delta_p^{-kq}(\Delta_p^{kq} z)$ as claimed.

Now, let ρ be a generator of the center of $\mathcal{G}(u,u)$. As $\Delta_p^q(u) \in Z(\mathcal{G}(u,u))$, there is some integer k such that $\rho^k = \Delta_p^q(u)$. That is ρ is a (pk,q)-regular element in $\mathcal{G}$. By applying π_p, we get that $\pi_p(\rho)$ is a (pk,q)-regular element of $B(W)$. By assumption, we have

$$\pi_p(\mathcal{G}(u,u)) = C_{B(W)}(\pi_p(\Delta_p^q)) \subset C_{B(W)}(\pi_p(\rho)).$$

We also have $C_{B(W)}(\pi_p(\Delta^q)) \supset C_{B(W)}(\pi_p(\rho))$ since $\pi_p(\Delta_p^q)$ is a power of $\pi_p(\rho)$. The element $\pi_p(\rho)$ is then a dk-regular braid in $B(W)$ with the same centralizer as a d-regular braid. Since d is maximal regarding to divisibility, we get that $k = \pm 1$. Thus $\rho = \Delta_p^{\pm q}(u)$ which shows the proposition. $\square$

Remark 9.2.47. The assumption that d is the gcd of the degrees of W which it divides is important, otherwise $Z(\mathcal{C})$ is generated by some root of Δ_p^q. For instance in the group $W = G_{37}$, the integers $d_1 = 5$ and $d_2 = 10$ are regulars. We have $h = 30$ and thus $(p_1, q_1) = (1, 6)$ and $(p_2, q_2) = (1, 3)$. The associated categories of periodic elements are then monoids, given by $C_M(\Delta^3)$ and $C_M(\Delta^6)$, respectively (M is the dual braid monoid of type G_{37}). Because d_1 and d_2 both divide 2 degrees of W, those centralizers are equal, and their centers are both equal to $\langle \Delta^3 \rangle^+$.

In [Gar23b, Theorem 3.31], we completely described the conjugacy of braided reflections in the group $B(W)^{\mu_d}$ in terms of atomic loops in the Springer groupoid. However, we saw in Section 7.3.2 that parabolic subgroups of rank 1 in a complex braid group were exactly the subgroups generated by distinguished braided reflections (see the discussion after Proposition 7.3.4). Applying Theorem 9.2.42 (parabolic subgroups up to conjugacy) then gives a more general proof of [Gar23b, Theorem 3.31]:

Corollary 9.2.48 (Braided reflections and atomic loops). *Let u be an object of $\mathcal{C}$, and let $s \in \mathcal{C}(u, -)$ be an atom. The atomic loop $\lambda(s)$ is a braided reflection in the group $B(W)^{\mu_d} \simeq \mathcal{G}(u, u)$. Conversely, any braided reflection in $\mathcal{G}(u, u)$ is conjugate in $\mathcal{G}$ to some atomic loop.*

Proof. Let $s = (a, b) \in \mathcal{C}(u, -)$. By construction, we have $s = \delta_b(u)$, and, for $k > 0$, we have $\delta_b^k(u) = ss^{\#} \cdots s^{(k-1)\#}$. Thus the atomic loop $\lambda(s)$ is equal to the first power $\delta_b^n(u)$ which lies in $\mathcal{C}(u, u)$. We claim that $\mathcal{G}_b(u, u) = \langle \lambda(s) \rangle$.

Let $x \in \mathcal{G}_b(u, u)$. Up to multiplying x by a big enough power of $\delta_b^n(u) = \lambda(s)$, we can assume that $x \in \mathcal{C}_b(u, u)$. Let us write $x = s_1 \cdots s_r$ the greedy normal form of x. We have $s_1 \preccurlyeq \delta_b(u) = s$ in $\mathcal{C}$, and thus $s_1 = s$ since s is an atom (and s_1 is nontrivial unless $x = 1_u$). Likewise, we deduce that $s_2 \preccurlyeq s^{\#} = \delta_b(v)$ where v is the target of s. By an immediate induction, we obtain $\lambda(s) \preccurlyeq x$ since x has target u. If $\lambda(s) \neq x$, then we can apply the same reasoning to $\lambda(s)^{-1}x \in \mathcal{C}_b(u, u)$ to get that $x = \lambda(a)^k$ for some $k > 0$.

By Theorem 9.2.42, we know that $\mathcal{G}_b(u, u) = \langle \lambda(s) \rangle$ is a parabolic subgroup of $\mathcal{G}(u, u) \simeq B(W_g)$. Since it is isomorphic to $\mathbb{Z}$, we obtain that $\lambda(s)$ is a distinguished braided reflection.

Conversely, let $\sigma \in \mathcal{G}(u, u)$ be a distinguished braided reflection. The subgroup $\langle \sigma \rangle \subset \mathcal{G}(u, u)$ is a parabolic subgroup, and by Theorem 9.2.42, there is some $f \in \mathcal{G}(u, v)$ such that $\langle \sigma \rangle^f = \langle \sigma^f \rangle = \mathcal{G}_\beta(v, v)$ for some $\mathcal{G}_\beta \in \mathcal{T}$. If we write $s := \delta_\beta(v)$, then we have $\lambda(s) \in \mathcal{G}_\beta(v)$ and $(\sigma^f)^k = \lambda(s)$ for some integer k. Let $a \in \mathcal{C}(v, -)$ be an atom such that $a \preccurlyeq s$. We know by the first part of the proof that $\lambda(a)$ is a distinguished braided reflection in $\mathcal{G}(v, v)$. Moreover, by construction, we have $\lambda(a) \in \mathcal{G}_\beta(v)$, and thus $\lambda(a) = (\sigma^f)^i$ for some integer i. But since $\lambda(a)$ and σ^f are both distinguished braided reflections, we have $\lambda(a) = \sigma^f$. If $b \preccurlyeq s$ is another atom of $\mathcal{C}$, then $\lambda(b) = \sigma^f = \lambda(a)$ with the same reasoning. Since atomic loops are greedy by Lemma 9.2.9, we deduce that $a = b$. That is, there is exactly one atom which left-divides s. This is possible if and only if s itself is an atom. Thus, $\lambda(s)$ is a distinguished braided reflection, and $\lambda(s) = \sigma^f$. That is, σ is conjugate by f to the atomic loop $\lambda(s)$. □

In [Gar23b, Corollary 3.38], we were also able to describe the conjugacy of atomic loops and their powers, taking advantage of the rigidity of atomic loops. Just like [DMM11, Proposition 2.2] can be seen as a consequence of the more general support-preservingness property in a Garside group, [Gar23b, Corollary 3.38] is a consequence of the support-preservingness of the shoal $\mathcal{T}'$ for the Springer groupoid.

Corollary 9.2.49. *Let $\lambda(s) \in \mathcal{C}(u,u)$ be an atomic loop, and let $f \in \mathcal{G}$. If $z := (\lambda(s)^n)^f \in \mathcal{C}$ for some $n \geqslant 1$, then $z = \lambda(s')^n$ for some atomic loop $\lambda(s')$ such that $\lambda(s') = \lambda(s)^f$.*

Proof. Let v be the source of z, and let $\mathcal{G}_b \in \mathcal{T}'$ be the categorical standard parabolic closure of z, so that $\mathcal{G}_b(v,v) = \mathrm{SPC}(z)$. By support-preservingness of the shoal $\mathcal{T}'$, we have

$$\mathrm{SPC}(z) = \mathrm{SPC}(\lambda(s))^f = \langle \lambda(s) \rangle^f = \langle \lambda(s)^f \rangle.$$

In particular, $\mathrm{SPC}(z)$ is a standard parabolic subgroup in $\mathcal{G}(v,v)$ which is isomorphic to $\mathbb{Z}$. If $y = \lambda(\delta_b(v))$, we obtain that $y = \lambda(s)^f$ by using the same proof as in Corollary 9.2.48. Thus $y = \lambda(s)^f = \lambda(s')$ is an atomic loop, and we have $z = \lambda(s')^m$ for some integer m. However, we have

$$z = (\lambda(s)^n)^f = (\lambda(s)^f)^n = \lambda(s')^n,$$

thus $n = m$ and the result is shown. □

As in [Gar23b], this result implies in turn that we can describe completely the super-summit sets of atomic loops and their powers.

Corollary 9.2.50 (Conjugacy of atomic loops). *Let $u \in \mathrm{Ob}(\mathcal{G})$, and let $\sigma \in \mathcal{G}(u,u)$ be a braided reflection. The super-summit set of σ in $\mathcal{G}$ consists of all the atomic loops to which σ is conjugate in $\mathcal{G}$. Furthermore, for $n \geqslant 1$, we have*

$$\mathrm{SSS}(\sigma^n) = \{\lambda(s)^n \mid \lambda(s) \in \mathrm{SSS}(\sigma)\}.$$

Proof. We already showed that σ is conjugate to some atomic loop $\lambda(s) \in \mathcal{G}$, and that $\lambda(s) \in \mathrm{SSS}(\lambda(s))$ for all atomic loops $\lambda(s)$. If $g \in \mathrm{SSS}(\lambda(s))$, then g is a positive conjugate of $\lambda(s)$ and Corollary 9.2.49 gives that g is an atomic loop.

We also have $\lambda(s)^n \in \mathrm{SSS}(\sigma^n)$. If $g \in \mathrm{SSS}(\sigma^n)$, then we also have that g is a positive conjugate of $\lambda(s)^n$. We get that g is of the form $\lambda(s')^n$ for some conjugate $\lambda(s')$ of $\lambda(s)$ by Corollary 9.2.49. □

Lastly, we also have the associated corollary on finite index subgroups in braid groups of regular centralizers in well-generated complex reflection groups.

Corollary 9.2.51. *Let W be a well-generated complex reflection group, and let $g \in W$ be a regular element. If $U \subset B(W_g)$ is a finite index subgroup, then $Z(U) \subset Z(B(W_g))$.*

Proof. Let d be the integer so that g is d-regular, and let $(\mathcal{G}, \mathcal{C}, \Delta)$ be the Springer groupoid attached to W, d. Since $B(W_g)$ is equivalent to $\mathcal{G}$, it is sufficient to show the result for $U \subset \mathcal{G}(u,u)$ for some object $u \in \mathrm{Ob}(\mathcal{G})$. Let $\sigma \in \mathcal{G}(u,u)$ be a braided reflection, and let $x \in Z(U)$. Since U has finite index, there is some $n \geqslant 1$ with $x\sigma^n = \sigma^n x$. We claim that $\sigma x = x\sigma$.

By Corollary 9.2.48, there is some morphism $f : u \to v$ in $\mathcal{G}$ such that $\sigma^f = \lambda(s)$ is an atomic loop. We define $x' := x^f$ and $U' := U^f \subset \mathcal{G}(v,v)$. By Theorem 9.2.49, since $\lambda(s)^{nx} = \lambda(s)^n$, we have $\lambda(s)x = x\lambda(s')$ for some atomic loop $\lambda(s')$ such that $\lambda(s')^n = \lambda(s)^n$. Since atomic loops are rigid, the equality $\lambda(s)^n = \lambda(s')^n$ implies $\lambda(s) = \lambda(s')$ by Lemma 3.2.13 and x' commutes with $\lambda(s)$. We obtain that x commutes with every braided reflection in $\mathcal{G}(u,u)$, and thus lies in $Z(\mathcal{G}(u,u))$. □

The reader will notice that this proof is a mere adaptation of the proof of Proposition 5.2.12, replacing atoms with atomic loops.

9.3 Parabolic subgroups of regular centralizers

In this section, we fix a finite dimensional complex vector space V of dimension n, along with an irreducible complex reflection group $W \subset \mathrm{GL}(V)$. We fix d a regular number for W. For the sake of readability, we choose a basepoint $* \in (X/W)^{\mu_d}$. We set $B(W)^{\mu_d} := \pi_1((X/W)^{\mu_d}, *)$ and $B(W) := \pi_1(X/W, *)$.

The embedding $(X/W)^{\mu_d} \hookrightarrow X/W$ identifies $B(W)^{\mu_d}$ with the centralizer in $B(W)$ of some d-regular braid ρ by Theorem 9.1.7 (Springer theory in braid groups). In this Section, we want to understand the parabolic subgroups of $B(W)^{\mu_d}$ in terms of the parabolic subgroups of $B(W)$.

Since $(X/W)^{\mu_d}$ is a subspace of X/W, we can easily associate a parabolic subgroup of $B(W)$ to a parabolic subgroup of $B(W)^{\mu_d}$. More precisely, let $\eta : [0,1] \to (V/W)^{\mu_d}$ be a capillary path starting at $*$. It defines a parabolic subgroup B_0 of $B(W)^{\mu_d}$. Moreover, η can also be seen as a capillary path in V/W. As such, it can be used to define a parabolic subgroup B_0 of $B(W)$. We will usually denote this parabolic subgroup by $B_0{\uparrow}$, but a priori, the group $B_0{\uparrow}$ could depend on the choice of the capillary path η. The goal of this section is to prove the following theorem, which proves in particular that the notation $\uparrow$ does not depend on a choice.

Theorem 9.3.1 (Parabolic subgroups of regular centralizers). *Let W be an irreducible complex reflection group, with d a regular number. Let $\rho \in B(W)$ be the d-regular braid such that $B(W)^{\mu_d}$ is identified in $B(W)$ with $C_{B(W)}(\rho)$. The map $B_0 \mapsto B_0{\downarrow} := B_0 \cap C_{B(W)}(\rho) = B_0 \cap B(W)^{\mu_d}$ induces a bijection between the sets*

$$\{B_0 \subset B(W) \mid B_0 \text{ parabolic and } (B_0)^\rho = B_0\} \text{ and } \{B_0 \subset B(W)^{\mu_d} \mid B_0 \text{ parabolic}\}.$$

The inverse bijection is given by $B_0 \mapsto B_0{\uparrow}$ (which is in particular a well-defined map which doesn't depend on the choice of a capillary path defining B_0).

This result is immediate in the case where $B(W)^{\mu_d} = B(W)$ (that is, if d divides all the degrees of W). A first corollary of this theorem is that it can be used to prove [GM22, Theorem 1.1 and 1.2] for regular centralizers.

Corollary 9.3.2. *In the situation of Theorem 9.3.1, if the parabolic subgroups of $B(W)$ are stable under intersection, then so are the parabolic subgroups of $B(W)^{\mu_d}$.*

Proof. Let $(B_i)_{i\in I}$ be a family of parabolic subgroups of $B(W)^{\mu_d}$. For each $i \in I$, $B_i{\uparrow}$ is a parabolic subgroup of $B(W)$, which is normalized by the regular braid ρ. By assumption, the intersection $\bigcap_{i\in I} B_i{\uparrow}$ is a parabolic subgroup of $B(W)$, which is normalized by ρ by construction. Thus, $B = (\bigcap_{i\in I} B_i{\uparrow}){\downarrow}$ is a parabolic subgroup of $B(W)$, and we have

$$B = \left(\bigcap_{i\in I} B_i{\uparrow}\right){\downarrow} = \bigcap_{i\in I} B_i{\uparrow} \cap B(W)^{\mu_d} = \bigcap_{i\in I} B_i.$$

whence the result. ☐

There are particular cases of Theorem 9.3.1 which are easy to prove in general.

Lemma 9.3.3. *In the situation of Theorem 9.3.1, the parabolic subgroups $\{1\}{\uparrow}$ and $B(W)^{\mu_d}{\uparrow}$ are well-defined (they do not depend on a choice of capillary path), and we have*

$$\begin{cases} B(W)^{\mu_d}{\uparrow} = B(W), \\ \{1\}{\uparrow} = \{1\}, \end{cases} \quad \text{and} \quad \begin{cases} B(W){\downarrow} = B(W)^{\mu_d}, \\ \{1\}{\downarrow} = \{1\}. \end{cases}$$

Proof. Let $\eta : [0,1] \to (V/W)^{\mu_d}$ be a capillary path such that the induced parabolic subgroup of $B(W)^{\mu_d}$ is itself. Let $g \in W$ be a d-regular element. The homeomorphism $V_g/W_g \simeq (V/W)^{\mu_d}$ sends the capillary path η to a capillary path η_1. Since the parabolic subgroup of $B(W_g)$ induced by η_1 is $B(W_g)$, Proposition 7.3.4 proves that $\eta_1(1) = W_g.0$. Thus, $\eta(1) = W.0 \in V/W$ and $B(W)^{\mu_d}\uparrow = B(W)$ again by Proposition 7.3.4. Likewise, let $\eta : [0,1] \to (V/W)^{\mu_d}$ be a capillary path such that the induced parabolic subgroup of $B(W)^{\mu_d}$ is trivial. This time we have $\eta(1) \in (X/W)^{\mu_d} \subset X/W$, and thus $\{1\}\uparrow = \{1\}$. The second statement is immediate. □

Proposition 9.3.4. *Theorem 9.3.1 holds when W has rank* 2

Proof. If W has rank 2, then $B(W)^{\mu_d}$ is either equal to $B(W)$ (in which case the result is clear), or to a cyclic group, say $B(W)^{\mu_d} := \langle \rho' \rangle$ with $\ell(\rho') > 0$. In this last case $B(W)^{\mu_d}$ is a complex braid group of rank 1, thus its only parabolic subgroups are $\{1\}$ and itself. By Lemma 9.3.3, it only remains to show that $B(W)$ and $\{1\}$ are the only parabolic subgroups of $B(W)$ that are normalized by ρ.

Let B_0 be a parabolic subgroup of $B(W)$ different from $\{1\}$ and $B(W)$. The group B_0 must have rank 1. Since parabolic subgroups of rank 1 are the subgroups generated by distinguished braided reflections, we have $B_0 = \langle \sigma \rangle$, where $\sigma \in B(W)$ is a distinguished braided reflection. Since σ is the only element of B_0 of length 1, we have

$$(B_0)^\rho = B_0 \Leftrightarrow \sigma^\rho = \sigma \Leftrightarrow \sigma \in C_{B(W)}(\rho) = B(W)^{\mu_d}.$$

Since $B(W)^{\mu_d} = \langle \rho' \rangle$, we then have an integer k such that $\rho'^k = \sigma$. However, since $\ell(\sigma) = 1$ and $\ell(\rho') > 0$, we deduce that $k = 1 = \ell(\rho')$ and $\rho' = \sigma$. In particular $\langle \sigma \rangle = B_0 = B(W)^{\mu_d}$.

However, since $B(W)^{\mu_d}$ contains a regular braid, it contains the full twist z_p. Inspecting the classification of irreducible complex reflection groups shows that, when W has rank 2, the full twist z_P can never be obtained as a power of a regular braid, which shows the claim. □

Now, we can show more difficult particular cases of Theorem 9.3.1.

The case of well-generated groups

In this section, we show that Theorem 9.3.1 holds when W is well-generated.

We use the notation of Section 9.1.2 and of Section 9.2. In particular, we fix h to be the highest degree of W, and set $p := d/d \wedge h$ and $q = h/d \wedge h$. We also consider the p-divided groupoid $(\mathcal{G}_p, \mathcal{C}_p, \Delta_p)$ with $\mathcal{G}_p \simeq B_p(W)$, along with the Springer groupoid $(\mathcal{G}, \mathcal{C}, \Delta_p)$, with $\mathcal{G} \simeq B_p^q(W)$.

Up to conjugacy, we can assume that the basepoint $W.*$ lies in $\mathcal{U}^{\mu_d}$ as defined in Section 9.1.2, and we can set u for the path connected component of $W.*$ in $\mathcal{U}^{\mu_d}$. In this case, the regular braid $\rho := \Delta_p^q(u) \in \mathcal{G}_p(u,u) \simeq B(W)$ is such that $\mathcal{G}(u,u)$ is identified with the centralizer $C_{B(W)}(\rho)$.

Lemma 9.3.5. *The map $B_0 \mapsto B_0\!\downarrow$ induces a bijection between the sets*

$$\{B_0 \subset B(W) \mid B_0 \text{ parabolic and } (B_0)^\rho = B_0\} \text{ and } \{B_0 \subset B(W)^{\mu_d} \mid B_0 \text{ parabolic}\}.$$

Proof. At the level of the Garside groupoids $\mathcal{G}$ and $\mathcal{G}_p$, we know that the map $H \mapsto H \cap \mathcal{G}(u,u)$ induces a bijection between $\{H \in \mathcal{P}_{\mathcal{T}}(\mathcal{G}_p(u,u)) \mid \phi_p^q(H) = H\}$ and $\mathcal{P}_{\mathcal{T}'}(\mathcal{G}(u,u))$. By Theorem 9.2.42 (parabolic subgroups up to conjugacy), we know that $\mathcal{T}$-parabolic subgroups of $\mathcal{G}_p(u,u)$ (resp. $\mathcal{T}'$-parabolic subgroups of $\mathcal{G}(u,u)$) are exactly the parabolic subgroups of $B(W) \simeq \mathcal{G}_p(u,u)$ (resp. of $B(W)^{\mu_d} \simeq \mathcal{G}(u,u)$). Thus the map $B_0 \mapsto B_0\!\downarrow$ is a bijection as claimed. □

It remains to show that the map $B_0 \mapsto B_0\uparrow$ is well-defined and is the inverse bijection of $B_0 \mapsto B_0\downarrow$. At the level of $\mathcal{G}_p$ and $\mathcal{G}$, the inverse bijection of $B_0 \mapsto B_0\downarrow$ sends a $\mathcal{T}'$-parabolic subgroup H of $\mathcal{G}(u,u)$ to the $\mathcal{T}$-parabolic closure of z_H in $\mathcal{P}_{\mathcal{T}}(\mathcal{G}_p(u,u))$ (where z_H is induced by the system of conjugacy representatives of 9.2.28 (system of conjugacy representatives). This group is also (by construction) the unique parabolic subgroup of $B(W)$ which is normalized by ρ and whose intersection with $B(W)^{\mu_d}$ is B_0.

Lemma 9.3.6. *Let B_0 be a parabolic subgroup of $B(W)^{\mu_d}$, and let $\widetilde{B_0}$ be the unique parabolic subgroup of $B(W)$ which is normalized by ρ and such that $\widetilde{B_0}\downarrow = B_0$.*
If η is a capillary path in $((X/W)^{\mu_d}, (V/W)^{\mu_d})$ defining B_0, and if B_1 is the parabolic subgroup of $B(W)$ defined by η, then we have $\widetilde{B_0} \subset B_1$.

Proof. By construction, we have $B_0 \subset B_1$. In particular, we have $z_{B_0} \in B_1$ and thus $\widetilde{B_0} \subset B_1$ as $\widetilde{B_0}$ is known to be the parabolic closure of z_{B_0} in $B(W)$. □

Lemma 9.3.7. *Let B_0 be a parabolic subgroup of $B(W)^{\mu_d}$. There is a capillary path η in $((X/W)^{\mu_d}, (V/W)^{\mu_d})$ defining B_0, and such that the parabolic subgroup of $B(W)$ defined by η is the unique parabolic subgroup of $B(W)$ which is normalized by ρ and such that $\widetilde{B_0}\downarrow = B_0$*

Proof. By Theorem 9.2.42, we can consider $f \in \mathcal{G}(u,v)$ such that $(B_0)^f = \mathcal{G}_\beta(v,v)$ is a $\mathcal{T}$-standard parabolic subgroup. We saw in the proof of Theorem 9.2.42 that $\mathcal{G}_\beta(u,u)$ and $(\mathcal{G}_p)_\beta(u,u)$ are parabolic subgroups of $\mathcal{G}(u,u)$ and $\mathcal{G}_p(u,u)$ respectively, defined with the same capillary path γ. If θ is a path in $(X/W)^{\mu_d}$ representing f in $\mathcal{G}$, then $\theta * \gamma$ is a capillary path in $((X/W)^{\mu_d}, (V/W)^{\mu_d})$, which induces the parabolic subgroup $(\mathcal{G}_\beta(u,u))^{f^{-1}} = B_0$. We also know that, seen as a capillary path in $(X/W, V/W)$, $\theta * \gamma$ induces the parabolic subgroup $\widetilde{B_0} := ((\mathcal{G}_p)_\beta(v,v))^{f^{-1}}$ of $\mathcal{G}_p(u,u) = B(W)$, which is the unique parabolic subgroup of $B(W)$ which is normalized by ρ and such that $\widetilde{B_0} \cap \mathcal{G}(u,u) = B_0$. □

Lemma 9.3.8. *Let B_0 be a parabolic subgroup of $B(W)^{\mu_d}$, and let η, η' be two capillary paths in $((X/W)^{\mu_d}, (V/W)^{\mu_d})$ defining B_0 as a parabolic subgroup of $B(W)^{\mu_d}$. Let also B_1, B_2 be the parabolic subgroups of $B(W)$ defined by η, η', respectively. The groups B_1, B_2 share the same rank*

Proof. The two endpoints of η, η' belong to the same stratum of the discriminant stratification of $(V/W)^{\mu_d}$. Let $g \in W$ be a d-regular element, and let $x \in X_g$ be such that $W.x = *$. Consider the respective lifts γ, γ' of η, η' in V starting from x. By construction, γ, γ' are capillary paths in (X_g, V_g).

Seeing B_0 as a parabolic subgroup of $B(W_g)$, we obtain that the image of B_0 in W_g is the parabolic subgroup stabilizing $\gamma(1)$, which is equal to the parabolic subgroup stabilizing $\gamma'(1)$. In particular, $\gamma(1)$ and $\gamma'(1)$ lie on the same hyperplanes of W_g. By Theorem 6.1.32 (Reflection arrangement of regular centralizer), $\gamma(1)$ and $\gamma'(1)$ lie on the same hyperplanes of W.

The images of B_1, B_2 in W are the respective stabilizers of $\gamma(1)$ and $\gamma'(1)$. Since $\gamma(1)$ and $\gamma'(1)$ belong to the same reflecting hyperplanes of W, we obtain that these images are equal, and that B_1 and B_2 are conjugate by an element of $P(W)$. In particular, they share the same rank. □

These lemmas are sufficient to finish the proof:

Proposition 9.3.9. *Theorem 9.3.1 holds when W is well-generated.*

Proof. Let $B_0 \subset B(W)^{\mu_d}$ be a parabolic subgroup. We claim that, for each capillary path η defining B_0, the parabolic subgroup of $B(W)$ induced by η is equal to the unique parabolic subgroup $\widetilde{B_0}$ of $B(W)$ normalized by ρ and such that $\widetilde{B_0} \cap B(W)^{\mu_d} = B_0$. This claim implies that $B_0\uparrow$ is well-defined and does not depend on the choice of η, while showing that it is also the inverse bijection of $B_0 \mapsto B_0\downarrow$.

Let $\widetilde{B_0}$ be the unique parabolic subgroup of $B(W)$ which is normalized by ρ and such that $\widetilde{B_0}\downarrow = B_0$. Let also η be a capillary path defining B_0. By Lemma 9.3.7, we can consider another capillary path γ which defines B_0, and which defines $\widetilde{B_0}$ seen as a capillary path in $(X/W, V/W)$. If B_1 denotes the parabolic subgroup of $B(W)$ defined by η, then $\widetilde{B_0}$ and B_1 share the same rank by Lemma 9.3.8. Since we also have $\widetilde{B_0} \subset B_1$ by Lemma 9.3.6, we have $\widetilde{B_0} = B_1$ by Corollary 7.3.5.

□

The case of groups in the infinite series

In this section, we show that Theorem 9.3.1 holds when W is a member of the infinite series. Since we already know the result when W is well-generated, we can restrict our attention to badly-generated groups in the infinite series.

In this section, we fix three integers $d, e, n \geqslant 2$, and we set $r := de$. We consider the groups $W := G(r, 1, n)$ and $W(e) := G(de, e, n)$. Let also $X := X_n(r)$ be the complement in $\mathbb{C}^n$ of the reflecting hyperplanes of W (or $W(e)$).

We showed in Section 6.2.4 that the covering map $X \twoheadrightarrow X/W$ factor through the covering map $X \twoheadrightarrow X/W(e)$ into a e-fold covering map $X/W(e) \twoheadrightarrow X/W$. Moreover, this covering map is actually induced by the isometric action of the group $W/W(e) \simeq \mathbb{Z}/e\mathbb{Z}$ on $X/W(e)$.

Let $k > 0$ be an integer. By inspecting the (co)degrees of W and of $W(e)$, we obtain that k is regular for $W(e)$ (resp. for W) if and only if k divides dn (resp. rn). We fix k to be a regular integer for $W(e)$ from now on. In particular k is also regular for W.

Since the space $(X/W(e))^{\mu_k}$ is stable under the action of $W/W(e)$, the covering map $X/W(e) \to X/W$ restricts to a covering map starting from $(X/W(e))^{\mu_k}$. Moreover, for $W(e).v \in X/W(e)$, we have

$$\zeta_k.(W(e).v) = W(e).(\zeta_k v) \Leftrightarrow \exists g \in W(e),\ g\ k\text{-regular} \mid v \in V(g, \zeta_k).$$

Since the regular elements of $W(e)$ are exactly the regular elements of W which belongs to $W(e)$, we obtain that $W(e).v \in (X/W(e))^{\mu_k}$ if and only if $W.v \in (X/W)^{\mu_k}$, and we have a commutative square

$$\begin{tikzcd}
(X/W(e))^{\mu_k} \arrow[r, two heads] \arrow[d, hook] & (X/W)^{\mu_k} \arrow[d, hook] \\
X/W(e) \arrow[r, two heads] & X/W
\end{tikzcd}$$

where the horizontal maps are covering maps, and the vertical ones are inclusions.

By Proposition 8.5.5 (Parabolic subgroups of $B_n^*(e)$) and Corollary 8.5.6, we have a bijection

$$\{B_0 \subset B(W(e)) \mid B_0 \text{ parabolic}\} \longrightarrow \{B_0 \subset B(W) \mid B_0 \text{ parabolic}\},$$

sending a parabolic subgroup B_0 induced by a capillary path η in $(X/W(e), V/W(e))$ to the parabolic subgroup of $B(W)$ induced by the image of η under the covering map $X/W(e) \twoheadrightarrow X/W$. The inverse bijection sends a parabolic subgroup B_0 induced by a capillary path η to $B_0 \cap B(W(e))$, which is the parabolic subgroup induced by the unique lift of η under the covering map $X/W(e) \twoheadrightarrow X/W$.

Lemma 9.3.10. *The bijection mentioned above restricts to a bijection*

$$\{B_0 \subset B(W(e)) \mid B_0 \text{ parabolic, } B_0^\rho = B_0\} \longrightarrow \{B_0 \subset B(W) \mid B_0 \text{ parabolic, } B_0^\rho = B_0\}.$$

Proof. If $B_0 \subset B(W)$ is such that $B_0^\rho = B_0$, then $(B_0 \cap B(W(e)))^\rho = B_0 \cap B(W(e))$ since $B(W(e))$ is normalized by ρ. Conversely, if $(B_0 \cap B(W(e)))^\rho = B_0 \cap B(W(e))$, then B_0^ρ is a parabolic subgroup of $B(W)$ whose intersection with $B(W(e))$ is equal to $B_0 \cap B(W(e))$. It is equal to B_0 by uniqueness. ☐

Now, by Proposition 9.3.9, we have another bijection:

Lemma 9.3.11. *There is a bijection*

$$\{B_0 \subset B(W) \mid B_0 \text{ parabolic, } B_0^\rho = B_0\} \longrightarrow \{B_0 \subset B(W)^{\mu_k} \mid B_0 \text{ parabolic}\},$$

sending B_0 to $B_0 \cap B(W)^{\mu_k}$. The inverse bijection sends a parabolic subgroup B_0 induced by a capillary path η in $((X/W)^{\mu_k}, (V/W)^{\mu_k})$ to the parabolic subgroup of $B(W)$ induced by the same capillary path.

Lemma 9.3.12. *There is a bijection*

$$\{B_0 \subset B(W)^{\mu_k} \mid B_0 \text{ parabolic}\} \longrightarrow \{B_0 \subset B(W(e))^{\mu_k} \mid B_0 \text{ parabolic}\},$$

sending B_0 to $B_0 \cap B(W(e))^{\mu_k}$. The inverse bijection sends a parabolic subgroup B_0 induced by a capillary path η in $((X/W(e))^{\mu_k}, (V/W(e))^{\mu_k})$ to the parabolic subgroup of $B(W)$ induced by the image of η under the covering map $X/W(e) \twoheadrightarrow X/W$.

Proof. By Corollary 7.2.9 (pseudoparabolic subgroups and group acting by isometries), we know that the parabolic subgroup of $B(W(e))^{\mu_k}$ are exactly the intersections with $B(W(e))^{\mu_k}$ of the parabolic subgroups of $B(W)^{\mu_k}$. Let $\mathcal{G}$ be the Springer groupoid associated with the regular number k of the well-generated group W. We know that the parabolic subgroups of $B(W)^{\mu_k}$ are exactly the $\mathcal{T}'$-parabolic subgroups of some $\mathcal{G}(u,u) \simeq B(W)^{\mu_k}$, where $\mathcal{T}'$ is the shoal for $\mathcal{G}$ constructed in Section 9.2.4. Seeing $B(W(e))^{\mu_k}$ as a finite index subgroup of $B(W)^{\mu_k}$, we can construct the associated groupoid of cosets, and the associated shoal $\mathcal{T}_e$ of standard parabolic subgroupoids. The $\mathcal{T}_e$-parabolic subgroups of $B(W(e))^{\mu_k}$ are then the intersections with $B(W(e))^{\mu_k}$ of the $\mathcal{T}'$-parabolic subgroups of $B(W)$ by Proposition 5.3.21 (parabolic subgroups of a finite index subgroups) (i.e. they are the same as the topological parabolic subgroups of $B(W(e))^{\mu_k}$). Applying the same reasoning as in the proof of Corollary 8.5.6, we obtain the desired bijection. ☐

Proposition 9.3.13. *Theorem 9.3.1 holds when W belongs to the infinite series.*

Proof. As we said, it is sufficient to restrict our attention to the situation we consider in this section. Let φ_1 (resp. φ_2, φ_3) denote the bijection of Lemma 9.3.10 (resp. 9.3.11, 9.3.12)

Let $B_0 \subset B(W(e))$ be a parabolic subgroup. The group $\varphi_1(B_0)$ is the unique parabolic subgroup B_1 of $B(W)$ such that $B_1 \cap B(W(e)) = B_0$. The group $\varphi_2(B_1)$ is $B_2 := B_1 \cap B(W)^{\mu_k}$. The group $\varphi_3(B_2)$ is $B_2 \cap B(W(e))^{\mu_k}$. The composition of these three bijections then sends B_0 to

$$\begin{aligned} B_2 \cap B(W(e))^{\mu_k} &= B_1 \cap B(W)^{\mu_k} \cap B(W(e))^{\mu_k} \\ &= B_1 \cap B(W(e)) \cap B(W(e))^{\mu_k} \\ &= B_0 \cap B(W(e))^{\mu_k}. \end{aligned}$$

Since $B(W(e))^{\mu_k} \subset B(W(e)) \cap B(W)^{\mu_k}$. In particular, $B_0 \cap B(W(e))^{\mu_k}$ is a parabolic subgroup of $B(W(e))^{\mu_d}$, and the map $B_0 \to B_0 \cap B(W(e))^{\mu_k}$ defined on the set of parabolic subgroups of $B(W(e))$ is a bijection.

Let now $B_0 \subset B(W(e))^{\mu_k}$ be a parabolic subgroup, and let η be a capillary path in the pair $((X/W(e))^{\mu_k}, (V/W(e))^{\mu_k})$ which induces B_0. The group $\varphi_3^{-1}(B_0)$ is the parabolic subgroup B_1 of $B(W)^{\mu_k}$ defined by the image $\overline{\eta}$ of η under the covering map $X/W(e) \twoheadrightarrow X/W$. The group $\varphi_2^{-1}(B_1)$ is the parabolic subgroup B_2 of $B(W)$ defined by $\overline{\eta}$. The group $\varphi_1^{-1}(B_2)$ is the parabolic subgroup of $B(W(e))$ defined using the unique lift η of $\overline{\eta}$ under the covering map $X/W(e) \twoheadrightarrow X/W$, that is, $B_0\!\uparrow$, which then does not depend on the choice of η, and gives an inverse to the bijection $B_0 \mapsto B_0\!\downarrow$.

□

The case of regular centralizers in well-generated groups

Theorem 9.3.1 is now proven for well-generated groups, for groups of rank 2, and for members of the infinite series. The only irreducible complex reflection group which does not fall into any of these categories if G_{31}, which is known to be a regular centralizer in the well-generated group G_{37}.

In this section, we show that Theorem 9.3.1 holds when we consider a group which is itself the centralizer of a regular braid inside of a well-generated group. Which is sufficient to conclude that Theorem 9.3.1 holds for every irreducible complex reflection group.

In this section, we fix a well-generated irreducible complex reflection group W, along with a r-regular element $g \in W$. We also fix d a regular integer for W_g. There is a r-regular braid $\delta \in B(W)$ such that $B(W_g) \simeq B(W)^{\mu_r}$ is identified with $C_{B(W)}(\delta)$ in $B(W)$. There is also a d-regular braid $\rho \in B(W_g)$ such that $B(W_g)^{\mu_d}$ is identified with $C_{B(W_g)}(\rho)$ in $B(W_g)$.

We have
$$(X/W)^{\mu_{r\vee d}} = ((X/W)^{\mu_r})^{\mu_d} \simeq (X_g/W_g)^{\mu_d},$$
and thus we have the following commutative square of topological spaces:

$$\begin{tikzcd} (X/W)^{\mu_{r\vee d}} \arrow[r, hook] \arrow[d, equal] & X/W \\ (X_g/W_g)^{\mu_d} \arrow[r, hook] & X_g/W_g \arrow[u, hook] \end{tikzcd}$$

Let $d' := d/d\wedge r$ and let $r' = r/d \wedge r$. By Lemma 9.1.22, there is a $(d \vee r)$-regular braid $q(\rho)$ in $B(W)$ such that $q(\rho)^{d'} = \delta$ and $q(\rho)^{r'} = \rho$. We have

$$C_{B(W)}(q(\rho)) \simeq B(W)^{\mu_{r\vee d}} \simeq B(W_g)^{\mu_d} \simeq C_{B(W_g)}(\rho).$$

By Proposition 9.3.9, we have a bijection

$$\{B_0 \subset B(W_g) \mid B_0 \text{ parabolic}\} \longrightarrow \{B_0 \subset B(W) \mid B_0 \text{ parabolic, } B_0^\delta = B_0\},$$

obtained by sending a parabolic subgroup B_0 induced by a capillary path η in $((X/W)^{\mu_r}, (V/W)^{\mu_r})$ to the parabolic subgroup of $B(W)$ induced by the same capillary path. The inverse bijection sends B_0 to $B_0 \cap B(W_g)$.

Lemma 9.3.14. *The bijection mentioned above restricts to a bijection*

$$\{B_0 \subset B(W_g) \mid B_0 \textit{ parabolic, } B_0^\rho = B_0\} \longrightarrow \{B_0 \subset B(W) \mid B_0 \textit{ parabolic, } B_0^\delta = B_0 = B_0^\rho\}.$$

Proof. If $B_0 \subset B(W)$ is such that $B_0^\rho = B_0 = B_0^\delta$, then $(B_0 \cap B(W_g))^\rho = B_0 \cap B(W_g)$ since $B(W_g)$ is normalized by $\rho \in B(W_g)$. Conversely, if $(B_0 \cap B(W_g))^\rho = B_0 \cap B(W_g)$, then B_0^ρ is a parabolic subgroup of $B(W)$, normalized by δ, whose intersection with $B(W_g)$ is equal to $B_0 \cap B(W_g)$. It is equal to B_0 by uniqueness. Note also that, since $q(\rho) = \rho^v \delta^u$, and since δ and ρ are powers of $q(\rho)$, a subgroup B_0 of $B(W)$ is normalized by $q(\rho)$ if and only if it is normalized by both ρ and δ. □

Now, again by Proposition 9.3.9, we have another bijection:

Lemma 9.3.15. *There is a bijection*

$$\{B_0 \subset B(W) \mid B_0 \textit{ parabolic, } B_0^{q(\rho)} = B_0\} \longrightarrow \{B_0 \subset B(W)^{\mu_{r\vee d}} \mid B_0 \textit{ parabolic}\},$$

obtained by sending a parabolic subgroup B_0 to $B_0 \cap B(W)^{\mu_{r\vee d}}$. The inverse bijection sends a parabolic subgroup B_0 induced by a capillary path η in $((X/W)^{\mu_{r\vee d}}, (V/W)^{\mu_{r\vee d}})$ to the parabolic subgroup of $B(W)$ induced by the same capillary path.

Corollary 9.3.16. *Let W be an irreducible well-generated complex reflection group, and let $g \in W$ be a regular element. Then Theorem 9.3.1 holds for W_g.*

Proof. We keep the above notation. Let φ_1 (resp. φ_2) be the bijection of Lemma 9.3.14 (resp. 9.3.15). Let $B_0 \subset B(W_g)$ be a parabolic subgroup normalized by ρ. The group $\varphi_1(B_0)$ is the unique parabolic subgroup $B_1 \subset B(W)$ which is normalized by $q(\rho)$ and such that $B_1 \cap B(W_g) = B_0$. We have $B_2 := \varphi_2(B_1) = B_1 \cap B(W)^{\mu_{r\vee d}}$. The composition of these two bijections then sends B_0 to

$$B_1 \cap B(W)^{\mu_{r\vee d}} = B_1 \cap B(W)^{\mu_r} \cap B(W)^{\mu_{r\vee d}} = B_0 \cap B(W)^{\mu_{r\vee d}}.$$

In particular, $B_0 \cap B(W)^{\mu_{r\vee d}}$ is a parabolic subgroup of $B(W)^{\mu_{r\vee d}}$, and the map $B_0 \mapsto B_0 \cap B(W)^{\mu_{r\vee d}}$ defined on the set of parabolic subgroups of $B(W_g)$ is a bijection.

Let now $B_0 \subset B(W)^{\mu_{r\vee d}}$ be a parabolic subgroup, and let η be a capillary path for the pair $((X/W)^{\mu_{r\vee d}}, (V/W)^{\mu_{r\vee d}})$ which induces B_0. The group $\varphi_2^{-1}(B_0)$ is the parabolic subgroup B_1 of $B(W)$ induced by η. We have $\varphi_1^{-1}(B_1) = B_1 \cap B(W_g)$. If $B_0{\uparrow}$ denotes the parabolic subgroup of $B(W_g)$ induced by η, then the image of $B_0{\uparrow}$ under φ_1 is equal to B_1 by definition. Thus we have $B_0{\uparrow} = B_1 \cap B(W_g)$ does not depend on the choice of η, and gives an inverse to the bijection $B_0 \mapsto B_0{\downarrow}$. □

Chapter 10

The particular case of the complex braid group $B(G_{31})$

In this last chapter, we apply the result of Chapter 9 to the complex braid group $B(G_{31})$. We use the Reidemeister-Schreier method for groupoids to compute particular presentations of this group using its associated Springer groupoid, as in [Gar23b]. Then we describe its parabolic subgroups up to conjugacy and we show that [GM22, Theorem 1.3] holds for this group, following my fifth paper [Gar24b]. This was the last result needed to extend the main theorems of [GM22] to every complex braid group.

Contents

The group G_{31} is an irreducible complex reflection group of rank 4 which is badly-generated. It is the only exceptional irreducible complex reflection group which is both badly-generated and has rank > 2. This makes the complex braid group $B(G_{31})$ hard to study directly.

However, we mentioned that G_{31} appears as a regular centralizer in the well-generated complex reflection group G_{37}. Using this, we can apply the results of Chapter 9 in this particular case, and obtain general results on $B(G_{31})$.

Throughout this Chapter, we consider the complex reflection group G_{37}, which is also the complexified real reflection group of type E_8. Its highest degree is $h := 30$. The integer $d = 4$ is regular for G_{37}, and the centralizer of a 4-regular element in G_{37} is a group of type G_{31} (see Example 6.1.31). We can set $p = d/d \wedge h = 2$ and $q := h/d \wedge h = 15$.

Let $I(G_{37})$ denote the set of simple elements of the dual braid monoid attached to G_{37}, and let $(\mathcal{B}_{31}, \mathcal{C}_{31}, \Delta_2)$ be the Springer groupoid attached to G_{37} and $d = 4$. We know by Theorem 9.1.14 that $\mathcal{B}_{31}$ is equivalent to $B(G_{31})$ as a groupoid.

Moreover, the groupoid $(\mathcal{B}_{31}, \mathcal{C}_{31}, \Delta_2)$ is endowed with a shoal $\mathcal{T}$ of standard parabolic subgroupoids such that, for $u \in \mathrm{Ob}(\mathcal{B}_{31})$, the isomorphism $\mathcal{B}_{31}(u,u) \simeq B(G_{31})$ identifies the $\mathcal{T}$-parabolic subgroups of $\mathcal{B}_{31}(u,u)$ with the parabolic subgroups of $B(G_{31})$ (Theorem 9.2.42 (parabolic subgroups up to conjugacy)). In particular, for any object u of $\mathcal{B}_{31}$, there is a set of atomic loops in $\mathcal{C}_{31}(u,u)$, and any braided reflection in $B(G_{31})$ is conjugate to some atomic loop in $\mathcal{B}_{31}$.

The positive integers $\lambda := 8, \mu := 1$ are such that $2\lambda - 15\mu = 1$. As $c^{15} = -\mathrm{Id}$ is central in G_{37}, we have

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}_{31}) &:= D_2^{15}(c) = \{u \in I(G_{37}) \mid u u^{c^8} = c \text{ and } \ell_T(u) = 4\}, \\ \mathcal{S}_2^{15} &:= D_4^{30}(c) = \{(a,b) \in (I(G_{37}))^2 \mid ab \in \mathrm{Ob}(\mathcal{C})\}, \\ R_2^{15} &:= D_6^{45}(c) = \{(x,y,z) \in (I(G_{37}))^3 \mid xyz \in \mathrm{Ob}(\mathcal{C})\}. \end{aligned}$$

By [Bes15, Theorem 13.3], we have $|O| = 88$ and $|\mathcal{S}| = 2691$ and $|\mathrm{Rel}| = 16359$ in this case.

By Lemma 9.2.3, the atoms of the Springer category $\mathcal{C}_{31}$ are exactly the elements of length 1. Since 4 is the gcd of degrees of G_{37} which it divides, we can apply Proposition 9.2.46 and Corollary 9.2.51. We get

Theorem 10.0.1. *The centers of $\mathcal{C}_{31}$ and $\mathcal{B}_{31}$ are cyclic and generated by Δ_2^{15}. If U is a finite index subgroup of $B(G_{31})$, then $Z(U) \subset Z(B(G_{31}))$. In particular, the center of the pure braid group $P(G_{31})$ is cyclic and generated by the full-twist.*

Proof. The only nontrivial part is that the full-twist is a generator of $Z(P(G_{31}))$. Let $B(G_{31}) \simeq \mathcal{B}_{31}(u,u)$, the center of $P(G_{31})$ is cyclic and generated by the smallest power of $\Delta_2^{15}(u)$ which lies in $P(G_{31})$. Since the collapse functor $\mathcal{C}_{31} \to M(c)$ sends $\Delta_2^{15}(u_0)$ to some element g with $g^2 = \Delta_2^{15}$, the smallest power of Δ_2^{15} lying in $P(G_{31})$ is the full-twist Δ_2^{60}. $\square$

10.1 Presentations of the complex braid group $B(G_{31})$

The goal of this section is to prove several presentations of $B(G_{31})$ starting from a presentation of the Springer groupoid $\mathcal{B}_{31}$. These presentations are all positive, homogeneous, and the generators are braided reflections.

Let u be an object of $\mathcal{B}_{31}$, we have an isomorphism $\mathcal{B}_{31}(u,u) \simeq B(G_{31})$, which sends atomic loops to braided reflections.

On the one hand, we build a conjectural presentation of $\mathcal{B}_{31}(u,u)$ with atomic loops as generators. On the other hand, the Reidemeister-Schreier method for groupoids (see Section 1.5.1) gives a presentation of $\mathcal{B}_{31}(u,u)$ which we know holds. We then prove that these two presentations are equivalent.

10.1.1 The method

In this section, we fix $u \in \mathrm{Ob}(\mathcal{B}_{31})$. We start by considering the submonoid L_u^+ of $\mathcal{C}_{31}(u,u)$ generated by atomic loops of u. Note that we have $L_u^+ \neq \mathcal{C}_{31}(u,u)$ in general.

Our first goal is to construct a (conjectural) group presentation using L_u^+. We consider the following algorithm:

Algorithm 10.1.1 Compute shortest right-multiple of atomic loops in L_u^+

Input: Two atomic loops $\lambda(s)$ and $\lambda(t)$ of u.
Output: If $\lambda(s), \lambda(t)$ admit a common right-multiple in L_u^+, then the output is a pair of words $\theta(\lambda(s), \lambda(t)), \theta(\lambda(t), \lambda(s))$ in L_u^+ such that $\lambda(s)\theta(\lambda(s), \lambda(t)) = \lambda(t)\theta(\lambda(t), \lambda(s))$. No output otherwise.
put $i := 1$
compute the set S_i of words of length i in L_u^+
while $\lambda(s)m_1 \neq \lambda(s)m_2$ in $\mathcal{C}_{31}$ for all $(m_1, m_2) \in S_i \times S_i$ **do**
  put $i := i + 1$
  compute the set S_i of words of length i in L_u^+
end while
put $S := \{(m_1, m_2) \in S_i \times S_i \mid \lambda(s)m_1 = \lambda(t)m_2 \in \mathcal{C}_{31}\}$
return $(m_1, m_2) \in S$ such that m_1 is the lowest possible in the lexicographic order, and m_2 is the lowest possible in the lexicographic order among the words m such that $(m_1, m) \in S$.

Algorithm 10.1.1, running on two atomic loops of u, terminates if and only if they admit a common right-multiple in L_u^+. The fact that it terminates on every pair of atomic loops of u (which we checked by computer using the data of Section 10.1.2) proves that all pairs of atomic loops of u admit a common right-multiple. We now consider the following presentation:

- The set of generators is a set $X_u := \{\lambda'(s)\}$ in bijection with atomic loops of u in $L_u^+ \subset \mathcal{C}_{31}(u, u)$.
- The set R_u is given by relations of the form

$$\lambda'(s)\theta(\lambda'(s), \lambda'(t)) = \lambda'(t)\theta(\lambda'(t), \lambda'(s)),$$

  where $\theta(\lambda'(s), \lambda'(t))$ is the same word as $\theta(\lambda(s), \lambda(t))$ but with letters in X_u.

We define $H_u := \langle X_u \mid R_u\rangle$ and $H_u^+ := \langle X_u \mid R_u\rangle^+$. By [DDGKM, Lemma II.4.3], the presentation of H_u^+ is *right-complemented* in the sense of [DDGKM, Definition II.4.1]. That is for $a \neq b \in X_u$, there is exactly one relation of the form $a \ldots = b \ldots$, namely $a\theta(a, b) = b\theta(b, a)$.

The goal of this section is to show the following theorem:

Theorem 10.1.1. *Let $u \in \mathrm{Ob}(\mathcal{C}_{31})$. The natural map from X_u to the set of atomic loops of u induces a group isomorphism $H_u \simeq \mathcal{B}_{31}(u, u)$. In particular, $\langle X_u \mid R_u\rangle$ gives a positive homogeneous presentations of $B(G_{31})$ with braided reflections as generators.*

We first notice that, by definition of R_u and X_u, the natural map from X_u to the set of atomic loops of u induces a group morphism $f_u : H_u \to \mathcal{B}_{31}(u, u)$. We only have to show that the said morphism is an isomorphism. We show this by case by case analysis on the objects of $\mathcal{C}_{31}$.

Lemma 10.1.2. *If f_u is an isomorphism, then $f_{\phi_2(u)}$ is also an isomorphism. In particular we only need to show that Theorem 10.1.1 holds for a system of representative of ϕ_2-orbits in $\mathcal{B}_{31}$.*

Proof. The automorphism ϕ_2 gives an isomorphism between $\mathcal{C}_{31}(u, u)$ and $\mathcal{C}_{31}(\phi(u), \phi(u))$. Because of Lemma 9.2.7, ϕ_2 induces a bijection between the sets of atomic loops of u and of $\phi(u)$. We obtain that ϕ_2 induces bijections between X_u and $X_{\phi(u)}$, and between R_u and $R_{\phi(u)}$, respectively. Thus $H_u \simeq H_{\phi(u)}$ which shows the claim. $\square$

Remark 10.1.3. The atoms of the monoid H_u^+ are in bijection with atomic loops of u. In particular H_u^+ is not isomorphic to the monoid $\mathcal{C}_{31}(u,u)$ in general. As a matter of fact, we will see that the monoid H_u^+ is never cancellative, and thus cannot be isomorphic to either L_u^+ or $\mathcal{C}_{31}(u,u)$. The isomorphism we consider only occurs at the level of groups.

Note that the monoid H_u^+ is homogeneous by definition. In particular we have a solution to the word problem in H_u^+, given by Algorithm 10.1.2.

Algorithm 10.1.2 Check equality between two words in H_u^+

Input: Two words m_1, m_2 in the atoms of H_u^+
Output: **true** if m_1 and m_2 represent the same element of H_u^+, **false** otherwise.
put $S := \varnothing$.
put $S_1 := \{m_1\}$.
while $S \neq S_1$ **do**
  put $S := S_1$.
  put S_1 the set of words obtained from elements of S_1 by applying one relation of R_u.
end while
if $m_2 \in S$ **then**
  return true
else
  return false
end if

Since H_u^+ is homogeneous, two words representing the same element have the same length, thus there is a finite number of words that represent the same element of H_u^+ and Algorithm 10.1.2 always terminates.

In order to prove Theorem 10.1.1 for the object u, we first compute a presentation of $\mathcal{B}_{31}(u,u)$ by the Reidemeister-Schreier method for groupoids (see Section 1.5.1). We start from a presentation of the groupoid $\mathcal{B}_{31}$, for instance that of Theorem 9.2.18 (presentation of Springer Groupoids).

As we want atomic loops to appear as generators, we need to choose the Schreier transversal accordingly. We use the following lemmas concerning atomic loops in the category $\mathcal{C}_{31}$.

Lemma 10.1.4. *Let s be an atom of $\mathcal{C}_{31}$, the atomic loop $\lambda(s)$ has length two in $\mathcal{C}_{31}$.*

Proof. Let $s := (a,b)$. We denote by u the source of s, and by v its target. The morphism $s^\#$ is given by $(a,b)^\# = (a^{c^8b^{-1}}, b)$. We know that

$$a^{c^7}b^{c^7}ab = c \Rightarrow c^7a^{c^7}b^{c^7}a = abc^7a = c^8b^{-1},$$

and so $a^{c^8b^{-1}} = a^{abc^7a} = a^{(ba^{c^8})c^7} = a^{vc^7}$. The morphism $s^{\#\#}$ is then given by $((a^{vc^7})^{avc^7}, b)$. Because of Lemma 6.1.33, a and a^{vc^7} commute, so $s^{\#\#} = (a^{(vc^7)^2}, b)$. Since $(vc^7)^2 = c^{15}$ is central, we have $s^{\#\#} = s$ as claimed. □

Lemma 10.1.5. *There is a Schreier transversal T rooted in u and containing all atoms with source u. In particular the atomic loops of u appear as generators of the presentation of $\mathcal{B}_{31}(u,u)$ induced by T.*

Proof. First, note that if a Schreier transversal T contains an atom s with source u, then Lemma 10.1.4 gives that $\gamma(s^{\#}) = ss^{\#} = \lambda(s)$ with the notation of Section 1.5.

Now, thanks to Corollary 9.2.5 (no pairs of parallel simples), all atoms with source u have different targets. We can thus consider a Schreier transversal T rooted in u and containing all atoms with source u. $\square$

Let T be a Schreier transversal rooted in u and containing all atoms with source u. Let $\langle S^* \mid R^* \rangle$ denote the presentation of $\mathcal{B}_{31}(u,u)$ obtained by the Reidemeister-Schreier method applied to T and to the presentation of Theorem 9.2.18. Of course, the presentation $\langle S^* \mid R^* \rangle$ is quite redundant. We first want to show that every element of S^* can be expressed as a word in the atomic loops. For this we repeatedly apply Tietze transformations, as in Algorithm 10.1.3.

Algorithm 10.1.3 Reduction of generators

Input: A group presentation $\langle S \mid R \rangle$ and a subset S' of S
Output: A group presentation $\langle S' \mid R' \rangle$ equivalent to the first by Tietze transformations, or no output.
while $S' \neq S$ **do**
  choose $r \in R$ a relator with only one letter a not belonging to $S' \cup S'^{-1}$
  replace in R every occurrence of a by its expression in S' using the relator r.
  remove the relator r from R
  remove the letter a from S.
end while
return the presentation $\langle S' \mid R \rangle$

The fact that this algorithm terminates for each object of $\mathcal{B}_{31}$, which is again checked by computer, proves the following result:

Proposition 10.1.6. *The atomic loops of u generate the group $\mathcal{B}_{31}(u,u)$. In particular the natural morphism $H_u \to \mathcal{B}_{31}(u,u)$ is surjective.*

By applying Algorithm 10.1.3 to the presentation $\langle S^* \mid R^* \rangle$, we obtain a presentation $\langle X_u \mid R'_u \rangle$ of the group $\mathcal{B}_{31}(u,u)$. In order to prove Theorem 10.1.1 for the object u, it is sufficient to prove that every relator of R'_u is in fact trivial in H_u. This will prove that the morphism $H_u \to \mathcal{B}_{31}(u,u)$ is injective.

Because the defining presentation of H_u^+ is right-complemented, we can consider a *right-reversing* algorithm in the sense of [DDGKM, Definition 4.21]. The main idea is that the relation $a\theta(a,b) = b\theta(b,a)$ implies $a^{-1}b = \theta(a,b)\theta(b,a)^{-1}$. We can use this type of relations in order to simplify words in $X_u \cup X_u^{-1}$.

Algorithm 10.1.4 doesn't always terminate: it may loop indefinitely. If it terminates, its output is a fraction, and we should check that it is trivial. Of course this may be quite long as algorithm 10.1.2 is far from optimal. This process can be sped up by "partially simplifying" at each step. The solution to the word problem given by Algorithm 10.1.2 allows for the computation of longest common divisors of elements of H_u^+. A general word in $X_u \cup X_u^{-1}$ can be written as a product of (short) fractions. We can simplify these fractions at each step of Algorithm 10.1.4.

We checked by computer that Algorithm 10.1.4 terminates on every relator in R'_u, that is every relator can be expressed as a right-fraction of elements of H_u^+. Finally, we use the following algorithm to prove that every relator, written as a fraction, is trivial in H_u.

Algorithm 10.1.4 Right-reversing in H_u^+ ([DDGKM, Algorithm II.4.33])

Input: A word w in $X_u \cup X_u^{-1}$
Output: A fraction fg^{-1} with $f, g \in H_u^+$ representing w in H_u, or no output.
while there is some subword of the form $a^{-1}b$ in w **do**
  put j the position in w of the first subword of the form $a^{-1}b$ in w
  if $a = b$ **then**
    remove the subword $a^{-1}b$ from w
  else
    replace $a^{-1}b$ with $\theta(a,b)\theta(b,a)^{-1}$ in w at position j
  end if
end while
return w.

Algorithm 10.1.5 Partial solution to the word problem in H_u

Input: A fraction fg^{-1} with $f, g \in H_u^+$.
Output: `true` If there is some $n \in H_u^+$ such that $fn = gn$. No output otherwise.
put $i := 1$
Compute the set S_i of words of length i in X_u
while $fn \neq gn$ for all $n \in S_i$ **do**
  put $i := i + 1$
  Compute the set S_i of words of length i in X_u
end while
return true

This algorithm is useful since the monoid H_u^+ may not be cancellative: we can have $fn = gn$ (and thus $fg^{-1} = 1$ in H_u) without having $f = g$ in H_u^+.

The fact that Algorithm 10.1.5 returns `true` for all relators of the presentation of $\mathcal{B}_{31}(u,u)$ finally proves that Theorem 10.1.1 holds for u.

10.1.2 Computational data

We consider the following elements in $\mathbb{C}^8$:

$$\begin{cases} \alpha_1 = \frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1), \\ \alpha_2 = (1,1,0,0,0,0,0,0), \\ \alpha_3 = (-1,1,0,0,0,0,0,0), \\ \alpha_4 = (0,-1,1,0,0,0,0,0), \\ \alpha_5 = (0,0,-1,1,0,0,0,0), \\ \alpha_6 = (0,0,0,-1,1,0,0,0), \\ \alpha_7 = (0,0,0,0,-1,1,0,0), \\ \alpha_8 = (0,0,0,0,0,-1,1,0), \\ \alpha_9 = (1,0,1,0,0,0,0,0), \end{cases} \qquad \begin{cases} \alpha_{10} = (0,0,-1,0,1,0,0,0), \\ \alpha_{11} = (1,0,0,0,0,1,0,0), \\ \alpha_{12} = \frac{1}{2}(-1,-1,-1,-1,-1,1,-1,1), \\ \alpha_{13} = (0,1,0,0,0,1,0,0), \\ \alpha_{14} = \frac{1}{2}(-1,-1,-1,-1,-1,-1,1,1), \\ \alpha_{15} = (0,1,0,0,0,0,1,0), \\ \alpha_{16} = \frac{1}{2}(1,-1,1,-1,-1,-1,1,1), \\ \alpha_{17} = \frac{1}{2}(1,-1,-1,1,-1,-1,1,1), \\ \alpha_{18} = \frac{1}{2}(-1,1,-1,-1,1,-1,1,1). \end{cases}$$

And for each $i \in [\![1, 18]\!]$, we consider the reflection s_i of $\mathbb{C}^8$ given by $s_i(x) = x - 2\frac{\langle x, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i$ (where $\langle ., .\rangle$ is the usual hermitian scalar product on $\mathbb{C}^8$). The set $s_1, \ldots, s_8$ generates a subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is isomorphic to the complex reflection group G_{37} and which contains all the s_i for $i \in [\![1, 18]\!]$. We will denote this group by G_{37} directly. A Coxeter element of G_{37} is given by $c = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$.

A system of representatives of ϕ_2-orbits of objects of the category $\mathcal{C}_{31} = M(G_{37})_2^{15} = M(G_{37}, c)_2^{15}$ is given by the following elements of G_{37}:

$$\begin{cases} u_1 = s_{10}s_{12}s_{13}s_{18} \\ u_2 = s_9 s_{10} s_{12} s_{16} \\ u_3 = s_3 s_{11} s_{12} s_{16} \\ u_4 = s_7 s_8 s_{14} s_{15} \end{cases} \quad \begin{cases} u_5 = s_3 s_7 s_{11} s_{12} \\ u_6 = s_3 s_4 s_7 s_{11} \\ u_7 = s_3 s_4 s_7 s_{17} \\ u_8 = s_3 s_7 s_{12} s_{17} \end{cases}.$$

As we want explicit relations between the different presentations we obtain for each representative, we give explicit isomorphisms between the different groups $\mathcal{B}_{31}(u_i, u_i)$. For this we use the following graph in $\mathcal{B}_{31}$:

$$\begin{tikzcd} & u_2 & u_8 & \\ u_3 & u_1 \arrow[l] \arrow[u] \arrow[r] \arrow[d] & u_5 \arrow[u] \arrow[r] \arrow[d] & u_7 \\ & u_4 & u_6 & \end{tikzcd}$$

Where each arrow is a simple morphism in $\mathcal{C}_{31}$ (because of Corollary 9.2.5 (no pairs of parallel simples), a simple morphism is uniquely determined by its source and target). For each $i, j \in [\![1, 8]\!]$, this graph induces a well defined isomorphism $\varphi_{i,j} : \mathcal{B}_{31}(u_i, u_i) \to \mathcal{B}_{31}(u_j, u_j)$ which preserves braided reflections.

For $i, j \in [\![1, 8]\!]$, we have by definition $\varphi_{i,j} = \varphi_{i,1}\varphi_{1,j}$, so we only need to describe morphisms of the form $\varphi_{i,1}$ and $\varphi_{1,i}$ for $i \in [\![1, 8]\!]$.

In the case of the orbit of u_1, we give expressions of the atomic loops in the generators $\sigma_1, \ldots, \sigma_8$ of the Artin group associated to G_{37}. Replacing $\sigma_1, \ldots, \sigma_8$ with $s_1, \ldots, s_8$ gives a set of elements in G_{37} which generate a copy of G_{31}. We also give a family of vectors in $\mathbb{C}^4$ such that the 2-reflections associated to the orthogonal hyperplanes of these vectors (in the usual hermitian product) generate a group isomorphic to G_{31}.

Furthermore, we know that the full-twist in $\mathcal{B}_{31}(u, u)$ is given by $\Delta_2^{60}(u)$ by Lemma 9.1.17. By Proposition 9.1.26, every root of the full-twist in $B(G_{31})$ is conjugate to a power of either a 20-th root or a 24-th root of the full-twist. Furthermore, the full-twist admits 20-th roots and 24-th roots. For each presentation we obtain, we give an explicit 20-th root (resp. 24-th root) of the full-twist as a word in the generators. If $\rho \in \mathcal{B}_{31}(u, u)$ is a 24-th root of the full-twist $\Delta_2^{60}(u)$, then ρ^6 is a 4-th root of $\Delta_2^{60}(u)$. By Theorem 3.4.4 (conjugacy of periodic elements) and Theorem 10.0.1, we get that $\rho^6 = \Delta_2^{15}(u)$ is a generator of $Z(\mathcal{B}_{31}(u, u))$.

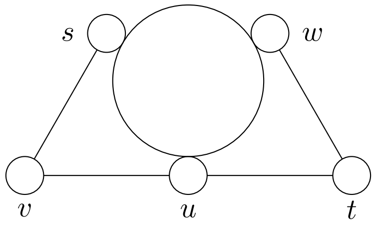
10.1.3 Presentation associated to representatives of the ϕ_2-orbits

Orbit of u_1

For the first orbit, we recover the presentation of $B(G_{31})$ conjectured in [BMR98, Table 3] and [BM04, Conjecture 2.4]. The object u_1 has 5 atomic loops s, t, u, v, w. The relations we obtain are as follows

$$\begin{cases} ts = st,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw. \end{cases}$$

This presentation is usually represented by the following diagram (corresponding to the Broué-Malle-Rouquier diagram for the reflection group G_{31}):



In the Artin group associated to G_{37}, the atomic loops s, t, u, v, w can be expressed as

$$\begin{aligned} s &:= (\sigma_1\sigma_4)^{\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8}, \\ t &:= (\sigma_4\sigma_2)^{\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7}, \\ u &:= (\sigma_4\sigma_2)^{\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_5\sigma_7\sigma_6\sigma_8\sigma_7\sigma_6}, \\ v &:= (\sigma_1\sigma_3)^{\sigma_3\sigma_4\sigma_5\sigma_4\sigma_6\sigma_7}, \\ w &:= (\sigma_2\sigma_3)^{\sigma_1\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_5\sigma_6\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_7\sigma_6}. \end{aligned}$$

In $\mathbb{C}^4$, the 2-reflections associated with the following roots (in the usual hermitian product) generate a subgroup of $\mathrm{GL}_4(\mathbb{C})$ which is isomorphic to G_{31}.

$$\alpha_s := \frac{1}{2}\begin{pmatrix} 2 \\ 1+i \\ -1-i \\ 0 \end{pmatrix},\ \alpha_t := \begin{pmatrix} 1 \\ -1 \\ i \\ -1 \end{pmatrix},\ \alpha_u := \begin{pmatrix} 1 \\ 0 \\ -1-i \\ -i \end{pmatrix},\ \alpha_v := \frac{1}{2}\begin{pmatrix} 2 \\ 1-i \\ 1-i \\ 0 \end{pmatrix},\ \alpha_w := \begin{pmatrix} 1 \\ i \\ -1 \\ -1 \end{pmatrix}.$$

The monoid $H^+_{u_1}$ given by the above presentation is not cancellative: we have $tuwtuw \neq uwtuwt$ and $stuwtuw = suwtuwt$ in $H^+_{u_1}$. Thus $H^+_{u_1}$ cannot be a Garside monoid.

The submonoid $L^+_{u_1}$ of $\mathcal{C}_{31}(u_1, u_1)$ generated by s, t, u, v, w is cancellative, but it doesn't admit right-lcms: we know that $utu = tut$ is the shortest common multiple of t and u. If right-lcms exists in $L^+_{u_1}$, then tut must be the right-lcm of t and u. We would then have

$$tut \preccurlyeq tuwtuw \Rightarrow t \preccurlyeq wtuw$$

in $L^+_{u_1}$ by cancellativity. We can check that $t \preccurlyeq wtuw$ doesn't hold in $L^+_{u_1}$: the element $t^{-1}wtuw$ does lie in $\mathcal{C}_{31}(u_1, u_1)$, but not in the submonoid $L^+_{u_1}$.

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_1)$	$stuvws$	$stuvw$

In particular, we get that $(stuvw)^6 = \Delta_2^{15}(u_1)$ generates $Z(\mathcal{B}_{31}(u_1, u_1))$.

Orbit of u_2

The object u_2 admits 7 atomic loops a, b, c, s, t, v, w, with relations as follows:

$$\begin{cases} sv = vb = bs,\ av = vc = ca, \\ wv = vw,\ st = ts,\ tv = vt,\ tb = bt, \\ was = swa = asw,\ wcb = bwc = cbw, \\ wtw = twt,\ ata = tat,\ aba = bab,\ tct = ctc, \\ swav = cabw. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_2}^+$ given by this presentation is not cancellative: we have $bwab \neq abwa$ and $cbwab = cabwa$ in $H_{u_2}^+$. Thus $H_{u_2}^+$ cannot be a Garside monoid.

The submonoid $L_{u_2}^+$ of $\mathcal{C}_{31}(u_2, u_2)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $aba = bab$ is the shortest common multiple of a and b. If right-lcms exists in $L_{u_2}^+$, then aba must be the right-lcm of a and b. We would then have

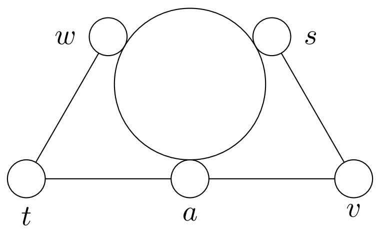
$$aba \preccurlyeq abwa \Rightarrow a \preccurlyeq wa,$$

which does not hold in $L_{u_2}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_2, u_2)$).

The relations defining $H_{u_2}^+$ give in particular $b = v^{-1}sv = s^v$ and $c = a^v$ in H_u. By deleting these generators, we get that $\mathcal{B}_{31}(u_2, u_2)$ is generated by a, s, t, v, w, with relations as follows:

$$\begin{cases} wv = vw,\ st = ts,\ vt = tv, \\ swa = was = asw, \\ twt = wtw,\ ata = tat,\ vav = ava,\ svs = vsv. \end{cases}$$

We recover the presentation of Section 10.1.3, summarized in the diagram



Again, we know that neither the monoid given by this presentation, neither the submonoid of $L_{u_2}^+$ generated by a, s, t, v, w are Garside monoids.

The morphisms $\varphi_{2,1}$ and $\varphi_{2,1}$ are given by

$$\varphi_{2,1} : \begin{cases} s \mapsto s, & a \mapsto u^w, \\ t \mapsto t, & b \mapsto s^v, \\ v \mapsto v, & c \mapsto u^{wv}, \\ w \mapsto w, \end{cases} \quad \text{and} \quad \varphi_{1,2} : \begin{cases} s \mapsto s, \\ t \mapsto t, \\ u \mapsto a^s, \\ v \mapsto b, \\ w \mapsto w. \end{cases}$$

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_2)$	$stwavs$	$stwav$

In particular, we get that $(stwav)^6 = \Delta_2^{15}(u_2)$ generates $Z(\mathcal{B}_{31}(u_2, u_2))$.

Orbit of u_3

The object u_3 admits 7 atomic loops d, e, f, t, u, v, w, with relations as follows:

$$\begin{cases} ue = ef = fu, \ wt = tf = fw, \\ df = fd, \ wv = vw, \ tv = vt, \ vf = fv, \\ udw = dwu = wud, \ dte = ted = edt, \\ utu = tut, \ uvu = vuv, \ dvd = vdv, \ eve = vev, \\ wtud = edwt. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_3}^+$ given by this presentation is not cancellative: we have $tudt \neq udtu$ and $wtudt = wudtu$ in $H_{u_3}^+$. Thus $H_{u_3}^+$ cannot be a Garside monoid.

The submonoid $L_{u_3}^+$ of $\mathcal{C}_{31}(u_3, u_3)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $utu = tut$ is the shortest common multiple of t and u. If right-lcms exists in $L_{u_3}^+$, then utu must be the right-lcm of u and t. We would then have

$$tut \preccurlyeq tudt \Rightarrow t \preccurlyeq dt,$$

which does not hold in $L_{u_3}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_3, u_3)$).

The relations defining $H_{u_3}^+$ give in particular $f := w^t$ and $e = f^u = w^{tu}$ in H_{u_3}. By deleting these generators, we get that $\mathcal{B}_{31}(u_3, u_3)$ is generated by d, t, u, v, w, with relations as follows:

$$\begin{cases} wv = vw, \ tv = vt, \\ udw = wud = dwu, \\ uvu = vuv, \ tut = utu, \ dvd = vdv, \ wtw = twt, \\ udtu = tudt. \end{cases}$$

The monoid given by this presentation is not cancellative: we have $tdwt \neq wtdw$ while $tdwtut = wtdwut$. The morphisms $\varphi_{3,1}$ and $\varphi_{1,3}$ are given by

$$\varphi_{3,1} : \begin{cases} t \mapsto t, & d \mapsto s^u, \\ u \mapsto u, & e \mapsto w^{tu}, \\ v \mapsto v, & f \mapsto w^t, \\ w \mapsto w, \end{cases} \quad \text{and} \quad \varphi_{1,3} : \begin{cases} s \mapsto d^w, \\ t \mapsto t, \\ u \mapsto u, \\ v \mapsto v, \\ w \mapsto w. \end{cases}$$

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_3)$	$wtudvw$	$tudvw$

In particular, we get that $(tudvw)^6 = \Delta_2^{15}(u_3)$ generates $Z(\mathcal{B}_{31}(u_3, u_3))$.

Orbit of u_4

The object u_4 admits 12 atomic loops $g, h, k, l, m, n, o, p, s, t, u, v$, with relations as follows:

$$\begin{cases} gk = hg = kh,\ gs = lg = sl,\ gn = tg = nt,\ gp = vg = pv,\ ht = mh = tm, \\ kn = mk = nm,\ lo = tl = ot,\ so = ns = on,\ ut = tp = pu,\ uv = nu = vn, \\ gm = mg,\ go = og,\ hn = nh,\ st = ts,\ tv = vt,\ np = pn, \\ hus = shu = ush,\ hvo = ohv = voh,\ kpo = okp = pok, \\ lmv = mvl = vlm,\ smp = mps = psm, \\ hph = php,\ svs = vsv,\ mom = omo, \\ gnp = utv,\ htvl = lomv,\ khnps = lgomp,\ knps = somp, \\ khnps = utvsm,\ khn = tgm,\ khpo = vgok,\ lgomp = utvsm, \\ lgo = nst,\ lgmp = pvsm,\ usht = mhps,\ uvsh = ohuv. \end{cases}$$

The last three lines of relations can be omitted, as they are implied by the others. The monoid $H_{u_4}^+$ given by this presentation is not cancellative: we have $hpsh \neq pshp$ and $mhpsh = mpshp$ in $H_{u_4}^+$. Thus $H_{u_4}^+$ cannot be a Garside monoid.

The submonoid $L_{u_4}^+$ of $\mathcal{C}_{31}(u_4, u_4)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $hph = php$ is the shortest common multiple of h and p. If right-lcms exists in $L_{u_4}^+$, then hph must be the right-lcm of h and p. We would then have

$$hph \preccurlyeq hpsh \Rightarrow h \preccurlyeq sh,$$

which does not hold in $L_{u_4}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_4, u_4)$).The relations defining $H_{u_4}^+$ give in particular

$$g = n^t = u^{vt},\ k = h^g = h^{(u^{vt})},\ l = g^s = u^{tvs},\ m = h^t,\ n = u^v,\ o = n^s = u^{vs},\ p = u^t$$

in H_{u_4}. By deleting these generators, we get that $\mathcal{B}_{31}(u_4, u_4)$ is generated by h, s, t, u, v, with relations as follows:

$$\begin{cases} vt = tv,\ st = ts, \\ ush = shu = hus, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ tht = hth, \\ shvs = vshv. \end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have $shvs = vshv$ and $vsv = svs$ are common right-multiples of v and s, but their longest common divisor is vs,which is not a common right-multiple of s and v. This also proves that the submonoid of $\mathcal{C}_{31}(u_4, u_4)$ generated by h, s, t, u, v does not admit right-lcms. The morphisms $\varphi_{4,1}$ and $\varphi_{1,4}$ are given by

$$\varphi_{3,1} : \begin{cases} g \mapsto u^{vt}, & o \mapsto u^{vs}, \\ h \mapsto w^s, & p \mapsto u^t, \\ k \mapsto u^{wtuwvsuvt}, & s \mapsto s, \\ l \mapsto u^{tvs}, & t \mapsto t, \\ m \mapsto w^{st}, & u \mapsto u, \\ n \mapsto u^v, & v \mapsto v, \end{cases} \quad \text{and} \quad \varphi_{1,3} : \begin{cases} s \mapsto s, \\ t \mapsto t, \\ u \mapsto u, \\ v \mapsto v, \\ w \mapsto h^u. \end{cases}$$

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_4)$	$tuvshv$	$tuvsh$

In particular, we get that $(tuvsh)^6 = \Delta_2^{15}(u_4)$ generates $Z(\mathcal{B}_{31}(u_4, u_4))$. We also have $((tuvsh)^t)^2 = (uvsht)^2 = \Delta_2^5(u_4)$ in this case.

Orbit of u_5

The object u_5 admits 10 atomic loops $b, f, g, n, p, s, t, u, v, w$, with relations as follows:

$$\begin{cases} ut = pu = tp, \ uv = nu = vn, \ gp = pv = vg, \\ gn = tg = nt, \ sv = vb = bs, \ wt = fw = tf, \\ st = ts, \ wv = vw, \ fv = vf, \ pn = np, \ tv = vt, \ tb = bt, \\ uws = suw = wsu, \ gfb = fbg = bgf, \ spf = fsp = pfs, \ wbn = nwb = bnw, \\ ufu = fuf, \ gsg = sgs, \ suw = wsu, \ sns = nsn, \ fnf = nfn, \\ utv = gpn, \ uwsv = bsnw, \ gnfb = wtbg, \ wsut = pufs, \ pfsv = bsgf. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_5}^+$ given by this presentation is not cancellative: we have $ufsu \neq fsuf$ and $pufsu = pfsuf$ in $H_{u_5}^+$. Thus $H_{u_5}^+$ cannot be a Garside monoid.

The submonoid $L_{u_5}^+$ of $\mathcal{C}_{31}(u_5, u_5)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $ufu = fuf$ is the shortest common multiple of f and u. If right-lcms exists in $L_{u_5}^+$, then ufu must be the right-lcm of u and f. We would then have

$$ufu \preccurlyeq ufsu \Rightarrow u \preccurlyeq su,$$

which does not hold in $L_{u_5}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_5, u_5)$).

The relations defining $H_{u_5}^+$ give in particular

$$b = s^v, \ f = w^t, \ g = u^{vt}, \ n = u^v, \ p = u^t$$

in H_{u_5}. By deleting these generators, we get that $\mathcal{B}_{31}(u_5, u_5)$ is generated by s, t, u, v, w, with the same relations as in $H_{u_1}^+$: we recover once again the obtained in Section 10.1.3. The morphisms $\varphi_{5,1}$ and $\varphi_{1,5}$ are given by

$$\varphi_{5,1} : \begin{cases} s \mapsto s, & b \mapsto s^v, \\ t \mapsto t, & f \mapsto w^t, \\ u \mapsto u, & g \mapsto u^{vt}, \\ v \mapsto v, & n \mapsto u^v, \\ w \mapsto w, & p \mapsto u^t, \end{cases} \quad \text{and} \quad \varphi_{1,5} : \begin{cases} s \mapsto s, \\ t \mapsto t, \\ u \mapsto u, \\ v \mapsto v, \\ w \mapsto w. \end{cases}$$

Fundamental roots of the full-twist are given by the same expressions as for the object u_1.

Orbit of u_6

The object u_6 admits 10 atomic loops $b, f, g, n, o, p, q, r, s, v$, with relations as follows:

$$\begin{cases} vg = pv = gp,\ qn = fq = nf,\ qr = gq = rg,\ so = ns = on,\ vb = sv = bs, \\ vf = fv,\ qb = bq,\ sr = rs,\ np = pn,\ og = go, \\ vro = ovr = rov,\ qop = pqo = opq,\ spf = fsp = pfs, \\ bgf = fbg = gfb,\ brn = nbr = rnb, \\ vqv = qvq,\ vnv = nvn,\ qsq = sqs,\ sgs = gsg,\ ngn = gng,\ pqo = opq, \\ bsgf = pvfb,\ bsro = onvr,\ fqsp = onpf,\ fqbr = rgnb,\ pvqo = rovg. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_6}^+$ given by this presentation is not cancellative: we have $nvrn \neq vrnv$ and $onvrn = ovrnv$ in $H_{u_6}^+$. Thus $H_{u_6}^+$ cannot be a Garside monoid.

The submonoid $L_{u_6}^+$ of $\mathcal{C}_{31}(u_6, u_6)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $nvn = vnv$ is the shortest common multiple of n and v. If right-lcms exists in $L_{u_6}^+$, then nvn must be the right-lcm of n and v. We would then have

$$nvn \preccurlyeq nvrn \Rightarrow n \preccurlyeq rn,$$

which does not hold in $L_{u_6}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_6, u_6)$).

The relations defining $H_{u_6}^+$ give in particular

$$b = s^v,\ f = q^n,\ o = n^s,\ p = v^g,\ r = q^g$$

in H_{u_6}. By deleting these generators, we get that $\mathcal{B}_{31}(u_6, u_6)$ is generated by g, n, q, s, v, with relations as follows:

$$\begin{cases} nsn = sns,\ vgv = gvg,\ vsv = svs,\ qnq = nqn,\ vnv = nvn, \\ qgq = gqg,\ ngn = gng,\ qsq = sqs,\ sgs = gsg,\ vqv = qvq, \\ gnvg = vgnv = nvgn,\ gqsg = qsgq = sgqs,\ nsgn = gnsg = sgns, \\ vqsv = svqs = qsvq,\ qnvq = nvqn = vqnv,\ vgqnsv = svgqns, \\ gqnsvgs = ngqnsvg. \end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have that $vgqnsv = svgqns$ and $vsv = svs$ are common right multiples of s and v, but their longest common left-divisor is sv, which is not a common right-multiple of s and v. This also proves that the submonoid of $\mathcal{C}_{31}(u_6, u_6)$ generated by g, n, q, s, v does not admit right-lcms. The morphisms $\varphi_{6,1}$ and $\varphi_{1,6}$ are given by

$$\varphi_{6,1} : \begin{cases} b \mapsto s^v, & p \mapsto u^t, \\ f \mapsto w^t, & q \mapsto t^{uwtv}, \\ g \mapsto u^{vt}, & r \mapsto u^{stuwtv}, \\ n \mapsto u^v, & s \mapsto s, \\ o \mapsto u^{vs}, & v \mapsto v, \end{cases} \quad \text{and} \quad \varphi_{1,6} : \begin{cases} s \mapsto s, \\ t \mapsto p^{vn}, \\ u \mapsto v^n, \\ v \mapsto v, \\ w \mapsto p^{vnf}. \end{cases}$$

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_6)$	$svgqns$	$svgqn$

In particular, we get that $(svgqn)^6 = \Delta_2^{15}(u_6)$ generates $Z(\mathcal{B}_{31}(u_6,u_6))$. We also have $svgqns = \Delta_2^3(u_6)$ in this case.

Orbit of u_7

The object u_7 admits 7 atomic loops g,k,m,n,o,p,s, with relations as follows:

$$\begin{cases} on = so = ns,\ kn = nm = mk, \\ go = og,\ gm = mg,\ pn = np, \\ pok = okp = kpo,\ psm = smp = mps, \\ omo = mom,\ gpg = pgp,\ gkg = kgk,\ gsg = sgs,\ gng = ngn, \\ kpon = somp. \end{cases}$$

The last line of relations can be omitted, as it is implied by the others. The monoid $H_{u_7}^+$ given by this presentation is not cancellative: we have $ompo \neq mpom$ and $sompo = smpom$ in $H_{u_7}^+$. Thus $H_{u_7}^+$ cannot be a Garside monoid.

The submonoid $L_{u_7}^+$ of $\mathcal{C}_{31}(u_7,u_7)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $omo = mom$ is the shortest common multiple of m and o. If right-lcms exists in $L_{u_7}^+$, then omo must be the right-lcm of m and o. We would then have

$$omo \preccurlyeq ompo \Rightarrow o \preccurlyeq po,$$

which does not hold in $L_{u_7}^+$ (it doesn't even hold in $\mathcal{C}_{31}(u_7,u_7)$). The relations defining $H_{u_7}^+$ give in particular $o = n^s$ and $k = n^m$ in H_{u_7}. By deleting these generators, we get that $\mathcal{B}_{31}(u_7,u_7)$ is generated by g,m,n,p,s, with relations as follows:

$$\begin{cases} gm = mg,\ pn = np, \\ smp = psm = mps, \\ sns = nsn,\ pgp = gpg,\ gsg = sgs,\ mnm = nmn,\ gng = ngn, \\ sgns = nsgn = gnsg. \end{cases}$$

The monoid given by this presentation does not admit right-lcms. We have that $sns = nsn$ and $sgns = nsgn$ are common right multiples of s and n, but their longest common left-divisor is ns, which is not a common right-multiple of s and n. This also proves that the submonoid of $\mathcal{C}_{31}(u_7,u_7)$ generated by g,m,n,p,s does not admit right-lcms. The morphisms $\varphi_{7,1}$ and $\varphi_{1,7}$ are given by

$$\varphi_{7,1} : \begin{cases} g \mapsto u^{vt}, & p \mapsto u^t, \\ k \mapsto u^{wtuwvsuvt}, & s \mapsto s, \\ m \mapsto w^{st}, \\ n \mapsto u^v, \\ o \mapsto u^{vs}, \end{cases} \quad \text{and} \quad \varphi_{1,7} : \begin{cases} s \mapsto s, \\ t \mapsto g^n, \\ u \mapsto p^{gn}, \\ v \mapsto p^g, \\ w \mapsto p^{gkpn}. \end{cases}$$

Fundamental roots of the full-twist are given by

Fundamental regular number d	20	24
d-th root of $\Delta_2^{60}(u_7)$	$gnmpsm$	$gnmps$

In particular, we get that $(gnmps)^6 = \Delta_2^{15}(u_7)$ generates $Z(\mathcal{B}_{31}(u_7,u_7))$.

Orbit of u_8

The object u_8 admits 6 atomic loops g, m, n, p, s, t, with relations as follows:

$$\begin{cases} tg = nt = gn, \\ ts = st, \ pn = np, \ gm = mg, \\ psm = smp = mps, \\ tpt = ptp, \ tmt = mtm, \ pgp = gpg, \ sns = nsn, \ sgs = gsg, \ nmn = mnm. \end{cases}$$

The monoid $H_{u_8}^+$ given by this presentation is not cancellative: we have $psgpsg \neq gpsgps$ and $mpsgpsg = mgpsgps$ in $H_{u_8}^+$. Thus $H_{u_8}^+$ cannot be a Garside monoid.

The submonoid $L_{u_8}^+$ of $\mathcal{C}_{31}(u_8, u_8)$ generated by the atomic loops is cancellative, but it doesn't admit right-lcms: we know that $pgp = gpg$ is the shortest common multiple of p and g. If right-lcms exists in $L_{u_8}^+$, then omo must be the right-lcm of m and o. We would then have

$$gpg \preccurlyeq gpsgps \Rightarrow g \preccurlyeq sgps,$$

which does not hold in $L_{u_8}^+$ (it does even hold in $\mathcal{C}_{31}(u_8, u_8)$, but $g^{-1}sgps$ is not generated positively by atomic loops). The relations defining $H_{u_8}^+$ give in particular $t = g^n$ in H_{u_8}. By deleting this generator, we get that $\mathcal{B}_{31}(u_8, u_8)$ is generated by g, m, n, p, s, with the same relations as in the end of Section 10.1.3. The morphisms $\varphi_{8,1}$ and $\varphi_{1,8}$ are given by

$$\varphi_{8,1} : \begin{cases} g \mapsto u^{vt} \\ m \mapsto w^{st}, \\ n \mapsto u^v, \\ p \mapsto u^t, \\ s \mapsto s, \\ t \mapsto t, \end{cases} \quad \text{and} \quad \varphi_{1,8} : \begin{cases} s \mapsto s, \\ t \mapsto t, \\ u \mapsto t^p, \\ v \mapsto n^{tp}, \\ w \mapsto m^{tmp}. \end{cases}$$

Fundamental roots of the full-twist are given by the same expressions as for the object u_7. Furthermore, we also have $(gnmps)^2 = \Delta_2^5(u_8)$ in this case.

Remark 10.1.7. As was pointed out to us by Jean Michel, the presentations with 5 generators that we give here are related by the Hurwitz action of the usual braid groups on the words giving 24-roots of the full-twist, as in [MM10, Section 6]. By identifying the atomic loops we consider with their image in $\mathcal{B}_{31}(u_1, u_1)$, we obtain for instance that (s, t, u, v, w), (s, t, w, a, v), (t, u, d, v, w) and (t, u, v, s, h) lie in the same Hurwitz orbit. The words (s, u, v, w, t) and (s, v, g, q, n) also lie in the same Hurwitz orbit, as well as (u, v, w, t, s) and (g, n, m, p, s).

10.2 Parabolic subgroups

In this section, we study parabolic subgroups of the complex braid group $B(G_{31})$. By Corollary 9.3.2, we know that the parabolic subgroups of $B(G_{31})$ are stable under intersection, which is [GM22, Theorem 1.1 and Theorem 1.2] in the case of $B(G_{31})$.

10.2.1 Diagrams for $B(G_{31})$

In this Section, we fix the object u_1 considered in Section 10.1.3. We saw in this section that that $\mathcal{B}_{31}(u_1, u_1)$ admits five atomic loops s, t, u, v, w. These atomic loops generate $\mathcal{B}_{31}(u_1, u_1) \simeq$

$B(G_{31})$ with the following presentation:

$$B(G_{31}) \simeq \mathcal{B}_{31}(u_1,u_1) = \left\langle s,t,u,v,w \;\middle|\; \begin{array}{l} st = ts,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw \end{array} \right\rangle. \tag{10.2.1}$$

Let us denote by $\overline{s},\overline{t},\overline{u},\overline{v},\overline{w}$ the images of s,t,u,v,w in the reflection group G_{31}. The elements $\overline{s},\overline{t},\overline{u},\overline{v},\overline{w}$ generate G_{31}, and a presentation is obtained from Presentation (10.2.1) by adding the relations $\overline{s}^2 = \overline{t}^2 = \overline{u}^2 = \overline{v}^2 = \overline{w}^2 = 1$. It is known that every parabolic subgroup of G_{31} is, up to conjugacy, generated by some subset S of $\{\overline{s},\overline{t},\overline{u},\overline{v},\overline{w}\}$. And a presentation is given by taking the relations in Presentation (10.2.1) which only involve the elements in S (plus the relations $r^2 = 1$ for $r \in S$). The following theorem states that the situation is the same at the level of the braid group $B(G_{31})$.

Theorem 10.2.1. *The lattice of parabolic subgroups of $\mathcal{B}_{31}(u_1,u_1)$ up to conjugacy is given by*

$$\begin{tikzcd}
& & \langle s,t,u,v,w\rangle \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\langle s,t,v\rangle \arrow[d, dash] \arrow[drrrr, dash] & & \langle s,u,v,w\rangle \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \langle t,u,v\rangle \arrow[d, dash] \arrow[dllll, dash] \\
\langle s,v\rangle \arrow[drr, dash] & & \langle s,u,w\rangle \arrow[d, dash] & & \langle t,v\rangle \arrow[dll, dash] \\
& & \langle s\rangle \arrow[d, dash] & & \\
& & \langle\varnothing\rangle & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & B(G_{31}) \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
B(A_2\times A_1) \arrow[d, dash] \arrow[drrrr, dash] & & B(G(4,2,3)) \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & B(A_3) \arrow[d, dash] \arrow[dllll, dash] \\
B(A_2) \arrow[drr, dash] & & B(G(4,2,2)) \arrow[d, dash] & & B(A_1\times A_1) \arrow[dll, dash] \\
& & B(A_1) \arrow[d, dash] & & \\
& & \{1\} & &
\end{tikzcd}$$

The lattice on the right, where A_n denotes the complex reflection group $G(1,1,n+1)$, describes the isomorphism type of the parabolic subgroups given on the left. For each such parabolic subgroup $\langle S\rangle \subset \mathcal{B}_{31}(u_1,u_1)$, a presentation is given by taking the relations in Presentation (10.2.1) which only involve elements of S.

Proof. First, we show that the considered subgroups are indeed parabolic. Let S be one of the following families of atomic loops in $\mathcal{B}_{31}(u_1,u_1)$:

$$\{s,t,v\}, \{s,u,v,w\}, \{t,u,v\}.$$

Let $\mathcal{T}$ denote the shoal of standard parabolic subgroupoid of $\mathcal{B}_{31}$ constructed in Proposition 9.2.29 (shoal for Springer groupoid). We denote by $\mathcal{G}_b$ the intersection of all the $\mathcal{T}$-standard parabolic subgroupoids of $\mathcal{B}_{31}$ which contain S. We check by direct computations that S is exactly the set of atomic loops of $\mathcal{B}_{31}(u_1,u_1)$ which lie in $\mathcal{G}_b(u_1,u_1)$. We then use the method and algorithms detailed in Section 10.1.1 on the presentation of $\mathcal{G}_b$ to show that S generates $\mathcal{G}_b(u_1,u_1)$ with the required presentation. As in 10.1.5, we choose a Schreier transversal T for $\mathcal{G}_b$ rooted in u_1 such that the elements of S appears as generators of the presentation of $\mathcal{G}_b(u_1,u_1)$ induced by T. We then prove that S generates $\mathcal{G}_b(u_1,u_1)$ by using Algorithms 10.1.3, and we prove that all the relators of the presentation of $\mathcal{G}_b(u_1,u_1)$ induced by T are deduced from the relators of the required presentation by using

The three groups $\langle s,t,v\rangle$, $\langle s,u,v,w\rangle$ and $\langle t,u,v\rangle$ are thus parabolic subgroups of $\mathcal{B}_{31}(u_1,u_1) \simeq B(G_{31})$ (and a presentation is given by taking the relations in Presentation (10.2.1) which only involve elements of S).

The subgroup $\langle s,t,v\rangle$ is a braid group of type $B(A_2\times A_1)$. It contains the subgroups $\langle s,t\rangle,\langle s,v\rangle,\langle s\rangle$ and $\langle\varnothing\rangle$ as parabolic subgroups by [GM22, Proposition 3.1] (we deduce in particular these groups have the desired presentation). These groups are then also parabolic subgroups of $\mathcal{B}_{31}(u_1,u_1)\simeq B(G_{31})$ by [GM22, Proposition 2.5]. Likewise, $\langle s,u,v,w\rangle$ is a parabolic subgroup of type $B(G(4,2,3))$, which contains $\langle s,u,w\rangle$ as a parabolic subgroup. The group $\langle s,u,w\rangle$ is then a parabolic subgroup of $\mathcal{B}_{31}(u_1,u_1)$ again by [GM22, Proposition 2.5], and it admits a presentation given by taking the relations in Presentation (10.2.1) which only involve elements of S.

The groups we consider are thus all parabolic subgroups of $\mathcal{B}_{31}(u_1,u_1)\simeq B(G_{31})$. Furthermore, the images of these groups in G_{31} form a complete set of representatives for the parabolic subgroups of G_{31} under conjugacy (see [OT92, Table C.12]). By [GM22, Proposition 2.6], the groups we consider then form a complete set of representatives for the parabolic subgroups of $B(G_{31})$ under conjugacy. Lastly, the lattice of parabolic subgroups of $B(G_{31})$ up to conjugacy is the same as the one of G_{31} by [GM22, Proposition 2.5], which terminates the proof.

□

We now plan to use this theorem to show that a standard parabolic subgroup in $\mathcal{B}_{31}$ is always generated by the atomic loops it contains. In Proposition 9.2.22 (isomorphism of standard parabolic subgroupoids), we constructed particular isomorphisms of standard parabolic subgroupoids in divided groupoids attached to well-generated complex reflection group. Since Springer groupoids are subgroupoids of fixed points in such divided groupoids, we obtain families of isomorphisms between standard parabolic subgroupoids in the shoal $\mathcal{T}$. Such isomorphisms will be called *ribbons*.

Lemma 10.2.2. *Let $\mathcal{C}_\beta$ be a standard parabolic subcategory of $\mathcal{B}_{31}$, and let $s\preccurlyeq\beta$ in $I(G_{37})$. The ribbon $\mathcal{C}_\beta\to\mathcal{C}_{\beta'}$ induced by s after Proposition 9.2.22 sends the atomic loops in $\mathcal{C}_\beta$ to atomic loops in $\mathcal{C}_{\beta'}$. Furthermore, every atomic loop in $\mathcal{C}_\beta$ is the image under ψ of an atomic loop of $\mathcal{C}_\beta$.*

Proof. By Corollary 9.2.5 (no pairs of parallel simples), Lemma 9.2.7 and Lemma 10.1.4, the atomic loops in $\mathcal{C}_{31}$ are exactly the composition of two atoms in $\mathcal{C}_{31}$ which happen to be endomorphisms. Since the atoms of $\mathcal{C}_{31}$ are exactly its elements of length 1, atomic loops are exactly the endomorphisms of length 2 in $\mathcal{C}_{31}$. Since the length functor on a standard parabolic subcategory is the restriction of the length functor of $\mathcal{C}$, the atomic loops in $\mathcal{C}_\beta$ (resp. $\mathcal{C}_{\beta'}$) are exactly its endomorphisms of length 2.

In the proof of Proposition 9.2.22, we saw that $\psi(a,b)=(a^s,s^{-1}bs^{c^\eta})$. Since $\ell_{\mathcal{R}}$ is invariant under conjugacy, we have

$$\ell(\psi(a,b))=\ell_{\mathcal{R}}(a^s)=\ell_{\mathcal{R}}(a)=\ell(a,b).$$

Thus, ribbons preserve the length function. In particular they preserve endomorphisms of length 2, that is, atomic loops. □

We then show that, up to isomorphism induced by ribbons, one can always assume that a standard parabolic subcategory contains the object u_1. The following lemma is shown through direct computations.

Lemma 10.2.3. *Let $\mathcal{C}_\beta$ be a standard parabolic subcategory of $\mathcal{B}_{31}$. There is a finite sequence of ribbons which sends $\mathcal{C}_\beta$ to a standard parabolic subcategory of $\mathcal{B}_{31}$ containing u_1 as an object.*

Proof. We show that there are two sequences $\beta = \beta_1, \dots, \beta_n$ and $s_1, \dots, s_{n-1}$ in $I(G_{37})$ such that

1. $s_i \preccurlyeq \beta_i$ and $\beta_{i+1} = s_i^{-1}\beta s_i^{c^8}$ for all $i \in [\![1, n-1]\!]$,
2. $\beta_n \preccurlyeq u_1$.

For $i \in [\![1, n-1]\!]$, the element s_i provides a ribbon $\psi_i : \mathcal{C}_{\beta_i} \to \mathcal{C}_{\beta_{i+1}}$. The sequence $\psi_{n-1} \circ \cdots \circ \psi_1$ gives the desired result.

The proof that such sequences exist is computational. We start with the set $B := \{\beta \in I(G_{37}) \mid \beta \preccurlyeq u_1\}$. Then, for every $\beta \in B$, and every $s \preccurlyeq \beta$, we add $s^{-1}\beta s^{c^8}$ to B. We iterate this process until no new element can be reached. The fact that every divisor of any $u \in \mathrm{Ob}(\mathcal{B}_{31})$ lies in B shows the claim. □

A useful consequence of this lemma is that standard parabolic subgroupoids are always connected. This can also be seen as a corollary of Theorem 9.2.41 (isomorphism of groupoids).

Lemma 10.2.4. *Every standard parabolic subgroupoid of $\mathcal{B}_{31}$ is connected. That is, if $\mathcal{G}_\beta$ is a standard parabolic subgroupoid of $\mathcal{B}_{31}$, and $u, v \in \mathrm{Ob}(\mathcal{G}_\beta)$, then $\mathcal{G}_\beta(u, v)$ is nonempty.*

Proof. By Lemma 10.2.3, every standard parabolic subgroupoid is isomorphic (through a sequence of ribbons) to a standard parabolic subgroupoid containing u_1. There are 20 such subgroupoids. Direct computations show that these 20 subgroupoids are all connected. □

We are now equipped to show that a standard parabolic subgroup in $\mathcal{B}_{31}$ is always generated by the atomic loops it contains.

Proposition 10.2.5. *Let $\mathcal{G}_\beta$ be a standard parabolic subgroupoid of $\mathcal{B}_{31}$, and let $u \in \mathrm{Ob}(\mathcal{G}_\beta)$. The group $\mathcal{G}_\beta(u, u)$ is generated by the atomic loops of $\mathcal{B}_{31}(u, u)$ which it contains.*

Proof. Let S be the set of atomic loops of $\mathcal{B}_{31}(u, u)$ which lie in $\mathcal{C}_\beta(u, u)$. By Lemma 10.2.3, there is an isomorphism of categories $\psi : \mathcal{C}_\beta \to \mathcal{C}_{\beta'}$ where $\mathcal{C}_{\beta'}$ is a standard parabolic subcategory with $u_1 \in \mathrm{Ob}(\mathcal{C}_{\beta'})$. By Lemma 10.2.2, ψ sends S to the set of atomic loops of $\psi(u)$ which lie in $\mathcal{C}_{\beta'}(\psi(u), \psi(u))$. Thus we can replace β by β' and u by $\psi(u)$ to assume that $u_1 \in \mathrm{Ob}(\mathcal{C}_\beta)$.

Now, by Lemma 10.2.4, we can consider a morphism $f : u \to u_1$. It is then sufficient to show that S^f generates $\mathcal{G}_\beta(u_1, u_1)$. Since Theorem 10.2.1 already gives a generating set R for $\mathcal{G}_\beta(u_1, u_1)$, it is sufficient to show that S^f generates R, which is done by a case-by-case analysis: every element of R is conjugate by an element of $\langle S^f \rangle$ to an element of S^f. □

10.2.2 Characterization of adjacency in the curve graph

As in [GM22], we define the *curve graph* Γ for $B(G_{31})$ as the graph whose vertices are irreducible parabolic subgroups of $B(G_{31})$, and where two such subgroups B_1 and B_2 are adjacent if $B_1 \neq B_2$ and either $B_1 \subset B_2$, $B_2 \subset B_1$ or $B_1 \cap B_2 = [B_1, B_2] = \{1\}$.

In this section, we fix the shoal $\mathcal{T}$ of standard parabolic subgroupoid of $\mathcal{B}_{31}$ constructed in Proposition 9.2.29 (shoal for Springer groupoid). We also fix the object u_1 considered in Section 10.1.3. By Theorem 9.2.42 (parabolic subgroups up to conjugacy), the parabolic subgroups of $B(G_{31})$ are exactly the $\mathcal{T}$-parabolic subgroups of $\mathcal{B}_{31}(u_1, u_1)$. The shoal $\mathcal{T}$ is also endowed with a system of conjugacy representatives. Thus, for $B_0 \subset B(G_{31})$ a parabolic subgroup, we can consider the element $z_{B_0} \in B_0$ such that $B_0 = \mathrm{PC}(z_{B_0})$.

Proposition 10.2.6. *Let $u \in \mathrm{Ob}(\mathcal{B}_{31})$. If $B_0 \subset \mathcal{B}_{31}(u,u)$ is an irreducible parabolic subgroup. Then z_{B_0} is a generator of $Z(B_0)$.*

Proof. Since $\mathcal{B}_{31}$ is a connected groupoid, we can consider a morphism $f \in \mathcal{B}_{31}(u,u_1)$. The group $B_1 := (B_0)^f$ is a parabolic subgroup of $B(G_{31}) = \mathcal{B}_{31}(u_1,u_1)$, and we have $z_{B_1} = (z_{B_0})^f$ by definition of a system of conjugacy representatives. By Theorem 10.2.1, there is some $g \in \mathcal{B}_{31}(u_1,u_1)$ so that $B_2 := (B_1)^g = \langle R\rangle$ is generated by one of the sets R of braided reflections given in Theorem 10.2.1. We also have $z_{B_2} = (z_{B_1})^g$. It remains to show that z_{B_2} is indeed a generator of the center of B_2. For all sets R given in Theorem 10.2.1, an expression of a generator of $\langle R\rangle$ as a word in R is known, and we check by direct computations that this element is indeed equal to $z_{\langle R\rangle}$. ☐

As a corollary (and by definition of a system of conjugacy representatives), we obtain the following characterization of conjugacy in irreducible parabolic subgroups of $B(G_{31})$:

Corollary 10.2.7. *Let $B_1, B_2 \subset B(G_{31})$ be two irreducible parabolic subgroups. If z_1, z_2 denotes the respective positive generators of the centers of B_1, B_2, then an element $b \in B(G_{31})$ conjugates B_1 to B_2 if and only if it conjugates z_1 to z_2.*

As in [GM22, Section 6.2], the adjacency in Γ is characterized very easily in terms of the elements z_{B_1} and z_{B_2}:

Theorem 10.2.8. *Two irreducible parabolic subgroups $B_1, B_2 \subset B(G_{31})$ are adjacent in Γ if and only if z_{B_1} and z_{B_2} are distinct and commute.*

Proof. Suppose that B_1 and B_2 are adjacent. We have $B_1 \neq B_2$ and $z_{B_1} \neq z_{B_2}$ by Proposition 5.2.30. If $B_1 \subset B_2$, since z_{B_2} is central in B_2 it follows that $[z_{B_1}, z_{B_2}] = 1$. If $B_2 \subset B_1$ the argument is the same. Finally, if $B_1 \cap B_2 = [B_1, B_2] = 1$, every element of B_1 commutes with every element of B_2, hence z_{B_1} commutes with z_{B_2} also in this case.

Conversely, suppose that z_{B_1} and z_{B_2} are distinct and that they commute. Since z_{B_i} is well-defined and depends only on B_i for $i = 1,2$, we have $B_1 \neq B_2$. Then, as in the proof of [GM22, Proposition 6.12], we simultaneously conjugate B_1 and B_2 to standard parabolic subgroups in $\mathcal{B}_{31}$.

Since B_1 is a parabolic subgroup, there is a morphism $f : u \to v$ in $\mathcal{B}_{31}$ such that $(B_1)^f$ is a standard parabolic subgroup by Theorem 9.2.42 (parabolic subgroups up to conjugacy). We can replace B_1 and B_2 by $(B_1)^f$ and $(B_2)^f$ to assume that B_1 is standard, and that z_{B_1} is positive by Proposition 5.2.32.

Now consider the left-fraction decomposition $a^{-1}b$ of z_{B_2}. By definition, we have $a \wedge b = 1$ and, left-multiplying by a^{-1}, we get $1 \wedge a^{-1}b = 1 \wedge z_{B_2} = a^{-1}$.

Since z_{B_1} is positive and commutes with z_{B_2}, we have that $z_{B_1} = (z_{B_1})^{z_{B_2}}$ is positive, hence recurrent. By Proposition 3.3.19, $(z_{B_1})^{1\wedge z_{B_2}} = (z_{B_1})^{a^{-1}}$ is a recurrent conjugate of z_{B_1}. Since z_{B_1} is positive, all its recurrent conjugates are positive by Proposition 3.3.8. It follows that, if we simultaneously conjugate z_{B_1} and z_{B_2} by a^{-1}, we replace z_{B_1} by a positive conjugate, and we replace z_{B_2} by $\mathrm{sw}(z_{B_2})$.

The obtained conjugates of z_{B_1} and z_{B_2} satisfy the same initial hypotheses; they commute and the first one is positive. Hence we can iterate the process described above, replacing z_{B_1} by a positive conjugate, $\mathrm{sw}(z_{B_2})$ by $\mathrm{sw}^2(z_{B_2})$ and so on. Since z_{B_2} is conjugate to a positive element by a series of iterated swaps, we finally obtain simultaneous conjugates of z_{B_1} and z_{B_2} which are both positive (and commute).

If we denote by f the common conjugator of z_{B_1} and z_{B_2} that we consider, we have, for $i = 1, 2$, $B_i' := (B_i)^f = \mathrm{PC}(z_{B_i})^f = \mathrm{PC}((z_{B_i})^f)$. Since $(z_{B_i})^f$ is positive by construction, it follows from Proposition 5.2.32 that B_1' and B_2' are both standard parabolic subgroups.

We can then assume, up to a simultaneous conjugation, that $B_1 = \mathcal{G}_{\beta_1}(u, u)$ and $B_2 = \mathcal{G}_{\beta_2}(u, u)$ are standard parabolic subgroups. Let R_1 (resp. R_2) be the set of atomic loops of $\mathcal{B}_{31}(u, u)$ which are contained in B_1 (resp. in B_2). We have $\langle R_1 \rangle = B_1$ and $\langle R_2 \rangle = B_2$ by Proposition 10.2.5. By listing the finite number of standard parabolic subgroups of $\mathcal{B}_{31}(u, u)$, we show through direct computations that z_{B_1} and z_{B_2} commute if and only if we have $R_1 \subset R_2$, or $R_2 \subset R_1$, or $xy = yx$ for all $(x, y) \in R_1 \times R_2$. This terminates the proof. $\square$

This finishes the proof of [GM22, Theorem 1.1, Theorem 1.2, Theorem 1.3] for the complex braid group $B(G_{31})$, which was the last open case among irreducible complex braid groups.

Part III

Appendices

Appendix A

Homology computations for Garside groupoids

In this chapter, which is taken almost verbatim from my second paper [Gar24a], we provide computational tools for studying homology of Garside groupoids, and their associated weak Garside groups. We apply these results to exceptional complex braid groups, in particular $B(G_{31})$.

Contents

A.1 Homology of a Category

In order to define an analogue of the Dehornoy-Lafont order complex for categories, one first needs to define the homology of a category in general. In particular we need notions of resolutions, free modules over a category and tensor product of modules over a category. Then we need, just as for monoids, to give a relation between the homology of a category and the homology of its enveloping groupoid under suitable assumptions.

A.1.1 Modules over a category

The representation theory of categories appears for instance in the study of quivers and correspondence functors (see [BT18, Section 2] and [Web07]). For convenience, we provide here a definition of modules and free modules over a category.

In this section, we fix a category $\mathcal{C}$.

Definition A.1.1 (Module over a category). A $\mathbb{Z}\mathcal{C}$*-module* (or $\mathcal{C}$-module) is a contravariant functor from $\mathcal{C}$ to the category **Ab** of abelian groups. Equivalently, a $\mathbb{Z}\mathcal{C}$-module is given by a contravariant additive functor $\mathbb{Z}\mathcal{C} \to \mathbf{Ab}$.

Let M be a $\mathbb{Z}\mathcal{C}$-module. Every $x \in \mathrm{Ob}(\mathcal{C})$ is mapped to an abelian group M_x. A morphism $f : x \to y$ induces a morphism $M(f) : M_y \to M_x$. For $a \in M_y$, we denote $f.a$ instead of $M(f)(a)$. This element lies in M_x.

Let $g \in \mathcal{C}(y, z)$. By definition of a contravariant functor, and because of our convention for composition of arrows in $\mathcal{C}$ (which is different from the composition in **Ab**), we have $M(fg) = M(f) \circ M(g)$. In our notation for $M(f)$, we get

$$\forall a \in M_z, \quad f.(g.a) = (fg).a.$$

Example A.1.2. One can always consider $\mathbb{Z}$ as a trivial $\mathcal{C}$-module by considering the functor mapping every object to $\mathbb{Z}$ and every morphism to the identity in $\mathbb{Z}$.

Remark A.1.3. Another point of view, usually adopted in the representation theory of quivers, is to consider the algebra A generated as a ring by all morphisms in $\mathcal{C}$, and with relations

$$fg = \begin{cases} f \circ g & \text{if the source of } g \text{ is the target of } f, \\ 0 & \text{otherwise.} \end{cases}$$

This point of view is in fact equivalent to ours. Indeed, considering a $\mathcal{C}$-module M, the abelian group $\bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} M_x$ naturally comes equipped with an A-module structure.

Proposition-Definition A.1.4. *We denote by $\mathbb{Z}\mathcal{C} - \mathbf{mod}$ the category of $\mathbb{Z}\mathcal{C}$-modules, where the morphisms are the natural transformations between functors (if M, N are $\mathcal{C}$-modules, we denote by $\mathrm{Hom}_{\mathcal{C}}(M, N)$ the corresponding set of morphisms in $\mathbb{Z}\mathcal{C} - \mathbf{mod}$). Since **Ab** is an abelian category, the category $\mathbb{Z}\mathcal{C} - \mathbf{mod}$ is an abelian category.*

As we want to consider free resolutions in the category $\mathbb{Z}\mathcal{C} - \mathbf{mod}$, we first define a notion of free module over $\mathcal{C}$. Consider the category $\mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$ of families of sets indexed by objects of $\mathcal{C}$. The morphisms between two families $\{S_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$ and $\{T_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$ are given by families of (set-theoretic) maps $\{\varphi_x : S_x \to T_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$.

The category $\mathbb{Z}\mathcal{C} - \mathbf{mod}$ is endowed with a forgetful functor to $\mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$, sending a functor M to the family $\{M_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$. We construct a free functor, adjoint to this forgetful functor.

Let $S := \{S_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$ be a family of sets. For $x \in \mathrm{Ob}(\mathcal{C})$, we define $F(S)_x$ as the free abelian group over the set

$$(\mathcal{C}S)_x := \{(g, s) \mid g : x \to y \text{ and } s \in S_y\}.$$

Then, for a morphism $f : x \to y$, the morphism $F(S)(f) : F(S)_y \to F(S)_x$ is defined on $(\mathcal{C}S)_y$ by

$$(g, s) \mapsto (fg, s) \in (\mathcal{C}S)_x,$$

and then extended to $F(S)_y$ by linearity.

If $\varphi = \{\varphi_x\}_{x\in \mathrm{Ob}(\mathcal{C})} : \{S_x\}_{x\in \mathrm{Ob}(\mathcal{C})} \to \{T_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ is a morphism in $\mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$. We get maps

$$\begin{array}{rccc} \mathcal{C}\varphi_x : & (\mathcal{C}S)_x & \longrightarrow & (\mathcal{C}T)_x \\ & (g,s) & \longmapsto & (g,\varphi(s)) \end{array}$$

which induce a morphism $F(\varphi)$ of $\mathbb{Z}\mathcal{C}$-modules between $F(S)$ and $F(T)$.

Lemma A.1.5. *The functor $F : \boldsymbol{Set}^{\mathrm{Ob}(\mathcal{C})} \to \mathbb{Z}\mathcal{C} - \boldsymbol{mod}$ constructed above is left-adjoint to the forgetful functor $\mathbb{Z}\mathcal{C} - \boldsymbol{mod} \to \boldsymbol{Set}^{\mathrm{Ob}(\mathcal{C})}$.*

Proof. Let $S = \{S_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ be a family of sets and let M be a $\mathbb{Z}\mathcal{C}$-module. If $\varphi : S \to \{M_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ is a morphism in $\mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$, then the formula

$$\forall x \in \mathrm{Ob}(\mathcal{C}), \sum gs \in F(S)_x,\ \eta_x\left(\sum gs\right) = \sum g\varphi(s) \in M_x$$

yields a natural transformation $\eta : F(S) \Rightarrow L$, which is uniquely determined by φ. Conversely, if $\eta : F(S) \Rightarrow M$ is a natural transformation, then defining $\varphi(s) := \eta_x(s)$ for $s \in S_x$ induces a morphism $\varphi : S \to \{M_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ in $\mathbf{Set}^{\mathrm{Ob}(\mathcal{C})}$.

The applications $\eta \mapsto \varphi$ and $\varphi \mapsto \eta$ are inverse bijections, which give the desired adjunction. □

Example A.1.6. Let $x_0 \in \mathrm{Ob}(\mathcal{C})$, the hom functor $\mathcal{C}(-, x_0)$ is the free functor over the family $\{M_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ where

$$M_x = \begin{cases} \varnothing & \text{if } x \neq x_0, \\ \{*\} & \text{if } x = x_0. \end{cases}$$

In this case, the adjunction formula can be seen as a consequence of the Yoneda Lemma.

Lemma A.1.7. *Free modules over $\mathcal{C}$, in the sense defined above, are projective objects in the category $\mathbb{Z}\mathcal{C} - \boldsymbol{mod}$*

Proof. Let $S = \{S_x\}_{x\in \mathrm{Ob}(\mathcal{C})}$ be a family of sets. Let also $\varepsilon : M \twoheadrightarrow N$ be an epimorphism of $\mathcal{C}$-modules, and let $\eta : F(S) \to N$ be a morphism of $\mathcal{C}$-module. We want to construct a morphism $\mu : F(S) \to M$ such that $\varepsilon\mu = \eta$.
By adjunction, the morphism η induces, for each $x \in \mathrm{Ob}(\mathcal{C})$, a map $\eta'_x : S_x \to N_x$. As $\mathcal{C} - \mathbf{mod}$ is a functor category, stating that ε is an epimorphism amounts to saying that, for all $x \in \mathrm{Ob}(\mathcal{C})$, ε_x is an epimorphism of abelian groups, in particular it is onto. We can thus construct a map $\mu'_x : S_x \to M_x$ such that $\varepsilon'_x \circ \mu'_x = \eta'_x$. Considering the morphism μ induced by μ' in the adjunction gives the desired result. □

Remark A.1.8. In Section A.2.1, we will construct a free resolution of the trivial module over a Gaussian category $\mathcal{C}$. This last lemma shows that such a free resolution is indeed a projective resolution suitable for homology computations.

A.1.2 Tensor product

In order to define the homology of a category $\mathcal{C}$ with coefficients in a $\mathcal{C}$-module M, we need to define a tensor product which extends the usual tensor product of modules. This definition is an immediate generalization of the construction given in [Mit72, Section 6] (which only considers $\mathbb{Z},\mathcal{C}$-bimodules and $\mathcal{C},\mathbb{Z}$-bimodules).

In this section, we fix three categories $\mathcal{C},\mathcal{D},\mathcal{E}$.

A $\mathcal{C},\mathcal{D}$-*bimodule* is a functor $\mathcal{C}\times\mathcal{D}^{op}\to\mathbf{Ab}$. The morphisms of bimodules between two $\mathcal{C},\mathcal{D}$-bimodules M and N will be denoted by $\mathrm{Hom}_{\mathcal{C},\mathcal{D}}(M,N)$.

Let M and N be a $\mathcal{C},\mathcal{D}$-bimodule and a $\mathcal{D},\mathcal{E}$-bimodule, respectively, and let $(x,z)\in\mathrm{Ob}(\mathcal{C}\times\mathcal{E}^{op})$. We define

$$P_{x,y} := \bigoplus_{z\in\mathrm{Ob}(\mathcal{D})} M_{x,z}\otimes_{\mathbb{Z}} M_{z,y}.$$

The image of (x,y) under the functor $M\otimes_{\mathcal{C}} N$ is then defined as the quotient of $P_{x,y}$ by all relations of the form $ad\otimes b = a\otimes db$ where $d\in\mathcal{D}(z,z'), a\in M_{x,z}, b\in M_{z',y}$.

A morphism $c\in\mathcal{C}(x,x')$ acts by $f.(a\otimes b) := (fa)\otimes b$. A morphism $e\in\mathcal{E}(y,y')$ acts by $(a\otimes b).e := a\otimes(be)$.

Like in the usual case, we recover an adjunction between the tensor product and the Hom functor. This adjointness property directly gives right-exactness of the tensor product.

Proposition A.1.9. *Let M,N,Q be a $\mathcal{C},\mathcal{D}$-bimodule, a $\mathcal{D},\mathcal{E}$-bimodule and a $\mathcal{C},\mathcal{E}$-bimodule, respectively. We have*

$$\mathrm{Hom}_{\mathcal{C},\mathcal{E}}(M\otimes_{\mathcal{C}} N, Q)\simeq \mathrm{Hom}_{\mathcal{D},\mathcal{E}}(N,\mathrm{Hom}_{\mathcal{C}}(M,Q)),$$

where $\mathrm{Hom}_{\mathcal{C}}(M,Q)$ denotes the $\mathcal{D},\mathcal{E}$ bimodule sending z,y to $\mathrm{Hom}_{\mathcal{C}}(M_{.,z},Q_{.,y})$.
This isomorphism is natural and induces an adjunction.

Proof. Let $\eta : M\otimes_{\mathcal{C}} N\to Q$ be a natural transformation of functors. Let also $(x,y)\in\mathrm{Ob}(\mathcal{C}\times\mathcal{E}^{op})$, $c\in\mathcal{C}(x',x)$ and $e\in\mathcal{E}(y,y')$. We have a commutative square

$$\begin{tikzcd}
(M\otimes_{\mathcal{C}} N)_{x,y} \arrow[r, "\eta_{x,y}"] \arrow[d, "(M\otimes_{\mathcal{C}}N)(c,e)"'] & Q_{x,y} \arrow[d, "Q(c,e)"] \\
(M\otimes_{\mathcal{C}} N)_{x',y'} \arrow[r, "\eta_{x',y'}"'] & Q_{x',y'}
\end{tikzcd},$$

which we summarize in the following formula : $\eta(ca\otimes be) = c.\eta(a\otimes b).e$ for all $a\in M$ and $b\in N$.

Let now $b\in N_{z,y}$ for some $z\in\mathrm{Ob}(\mathcal{D})$ and some $y\in\mathrm{Ob}(\mathcal{E})$. For $x\in\mathrm{Ob}(\mathcal{C})$, we have a morphism

$$\begin{array}{rccc}\varphi(b)_x : & M_{x,z} & \longrightarrow & Q_{x,y}\\ & a & \longmapsto & \eta(a\otimes b)\end{array}$$

which induces a natural transformation $\varphi(b) : M_{.,z}\Rightarrow Q_{.,y}$.

Conversely, for $\psi\in\mathrm{Hom}_{\mathcal{D},\mathcal{E}}(N,\mathrm{Hom}_{\mathcal{C}}(M,Q))$, we have a natural transformation $\varepsilon : M\otimes_{\mathcal{C}} N\Rightarrow Q$ given by

$$\begin{array}{rccc}\varepsilon_{x,y} : & (M\otimes N)_{x,y} & \longrightarrow & Q_{x,y}\\ & a\otimes b & \longmapsto & \psi(b)_x(a)\end{array}$$

which gives the inverse bijection. □

Corollary A.1.10 (Tensor product is right exact). *Let N be a $\mathcal{D},\mathcal{E}$-bimodule, then the functor $-\otimes_{\mathcal{D}} N : \mathcal{C} - \boldsymbol{mod} - \mathcal{D} \to \mathcal{C} - \boldsymbol{mod} - \mathcal{E}$ is a right-exact functor.*

In particular we can define $\mathrm{Tor}_n^{\mathbb{Z}\mathcal{D}}(-,N)$ to be the n-th left-derived functor of the tensor product $-\otimes_{\mathcal{D}} N$.

Definition A.1.11 (Homology of a category). Let $\mathcal{C},\mathcal{D}$ be categories, and let M be a $\mathcal{C},\mathcal{D}$-bimodule. The n-th *homology group* of $\mathcal{C}$ with coefficients in M is defined as

$$H_n(\mathcal{C},M) := \mathrm{Tor}_n^{\mathbb{Z}\mathcal{C}}(\mathbb{Z},M).$$

It is endowed with a structure of (right)-$\mathcal{D}$-module.

In the case where $\mathcal{C}=G$ is a group, and A is a $\mathbb{Z}G$-module, we recover the classical definition of the homology of G with coefficients in A. In this case, one can check that $\mathbb{Z}\otimes_{\mathbb{Z}G}-$ is isomorphic to the *functor of coinvariants*, sending a $\mathbb{Z}G$-module A to

$$A_G := A \Big/ \langle g.a - a \mid g\in G, a\in A\rangle$$

(see for instance [Wei94, Section 6.1]).

Back to the general case, if we specialize the above definition to the case where $\mathcal{D}=\mathbb{Z}$, then the tensor product $\mathbb{Z}\otimes_{\mathbb{Z}\mathcal{C}} M$ is constructed by considering the quotient of the direct sum

$$P := \bigoplus_{z\in \mathrm{Ob}(\mathcal{C})} M_z$$

by all relations of the form $f.a = a$ for $a\in M_z$ and $f\in\mathcal{C}(-,z)$. We see that this construction is reminiscent of the functor of coinvariants in group homology.

Remark A.1.12. The arguments of [Wei94, Section 6.5] can be adapted to give a notion of "bar resolution" for categories. The bar resolution induces in turn a canonical resolution of the trivial $\mathbb{Z}\mathcal{C}$-module.

A.1.3 Category, groupoid and group

Let $\mathcal{C}$ be a category. We can always consider the enveloping groupoid $\mathcal{G}$ of $\mathcal{C}$ as in Definition 1.2.12. However, there is no general reason that the homology of $\mathcal{C}$ should be the same as that of $\mathcal{G}$, as in the following example:

Example A.1.13. ([McD79, Theorem 1]) Every path-connected space has the same weak homotopy type as the classifying space of some discrete monoid. This is false if we replace "monoid" by "group", since the classifying space of a discrete group is always a $K(\pi,1)$ space.

In this section, we fix a category $\mathcal{C}$, with enveloping groupoid $\mathcal{G}$.

As $\mathcal{G}$ and $\mathcal{C}$ have the same set of objects, one can consider $\mathbb{Z}\mathcal{G}$ as a $\mathcal{G},\mathcal{C}$-bimodule, sending a pair of objects (x,y) to $\mathbb{Z}\mathcal{G}(x,y)$.

We have a "scalar restriction" functor $\mathbb{Z}\mathcal{G}-\mathbf{mod}\to\mathbb{Z}\mathcal{C}-\mathbf{mod}$ coming from the canonical functor $\mathcal{C}\to\mathcal{G}$. We also have an "inversion of scalars" functor $\mathbb{Z}\mathcal{C}-\mathbf{mod}\to\mathbb{Z}\mathcal{G}-\mathbf{mod}$ given by $\mathbb{Z}\mathcal{G}\otimes_{\mathcal{C}}-$.

Lemma A.1.14. *The scalar inversion functor $\mathbb{Z}\mathcal{C} - \boldsymbol{mod} \to \mathbb{Z}\mathcal{G} - \boldsymbol{mod}$ is left-adjoint to the scalar restriction functor $\mathbb{Z}\mathcal{G} - \boldsymbol{mod} \to \mathbb{Z}\mathcal{C}$.*

Proof. This comes from the tensor-hom adjunction of Proposition A.1.9. Let M and Q respectively be a $\mathcal{C}$-module and a $\mathcal{G}$-module. We have

$$\mathrm{Hom}_{\mathcal{G}}(\mathbb{Z}\mathcal{G} \otimes_{\mathcal{C}} M, Q) \simeq \mathrm{Hom}_{\mathcal{C}}(M, \mathrm{Hom}_{\mathcal{G}}(\mathbb{Z}\mathcal{G}, Q)).$$

The $\mathcal{C}$-module $\mathrm{Hom}_{\mathcal{G}}(\mathbb{Z}\mathcal{G}, Q)$ is isomorphic to the scalar restriction functor (by the Yoneda Lemma). $\square$

This lemma is somewhat reminiscent of the Frobenius reciprocity. Except here, instead of inducing from a subgroup to an ambient group, we start from a category and we induce to its enveloping groupoid. This adjointness property implies in particular that the functor $\mathbb{Z}_{\mathcal{G}} \otimes_{\mathcal{C}} -$ is right-exact.

We want a condition under which the scalar inversion functor is not only right-exact (which always holds), but also left-exact. This would ensure that this functor preserves homology. A good such condition is given by the notion of an Ore category (Definition 1.3.2). This classical notion allows for a convenient description of the enveloping groupoid $\mathcal{G}(\mathcal{C})$ in terms of left-fractions (Theorem 1.3.5). This convenient description of the morphisms $\mathcal{G}(\mathcal{C})$ in turn has a remarkable consequence on modules of the form $\mathbb{Z}\mathcal{G}(x, -) \otimes_{\mathcal{C}} -$ for $x \in \mathrm{Ob}(\mathcal{C})$. This in turn induces the exactness of the scalar inversion functor $\mathbb{Z}\mathcal{G} \otimes_{\mathcal{C}} -$.

Proposition A.1.15. *Let $x \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is a left-Ore category, then the right $\mathcal{C}$-module $\mathbb{Z}\mathcal{G}(x, -) \otimes_{\mathcal{C}} -$ is a direct limit of free $\mathcal{C}$-modules. In particular it is a flat module.*

Proof. Our argument is a categorical rephrasing of the proof of [Squ94, Theorem 2.3].

For every morphism $f \in \mathcal{C}(y, x)$, precomposition by f^{-1} induces a morphism of right $\mathcal{C}$-module $\varphi_f : \mathbb{Z}\mathcal{C}(y, -) \to \mathbb{Z}\mathcal{G}(x, -)$. Since $\mathcal{C}$ is a left-Ore category, every morphism in $\mathcal{G}(x, -)$ can be described as a fraction. This means that $\mathcal{G}(x, -)$ is the union of the images of the morphisms φ_f for $f \in \mathcal{C}(-, x)$. Furthermore, the system given by the φ_f is a directed system because of the existence of left-multiples.

Just like in the usual case, the functor Tor commutes with inductive limits, which gives the flatness of $\mathbb{Z}\mathcal{G}(x, -)$. $\square$

Theorem A.1.16 (Exactness of scalar inversion functor). *Let $\mathcal{C}$ be a left-Ore category with enveloping groupoid $\mathcal{G}$. The $\mathcal{G}, \mathcal{C}$-bimodule $\mathbb{Z}\mathcal{G}$ is a flat $\mathcal{C}$-module.*

Proof. We already know that, as a left-adjoint, the functor $\mathbb{Z}\mathcal{G} \otimes_{\mathcal{C}} -$ is right-exact. We only have to show that it is left-exact, that is it preserves kernels.
Let M be a $\mathcal{C}$-module, the abelian group $\mathbb{Z}\mathcal{G}(x, -) \otimes_{\mathcal{C}} M$ is a quotient of the direct sum

$$\bigoplus_{y \in \mathcal{C}} \mathbb{Z}\mathcal{G}(x, y) \otimes_{\mathbb{Z}} M_y.$$

We see that the abelian group $\mathbb{Z}\mathcal{G}(x, -) \otimes_{\mathcal{C}} M$ is the image of x under the functor $\mathbb{Z}\mathcal{G} \otimes_{\mathcal{C}} M$. As kernels in a functor category are computed objectwise, the flatness of $\mathbb{Z}\mathcal{G}(x, -)$ induces the flatness of $\mathbb{Z}\mathcal{G}$ as claimed. $\square$

This is the exactness property we were looking for. A first consequence is that, under the assumption that $\mathcal{C}$ is left-Ore, the scalar inversion functor preserves homology.

Corollary A.1.17 (Scalar inversion preserves homology). *Assume that $\mathcal{C}$ is a left-Ore category. For every $\mathcal{C}$-module M and every $n \in \mathbb{Z}_{\geqslant 0}$ we have $H_n(\mathcal{C}, M) = H_n(\mathcal{G}(\mathcal{C}), \mathbb{Z}\mathcal{G} \otimes_{\mathcal{C}} M)$.*

Proof. Let M be a $\mathcal{C}$-module. By definition we have $H_n(\mathcal{C}, M) = \mathrm{Tor}_n^{\mathbb{Z}\mathcal{C}}(\mathbb{Z}, M)$. By Theorem A.1.16, this is equal to $\mathrm{Tor}_n^{\mathbb{Z}\mathcal{G}}(\mathbb{Z}, \mathbb{Z}\mathcal{G} \otimes_{\mathbb{Z}\mathcal{C}} M) = H_n(\mathbb{Z}\mathcal{G}, M)$. □

Now that we have equality between the homology of a category and that of its enveloping groupoid (under suitable assumptions), we want equality between the homology of a groupoid and that of a group to which it is equivalent as a category.

Let $\mathcal{G}$ be a groupoid, that we assume to be connected from now on. We also fix an object $x_0 \in \mathrm{Ob}(\mathcal{G})$ and set $G := \mathcal{G}(x_0, x_0)$.

We denote by ι the inclusion functor $G \to \mathcal{G}$. The choice, for every $x \in \mathrm{Ob}(\mathcal{G})$, of a morphism $u_x \in \mathcal{G}(x_0, x)$ induces a functor $\pi : \mathcal{G} \to G$, sending $f : x \to y$ to $u_x f u_y^{-1} \in G$. The functors ι and π are quasi-inverse equivalences of categories. Indeed $\pi \circ \iota$ is the identity morphism of G, and $\iota \circ \pi$ is fully faithful, and essentially surjective since $\mathcal{G}$ is a connected groupoid. The equivalences ι and π induce in turn equivalences between the categories of $\mathcal{G}$-modules and of G-modules.

In practice, if M is a G-module, then the induced $\mathcal{G}$-module sends every object to M and a morphism f acts by $u_x f u_y^{-1}$. We also denote this module by M.

Proposition A.1.18. *Let $\mathcal{G}$ be a connected groupoid, equivalent to a group G. For every G-module M and every $n \in \mathbb{Z}_{\geqslant 0}$ we have $H_n(G, M) = H_n(\mathcal{G}, M)$.*

Proof. The functor $\pi : \mathcal{G} \to G$ induces an equivalence of categories $\pi_* : \mathbb{Z}G - \mathbf{mod} \to \mathbb{Z}\mathcal{G}$, which is in particular an exact functor. For a G-module M, we get

$$\begin{aligned} H_n(G, M) &= \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M) \\ &= \mathrm{Tor}_n^{\mathbb{Z}\mathcal{G}}(\pi_*(\mathbb{Z}), \pi_*(M)) \\ &= \mathrm{Tor}_n^{\mathbb{Z}\mathcal{G}}(\mathbb{Z}, M) = H_n(\mathcal{G}, M), \end{aligned}$$

as claimed. □

A.2 The Dehornoy-Lafont order complex for categories

One of the uses of a Garside structure on a category that we did not discuss here before is that it gives rise to convenient resolutions allowing for the computation of the homology of the category (see [DDGKM, Section III.3.4]). These resolutions, are generalizations of previous works on Garside monoids (see [DL03] and [CMW04]).

However, the order complex of Dehornoy and Lafont, which we will call the Dehornoy-Lafont complex from now on, has not been adapted to the case of a category in [DDGKM]. This complex (in the case of a monoid) is smaller in size, so it is usually more suitable for practical purposes. We propose in this section a generalization of this complex to a categorical setting.

A.2.1 The complex

In the case of a monoid, the Dehornoy-Lafont complex was introduced in [DL03, Section 4] under the name "order resolution". It is based on a well-ordering of some generating system of the monoid. Although we plan to apply this new complex to a Garside category, the definition can be formulated in a slightly more general case mimicking the definition of Gaussian monoid. The proofs and construction are direct adaptations of the arguments of [DL03, Section 4] to the case of a category.

Definition A.2.1 (Gaussian category). [DDGKM, Definition II.2.9 and Definition II.2.20] Let $\mathcal{C}$ be a category, and let f, g be two morphisms with the same target. The category $\mathcal{C}$ admits *conditional left-lcms* if any two elements of $\mathcal{C}$ that admit a common left-multiple admit a left-lcm. Following [DL03, Section 1.1], a right-cancellative right-Noetherian category which admits conditional left-lcms is called *locally left-Gaussian.* If $\mathcal{C}$ furthermore admits left-lcms, $\mathcal{C}$ is called *left-Gaussian.*

Notice that Garside categories are particular cases of Gaussian categories. This is why we can apply the Dehornoy-Lafont complex to Garside categories.

In this section, we fix a locally left-Gaussian category $\mathcal{C}$, along with a finite set of morphisms $\mathcal{A}$, which generates $\mathcal{C}$. We also fix, for every object x of $\mathcal{C}$, a linear ordering $<$ on the set $\mathcal{A}(-,x)$ of elements of $\mathcal{A}$ with target x. In practice this amounts to fixing a linear ordering on the set $\mathcal{A}$, that we then restrict to the sets $\mathcal{A}(-,x)$.

For every morphism $f \in \mathcal{C}$, the set of elements of $\mathcal{A}$ dividing f on the right is ordered by $<$. Since $\mathcal{A}$ is finite, one can define $\mathrm{md}(f)$ to be the $<$-least right-divisor of f in $\mathcal{A}$.

Definition A.2.2 (Cells). Let n be an integer. an *n-cell* is an n-tuple $[\alpha_1, \ldots, \alpha_n]$ of elements of $\mathcal{A}$ sharing the same target such that $\alpha_1 < \ldots < \alpha_n$ and

$$\forall i \in [\![1, n]\!], \alpha_i = \mathrm{md}(\mathrm{lcm}(\alpha_i, \ldots, \alpha_n)).$$

We say that an n-cell $[\alpha_1, \ldots, \alpha_n]$ has source $x \in \mathrm{Ob}(\mathcal{C})$ if x is the source of $\mathrm{lcm}(\alpha_1, \ldots, \alpha_n)$. For $x \in \mathrm{Ob}(\mathcal{C})$, we define $(\mathcal{X}_n)_x$ to be the set of n-cells with source x.

We define C_n to be the free $\mathbb{Z}\mathcal{C}$-module associated to the family $\{(\mathcal{X}_n)_x\}_{x \in \mathrm{Ob}(\mathcal{C})}$

In particular we see that, for each object x, we have $(\mathcal{X}_0)_x = \{[\varnothing]\}$ and $(\mathcal{X}_1)_x = \mathcal{A}(x,-)$. To avoid confusion, we will alternatively denote by $[\varnothing]_x$ the only element of $(\mathcal{X}_0)_x$.

Let x be an object of $\mathcal{C}$. By definition of a free module over a category (see Section A.1.1), $(C_n)_x$ is generated as an abelian group by elements of the form $f[A]$, where $f \in \mathcal{C}(x,y)$ and $[A]$ is an n-cell with source y. We call such elements *elementary n-chains.*

Like in the case of a monoid, the following preodering on elementary n-chains will allow us to use induction arguments.

Definition A.2.3. We denote by A_1 the first element of a nonempty tuple $[A]$. Let $f[A]$ and $g[B]$ be elementary n-chains with same source. We say that $f[A] \sqsubset g[B]$ holds if we have either $f\,\mathrm{lcm}(A) \prec g\,\mathrm{lcm}(B)$, or $n > 0$, $f\,\mathrm{lcm}(A) = g\,\mathrm{lcm}(B)$ and $A_1 < B_1$.
If $\sum f_i[A_i]$ is an arbitrary n-chain, we say that $\sum f_i[A_i] \sqsubset g[B]$ holds if $f_i[A_i] \sqsubset g[B]$ holds for every i.

Lemma A.2.4. *For every n, the relation $\sqsubset$ on n-dimensional elementary chains with same source is compatible with multiplication on the left, and it has no infinite decreasing sequence.*

Proof. Assume $f[A] \sqsubset g[B]$, and let h be a morphism in $\mathcal{C}$. Then $f\,\mathrm{lcm}(A) \prec g\,\mathrm{lcm}(B)$ implies $hf[A] \prec hg[B]$, and $f\,\mathrm{lcm}(A) = g\,\mathrm{lcm}(B)$ implies $hf\,\mathrm{lcm}(A) = hg\,\mathrm{lcm}(B)$. We have $hf[A] \sqsubset hg[B]$ in each case.

Let now $\ldots \sqsubset f_2[A_2] \sqsubset f_1[A_1]$ be a decreasing sequence. We have a decreasing sequence $\ldots \preccurlyeq f_2\,\mathrm{lcm}(A_2) \preccurlyeq f_1\,\mathrm{lcm}(A_1)$. The left-Noetherianity of $\mathcal{C}$ gives that this sequence is stationary: there is some i_0 such that we have $f_i\,\mathrm{lcm}(A_i) = f_{i+1}\,\mathrm{lcm}(A_{i+1})$ for $i \geqslant i_0$. By definition of $\sqsubset$, we must have $(A_{i+1})_1 < (A_i)_1$ for $i \geqslant i_0$. The sequence $(A_j)_1$ for $j \geqslant i_0$ is $<$-decreasing. But $(A_i)_1$ right-divides $\mathrm{lcm}(A_i)$ and $f_i\,\mathrm{lcm}(A_i)$. So the sequence $(A_j)_1$ for $j \geqslant i_0$ is a $<$-decreasing sequence of divisors of $f_{i_0}\,\mathrm{lcm}(A_{i_0})$. The sequence $(A_j)_1$ thus is stationary, and the sequence $f_i[A_i]$ is stationary. □

We will now define the differential map $\partial_n : C_n \to C_{n-1}$ along with a contracting homotopy $s_n : C_n \to C_{n+1}$ and a so-called reduction map $r_n : C_n \to C_n$. The map ∂_n is $\mathbb{Z}\mathcal{C}$-linear, whereas s_n and r_n are only $\mathbb{Z}$-linear.

Definition A.2.5. Let $f[A]$ be an elementary chain. We say that $f[A]$ is *irreducible* if either $f[A] = 1_x[\varnothing]_x$ or $A_1 = \mathrm{md}(f\,\mathrm{lcm}(A))$. Otherwise we say that $f[A]$ is *reducible*.

The construction of ∂_*, r_* and s_* uses induction on n. The induction hypothesis, denoted (H_n) is the conjunction of the following two statements, where $r_n := s_{n-1} \circ \partial_n$:

$$
\begin{array}{ll}
(P_n) & \partial_n(r_n(f[A])) = \partial_n(f[A]) \\
(Q_n) & r_n(f[A]) \begin{cases} = f[A] & \text{if } f[A] \text{ is irreducible} \\ \sqsubset f[A] & \text{if } f[A] \text{ is reducible} \end{cases}
\end{array}
$$

In degree 0, the construction is usual and straightforward: we define $\partial_0 : C_0 \to \mathbb{Z}$ and $s_{-1} : \mathbb{Z} \to C_0$ by

$$\forall x \in \mathrm{Ob}(\mathcal{C}), \quad \partial_0([\varnothing]_x) := 1 \text{ and } s_{-1}(1) = [\varnothing]_x.$$

Lemma A.2.6. *Property (H_0) is satisfied.*

Proof. The mapping $r_0 := s_{-1} \circ \partial_0$ is $\mathbb{Z}$-linear with

$$r_0(f[\varnothing]_x) = s_1(\partial_0(f[\varnothing]_x)) = [\varnothing]_y$$

for $f \in \mathcal{C}(y, x)$. Hence we obtain

$$\partial_0(r_0(f[\varnothing]_x)) = \partial_0([\varnothing]_y) = 1, \quad \partial(f[\varnothing]_x) = f.1 = 1$$

because of the structure of the trivial $\mathcal{C}$-module $\mathbb{Z}$. Thus (P_0) holds. For (Q_0), we know that an elementary 0-chain $f[\varnothing]_x$ is irreducible if and only if $f = 1_x$, in which case we have $r_0(f[\varnothing]_x) = r_0([\varnothing]_x) = [\varnothing]_x$. Otherwise, we have $r_0(f[\varnothing]_x) = [\varnothing]_y \sqsubset f[\varnothing]_x$ by definition. Thus (Q_0) also holds. □

We now assume that (H_n) is satisfied, in particular we assume that both ∂_n and r_n have been constructed. We must now define

$$\partial_{n+1} : C_{n+1} \to C_n, \quad s_n : C_n \to C_{n+1}, \quad r_{n+1} = s_n \circ \partial_{n+1} : C_{n+1} \to C_{n+1}n$$

and show that (H_{n+1}) is satisfied. In the sequel, we use the notation $[\alpha, A]$ to write $(n+1)$-cells. By definition, saying that $[\alpha, A]$ is a $(n+1)$-cell amounts to saying that A is an n-cell and $\alpha = \mathrm{md}(\mathrm{lcm}(\alpha, A))$. We denote by $\alpha_{/A}$ the morphism defined by the equation $(\alpha_{/A})\,\mathrm{lcm}(A) = \mathrm{lcm}(\alpha, A)$.

Definition A.2.7 (Dehornoy-Lafont order complex).

- We define the morphism of $\mathbb{Z}\mathcal{C}$-module $\partial_{n+1} : C_{n+1} \to C_n$ by

$$\partial_{n+1}([\alpha, A]) := \alpha_{/A}[A] - r_n(\alpha_{/A}[A]).$$

- We inductively define the $\mathbb{Z}$-linear map $s_n : C_n \to C_{n+1}$ by

$$s_n(f[A]) := \begin{cases} 0 & \text{if } f[A] \text{ is irreducible,} \\ g[\alpha, A] + s_n(gr_n(\alpha_{/A}[A])) & \text{otherwise, with } \alpha = \operatorname{md}(f \operatorname{lcm}(A)) \text{ and } f = g\alpha_{/A}. \end{cases}$$

- Finally, we define $r_{n+1} : C_{n+1} \to C_{n+1}$ by $r_{n+1} = s_n \circ \partial_{n+1}$.

We can first notice that, under these definitions, ∂_{n+1}, s_n and r_n all preserve the source of n-cells.

The definition of ∂_{n+1} is direct (since r_n has been constructed in order to satisfy property (H_n)). The definition of s_n is inductive and we must check that it is well founded. Let $f[A]$ be a reducible chain. The chain $\alpha_{/A}[A]$ appearing in the definition of s_n is also reducible since $\alpha < A_1$ holds by definition. Thus (Q_n) gives $r_n(\alpha_{/A}[A]) \sqsubset \alpha_{/A}[A]$ and

$$gr_n(\alpha_{/A}[A]) \sqsubset g\alpha_{/A}[A] = f[A].$$

Since the relation $\sqsubset$ admits no infinite decreasing sequence, the definition of s_n (and of r_{n+1}) is then well founded.

We can also check that ∂_* induces a chain complex. Let $[\alpha, A]$ be a $(n+1)$-cell, we have

$$\partial_n\partial_{n+1}[\alpha, A] = \partial_n(\alpha_{/A}[A]) - \partial_n(r_n(\alpha_{/A}[A])) = 0$$

because of (P_n).

The following lemma is useful for showing that (H_n) implies (P_{n+1}), but it also contains the information that (s_n) provides a contracting homotopy for the Dehornoy-Lafont complex (see the proof of Proposition A.2.12).

Lemma A.2.8. *Let $f[A]$ be an elementary n-chain. Assuming (H_n) we have*

$$\partial_{n+1}s_n(f[A]) = f[A] - r_n(f[A]).$$

Proof. We use a $\sqsubset$-induction on $f[A]$. If $f[A]$ is irreducible, then by (Q_n) we have

$$\partial_{n+1}s_n(f[A]) = 0 = f[A] - r_n(f[A]).$$

Assume now that $f[A]$ is reducible. With the notation of Definition A.2.7, we obtain

$$\partial_{n+1}s_n(f[A]) = g\partial_{n+1}([\alpha, A]) - \partial_{n+1}s_n(gr_n(\alpha_{/A}[A])).$$

By (Q_n), we have $gr_n(\alpha_{/A}[A]) \sqsubset f[A]$, so the induction hypothesis gives us

$$\partial_{n+1}s_n(gr_n(\alpha_{/A}[A])) = gr_n(\alpha_{/A}[A]) - r_n(gr_n(\alpha_{/A}[A])).$$

Applying (P_n) we deduce

$$\begin{aligned} r_n(gr_n(\alpha_{/A}[A])) &= s_{n-1}(g\partial_n(r_n(\alpha_{/A}[A]))) \\ &= s_{n-1}(g\partial_n(\alpha_{/A}[A])) \\ &= r_n(g\alpha_{/A}[A]) = r_n(f[A]). \end{aligned}$$

And so

$$\partial_{n+1}s_n(f[A]) = g\alpha_{/A}[A] - gr_n(\alpha_{/A}[A] + gr_n(\alpha_{/A}[A]) - r_n(f[A]) = f[A] - r_n(f[A]),$$

as expected. □

Lemma A.2.9. *Assuming (H_n), (P_{n+1}) is satisfied.*

Proof. Let $[\alpha, A]$ be an elementary $(n+1)$-chain. We find

$$\begin{aligned} \partial_{n+1}(r_{n+1}(f[A])) &= \partial_{n+1}s_n\partial_{n+1}(f[A]) \\ &= \partial_{n+1}(f[A]) - r_n(\partial_{n+1}(f[A])) \\ &= \partial_{n+1}(f[A]) - s_{n-1}\partial_n\partial_{n+1}(f[A]) \\ &= \partial_{n+1}(f[A]). \end{aligned}$$

□

Before we show that (H_n) implies (H_{n+1}), we need the following lemma, which substantiates the behavior of s_n relative to $\sqsubset$.

Lemma A.2.10. *Let $f[\alpha, A]$ be a reducible $(n+1)$-chain. For each reducible n-chain $g[B]$ satisfying $g\,\mathrm{lcm}(B) \preccurlyeq f\,\mathrm{lcm}(\alpha, A)$, we have $s_n(g[B]) \sqsubset f[\alpha, A]$.*

Proof. We use $\sqsubset$-induction on $g[B]$. By definition we have

$$s_n(g[B]) = h[\beta, B] + s_n\left(\sum h_i[C_i]\right),$$

with $\beta = \mathrm{md}(g\,\mathrm{lcm}(B))$, $g\,\mathrm{lcm}(B) = h\,\mathrm{lcm}(\beta, B)$ and $\sum h_i[C_i] = hr_n(\beta_{/B}[B])$. Furthermore, since we know that $hr_n(\beta_{/B}[B]) \sqsubset g[B]$, we always have $h_i[C_i] \sqsubset g[B]$, hence in particular, $h_i\,\mathrm{lcm}(C_i) \preccurlyeq g\,\mathrm{lcm}(B) \preccurlyeq f\,\mathrm{lcm}(\alpha, A)$. The induction hypothesis gives $s_n(h_i[C_i]) \sqsubset f[\alpha, A]$ if $h_i[C_i]$ is reducible. If $h_i[C_i]$ is irreducible, then the contribution of $s_n(h_i[C_i]) = 0$ to the sum defining $s_n(hr_n(\beta_{/B}[B])$ is trivial. In both cases, it only remains to compare $h[\beta, B]$ and $f[\alpha, A]$.

Two cases are possible. Assume first $g\,\mathrm{lcm}(B) \prec f\,\mathrm{lcm}(\alpha, A)$. By construction, we have $h\,\mathrm{lcm}(\beta, B) = g\,\mathrm{lcm}(B)$, so we deduce $h\,\mathrm{lcm}(\beta, B) \prec f\,\mathrm{lcm}(\alpha, A)$ and therefore $h[\beta, B] \sqsubset f[\alpha, A]$.

Assume now $g\,\mathrm{lcm}(B) = f\,\mathrm{lcm}(\alpha, A)$. By construction, β is the $<$-least right-divisor of $g\,\mathrm{lcm}(B)$, hence of $f\,\mathrm{lcm}(\alpha, A)$. The hypothesis that $f[\alpha, A]$ is reducible means that α is a right-divisor of the latter element, but is not its least right-divisor, so we must have $\beta < \alpha$. This gives $h[\beta, B] \sqsubset f[\alpha, A]$ by definition. □

Lemma A.2.11. *Assuming (H_n), (H_{n+1}) is satisfied.*

Proof. By Lemma A.2.9, it only remains to prove (Q_{n+1}). Let $f[\alpha, A]$ be an elementary $(n+1)$-chain. By definition, we have

$$r_{n+1}(f[\alpha, A]) = s_n(f\alpha_{/A}[A]) - s_n\left(\sum g_i[B_i]\right),$$

with $\sum g_i[B_i] = fr_n(\alpha_{/A}[A])$. If $f[\alpha, A]$ is irreducible, then we have $\alpha = \mathrm{md}(x\,\mathrm{lcm}(\alpha, A))$. The definition of s_n gives

$$s_n(f\alpha_{/A}[A]) = f[\alpha, A] + s_n\left(\sum g_i[B_i]\right),$$

and we deduce $r_{n+1}(f[\alpha, A]) = f[\alpha, A]$.

Assume now that $f[\alpha, A]$ is reducible. First, we have $f\alpha_{/A}\,\mathrm{lcm}(A) = f\,\mathrm{lcm}(\alpha, A)$, so Lemma A.2.10 gives $s_n(f\alpha_{/A}[A]) \sqsubset f[\alpha, A]$. Since $\alpha = \mathrm{md}(\mathrm{lcm}(\alpha_{/A}, A) < A_1$, the chain $\alpha_{/A}[A]$ is reducible, so property (Q_n) gives $r_n(\alpha_{/A}[A]) \sqsubset \alpha_{/A}[A]$. Hence Lemma A.2.4 gives $xr_n(\alpha_{/A}[A]) \sqsubset f\alpha_{/A}[A]$, i.e, $g_i[B_i] \sqsubset f\alpha_{/A}[A]$ for all i. This implies in particular $g_i\,\mathrm{lcm}(B_i) \preccurlyeq f\alpha_{/A}\,\mathrm{lcm}(A) = f\,\mathrm{lcm}(\alpha, A)$. Applying Lemma A.2.10 to $g_i[B_i]$ gives $s_n(g_i[B_i]) \sqsubset f[\alpha, A]$. We deduce that $r_{n+1}(f[\alpha, A]) \sqsubset f[\alpha, A]$, which is property (Q_{n+1}). □

Thus, the induction hypothesis is maintained, and the construction can be carried out. We can now state the main result of this section: the Dehornoy-Lafont complex for a Gaussian category provides a free resolution of the trivial module.

Proposition A.2.12. *Let $\mathcal{C}$ be a locally left-Gaussian category. The complex (C_*, ∂_*) is a free resolution of the trivial $\mathbb{Z}\mathcal{C}$-module $\mathbb{Z}$.*

Proof. We already have seen that (C_*, ∂_*) is a complex of $\mathbb{Z}\mathcal{C}$-modules. The formula of Lemma A.2.8 rewrites into

$$\partial_{n+1}s_n + s_{n+1}\partial_n = 1,$$

which shows that s_* is a contracting homotopy. □

Combining this proposition and Theorem A.1.16 (exactness of scalar inversion functor), we get an analogue for groupoids. This is the result we will use in Section A.3.

Proposition A.2.13. *Let $\mathcal{C}$ be a cancellative left-Gaussian category. The complex $(\mathbb{Z}\mathcal{G}(\mathcal{C}) \otimes_{\mathbb{Z}\mathcal{C}} C_*, \partial_*)$ is a free resolution of the trivial $\mathbb{Z}\mathcal{G}$-module $\mathbb{Z}$.*

A.2.2 Reduction of computations

In this section, we fix $\mathcal{C}$ be a locally left-Gaussian category, along with a finite set $\mathcal{A}$ of morphisms of $\mathcal{C}$ which generates $\mathcal{C}$.

The Dehornoy-Lafont complex depends on the one hand on the structure of $\mathcal{C}$ as a category, and on the other hand on the linear order chosen on $\mathcal{A}$. Here we propose a computationally efficient solution for constructing an order on $\mathcal{A}$ yielding few 2-cells. This order is not optimal a priori (even for minimizing the number of 2-cells), but it gives good results in practice.

Let x be an object of $\mathcal{C}$. We first consider the set L_x of all elements of $\mathcal{C}(-, x)$ which are the lcm of a pair of distinct elements of $\mathcal{A}$. Our strategy is, for each $\ell \in L_x$, to try to reduce the number of two cells $[a, b]$ with $a \vee b = \ell$.

Let ℓ be in L_x. One can consider $\mathcal{A}_\ell$ the set of elements of $\mathcal{A}$ which right-divide ℓ. This set is included in $\subset \mathcal{A}(-,x)$ by definition. For $a \in \mathcal{A}_\ell$, we set $n(a,\ell)$ to be the cardinality of the following set:

$$\{b \in \mathcal{A}_\ell \mid a \vee b = \ell\}.$$

If a is the $<$-minimum of $\mathcal{A}_\ell$, then there are precisely $n(a,\ell)$ 2-cells of the form $[a,b]$ with $a \vee b = \ell$. In particular we deduce the following lemma:

Lemma A.2.14. *Let x be an object of $\mathcal{C}$. A lower (resp. upper) bound for the number of 2-cells made of elements of $\mathcal{A}$ with target x is given by*

$$\sum_{\ell \in L_x} \min_{a \in \mathcal{A}_\ell} n(a,\ell) \quad \left(\textit{resp.} \sum_{\ell \in L_x} \max_{a \in \mathcal{A}_\ell} n(a,\ell) \right).$$

In practice, these bounds may or may not be reached.

Definition A.2.15. Let x be an object of $\mathcal{C}$ and let $\ell \in L_x$. For $a \in \mathcal{A}_\ell$, the *condition* on $\mathcal{A}(-,x)$ associated to a and ℓ is the set-theoretic relation

$$\{(a,b) \mid b \in \mathcal{A}_\ell\} \subset \mathcal{A}(-,x) \times \mathcal{A}(-,x).$$

We say that such a condition is *optimal* if we furthermore have

$$n(a,\ell) = \min_{b \in \mathcal{A}_\ell} n(b,\ell).$$

We say that a family of conditions is *compatible* if their union is a subrelation of an order on $\mathcal{A}(-,x)$.

We can first check that the reflexive closure of a condition is always an order. It is also obvious that two different conditions associated to a same element $\ell \in L_x$ are never compatible: it would mean that $\mathcal{A}_\ell$ has two distinct minima.

Testing if a family of conditions is compatible only amounts to testing whether or not its reflexive transitive closure is antisymmetric. This is easy to test in practice due to the form of conditions as set-theoretic relations.

In order to get an adequate order, we try and find a maximal family of compatible conditions yielding few 2-cells. We propose the following procedure (a detailed `GAP` code is available at `https://github.com/ogarnier/dehornoy_lafont_computations.git`) :

1. Set $C := \varnothing$.
2. Compute $Comp(C)$ the set of conditions on $\mathcal{A}(-,x)$ which are compatible with C.
3. While $Comp(C) \neq \varnothing$ do
   - Choose a condition (a,ℓ) in $Comp(C)$ which minimizes the quantity

     $$n(a,\ell) - \min_{b \in \mathcal{A}_\ell} n(b,l).$$

   - Add (a,ℓ) to C.

The result of this procedure is a maximal family of compatible conditions. This family then induces an order which can be refined into a linear order over $\mathcal{A}(-,x)$. Since a choice is made at each step of this procedure, it is very hard to check whether or not the resulting order is optimal for minimizing the number of 2-cells. The condition chosen at the n-th step could for instance be incompatible with an optimal condition at the $n+1$-th step, which would force us to fall back on a less good condition. This would give us an order with more 2-cells than if we reversed the steps $n+1$ and n. Nevertheless, we will see in the next section that this procedure gives rather good results in practice.

Another computational issue arising from the Dehornoy-Lafont complex is the computation of the differential. Since this differential is defined recursively and using the auxiliary morphisms r_n and s_n, its calculation may lead to a lot of redundancy.

A first solution is to stock the results of ∂_n applied on the cells and then use the $\mathbb{Z}\mathcal{C}$-linearity of ∂_n. Unfortunately, one cannot do the same for s_n and r_n as they are not $\mathbb{Z}\mathcal{C}$-linear, but only $\mathbb{Z}$-linear. One would theoretically have to stock the results of r_n and s_n applied to every elementary chain, and not only to cells. But, in practice, r_n needs only be calculated on chains of the form $\alpha_{/A}[A]$, where $[\alpha, A]$ is a cell (see Definition A.2.7). So we can also store the results of r_n on chains of this form to avoid redundant computations.

A.3 Homology computations for exceptional complex braid groups

We are now going to use the Dehornoy-Lafont complex to compute the homology of exceptional complex braid groups.

As the construction of the Dehornoy-Lafont complex relies on some underlying Gaussian category (or monoid), we will use various Garside structures for exceptional braid groups.

Let W be a complex reflection group. Once we have computed the Dehornoy-Lafont complex attached to some Garside structure on $B(W)$, we will use it to compute the homology of $B(W)$ with coefficients in the following modules M:

- $M = \mathbb{Z}$ is the trivial $B(W)$-module.
- $M = \mathbb{Z}$, where the braid reflections of $B(W)$ act by -1. We denote this module by $\mathbb{Z}_\varepsilon$, we call it the *sign representation* of $B(W)$.
- $M = k[t, t^{-1}]$, where k is a field and the braid reflections of $B(W)$ act by t. We consider the case where $k = \mathbb{Q}$ or k is some finite field.

As pointed out in [Cal05], the homology $H_*(B(W), k[t, t^{-1}])$ where k is a field can be identified with the homology of the Milnor fiber of the singularity corresponding to W.

A.3.1 Isodiscriminantality

Let W and W' be two irreducible complex reflection groups. By the Chevalley-Shephard-Todd Theorem, one can choose a family of homogeneous polynomials $f_1, \dots, f_n$ such that the algebra $\mathbb{C}[X_1, \dots, X_n]^W$ of W-invariant polynomials is a polynomial algebra generated by $f_1, \dots, f_n$. The sequence $f_1, \dots, f_n$ is called a system of basic invariants for W. The polynomial map $f = (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ induces a map $\widehat{f} : \mathbb{C}^n/W \to \mathbb{C}^n$ which sends X/W to the complementary of an algebraic hypersurface $\mathcal{H}$. The hypersurface $\mathcal{H}$ is the image under f of the union of the reflecting hyperplanes.

Consider $W, W' \leqslant \mathrm{GL}_n(\mathbb{C})$ two complex reflection groups with two systems of basic invariants $f_1, \ldots, f_n$ and $f'_1, \ldots, f'_n$ for which the associated discriminant hypersurfaces are the same. This choice induces a homeomorphism between the associated regular orbit spaces, and an isomorphism $B(W) \simeq B(W')$ which sends braid reflections to braid reflections. This last point gives that the actions of $B(W)$ and $B(W')$ on $\mathbb{Z}_\varepsilon$ and $k[t, t^{-1}]$ are the same and we only need to compute the associated homology for one representative of the isodiscriminantality class.

A.3.2 Coxeter groups and Artin groups

The first case we are going to consider is that of complexified real reflection groups. We refer to [Bou81, Section IV.1] for classical results about Coxeter groups and real reflection groups.

Consider (W, S) a Coxeter system of spherical type. For $s, t \in S$, we denote by $m_{s,t}$ the order of st in W. The *Artin group* associated to (W, S) is defined by the following presentation:

$$A(W) := \langle S \mid \langle s, t\rangle^{m_{s,t}} = \langle t, s\rangle^{m_{t,s}} \ \forall s \neq t \rangle ,$$

where $\langle x, y\rangle^m$ denotes the product $xyxy...$ with m terms.

It is known (see [Bri71]) that the braid group of W seen as a complex reflection group is isomorphic to $A(W)$ (and this isomorphism sends the elements of S to braid reflections).

The presentation of $A(W)$ also gives rise to a monoid, denoted by $M(W)$. The monoid $M(W)$ is always locally left-Gaussian, and since (W, S) is of spherical type, it is Gaussian (see [DP99, Example 1]). The monoid $M(W)$ is the *Artin monoid* associated to the Coxeter system (W, S).

The homology of Artin monoids has already been studied by Salvetti (see [Sal94]), and by Squier (see [Squ94]). The approach of the latter was then generalized in [DL03, Section 4] into the order complex.

As we want to use solely the Dehornoy-Lafont complex in our computations, we give its construction in the case of a Coxeter group.

If S' is a subset of S, then the subgroup W' generated by S' in W is also a spherical Coxeter group. The Artin monoid M' associated to (W', S') is the submonoid generated by S' in M. In particular the right-lcm of the elements of S' lies inside M'. So the atoms right-dividing this lcm are precisely the elements of S'. We get the following lemma:

Lemma A.3.1. *Let (S, W) be a Coxeter system of spherical type, with $S = \{s_1, \ldots, s_n\}$. For k be a positive integer, the k-cells for the Dehornoy-Lafont complex associated to (S, W) are given by*

$$X_k := \{[s_{i_1}, \ldots, s_{i_k}] \mid 1 \leqslant i_1 < \ldots < i_k \leqslant n\} .$$

In particular the cardinality of X_k is $\binom{n}{k}$ and does not depend on the choice of an order of the atoms.

This is an extreme case where the tools of subsection A.2.2 don't apply. Indeed the lcm of two distinct atoms s, t is always of the form $\ell = \langle s, t\rangle^{m_{s,t}}$, with $\mathcal{A}_\ell = \{s, t\}$. This means that the two bounds of Lemma A.2.14 are equal in this case.

The classical Artin monoid has the advantage of providing relatively few cells. Even for the largest case (which is $E_8 \simeq G_{37}$), we have at most $\binom{8}{4} = 70$ cells (in rank 4). On the other hand, the differential is often long to compute because of recursion: elements of the form $\alpha_{/A}$ may have great length, because the simple elements in the Artin monoids can have length up to 120 in the case of E_8.

As seen in the tables below, the case of the Artin monoids covers the exceptional groups which are complexified real reflection groups:

$$B(G_{23}) \simeq A(H_3),\ B(G_{28}) \simeq A(F_4),\ B(G_{30}) \simeq A(H_4),$$
$$B(G_{35}) = A(E_6),\ B(G_{36}) = A(E_7),\ B(G_{37}) \simeq A(E_8).$$

Furthermore, some exceptional groups are known to be isodiscriminantal to complexified real reflection groups (see [OS88a, Theorem 2.25] and [Ban76, Section 2]). Therefore the study of Artin monoids also gives the homology of the following groups:

$$B(G_4),\ B(G_5),\ B(G_6),\ B(G_8),\ B(G_9),\ B(G_{10}),\ B(G_{14}),\ B(G_{16}),$$
$$B(G_{17}),\ B(G_{18}),\ B(G_{20}),\ B(G_{21}),\ B(G_{25}),\ B(G_{26}),\ B(G_{32}).$$

A.3.3 Exceptional groups of rank two

The next case we consider is that of exceptional groups of rank two. Some of these groups are known to be isodiscriminantal to complexified real groups, which we already considered in Section A.3.2. This leaves the following groups:

$$B(G_7),\ B(G_{11}),\ B(G_{12}),\ B(G_{13}),\ B(G_{15}),\ B(G_{19}),\ B(G_{22}).$$

Among these, $B(G_7)$, $B(G_{11})$ and $B(G_{19})$ are isodiscriminantal (see [Ban76, Section 2]). In order to study these various groups, we use *ad hoc* Garside monoids, which are all detailed in [Pic00, Examples 11, 12, 13]. Most of these monoids are *circular monoids*:

$$B(G_7) \simeq B(G_{11}) \simeq B(G_{19} = \langle a, b, c \mid abc = bca = cab\rangle,$$

$$B(G_{12}) = \langle a, b, c \mid abca = bcab = cabc\rangle,$$
$$B(G_{22}) = \langle a, b, c \mid abcab = bcabc = cabca\rangle.$$

These are group presentations, which can also be seen as monoid presentations. We amalgamate such a group presentation and the underlying monoid presented by the same data. In these three monoids, one can see that the atoms play a symmetric role: changing the ordering on the atoms does not affect the number of cells. Furthermore, as the lcm of two distinct atoms is always the same, we get that there are only 1-cells and 2-cells (n cells of rank 1 and $n-1$ cells of rank 2 if n is the number of atoms).

For $B(G_{13})$, we use the following monoid:

$$B(G_{13}) = \langle a, b, c \mid acabc = bcaba,\ bcab = cabc,\ cabca = abcab\rangle.$$

Using the notations of Section A.2.2, we have

$$\ell_1 := b \vee c = bcab,\ \ell_2 := a \vee b = a \vee c = abcab.$$

For ℓ_1, we have $\mathcal{A}_{\ell_1} = \{b, c\}$ and $n(b, \ell_1) = n(c, \ell_1) = 1$, so there is no use in setting either $b < c$ or $c < b$. For ℓ_2, we have $\mathcal{A}_{\ell_2} = \{a, b, c\}$ and $n(a, \ell_2) = 2, n(b, \ell_2) = n(c, \ell_2) = 1$. So by considering the order $c < a < b$, we get one 1-cell, two 2-cells, and zero 3-cells. If we instead consider the order $a < b < c$, we get three 2-cells instead of two.

Lastly, for $B(G_{15})$, we use the monoid

$$B(G_{15}) = \langle a, b, c \mid abc = bca,\ cabcb = abcbc \rangle \,.$$

We have

$$\ell_1 = a \vee c = abc, \quad \ell_2 = b \vee c = b \vee a = abcbc.$$

For ℓ_1, we have $\mathcal{A}_{\ell_1} = \{a, c\}$ and $n(a, \ell_1) = n(c, \ell_1) = 1$, so there is again no use in setting a priori that either $a < c$ or $c < a$. For ℓ_2, we have $\mathcal{A}_{\ell_2} = \{a, b, c\}$ and $n(a, \ell_2) = n(c, \ell_2) = 1$, $n(b, \ell_2) = 2$. So we consider the order $c < a < b$ in order to get as few cells as possible.

A.3.4 Well-generated exceptional groups

At this point, we still have six groups to consider, and five of them are "well-generated" in the sense of [Bes15, Section 2]:

$$B(G_{24}),\ B(G_{27}),\ B(G_{29}),\ B(G_{33}),\ B(G_{34}).$$

The monoid we use for these groups is the *dual braid monoid* (see [Bes15, Section 8]). The main problem with these monoids is that they have many atoms, and so they give rise to relatively big complexes. This is where the methods of Section A.2.2 are most useful. The bounds of Lemma A.2.14 for the number of 2-cells are respectively given by

	$B(G_{24})$	$B(G_{27})$	$B(G_{29})$	$B(G_{33})$	$B(G_{34})$
Lower bound	38	60	120	213	630
Upper bound	40	65	158	302	1071

Applying the method of Section A.2.2 to get a convenient order, we see in Table A.1 that the lower bounds are not always reached, but we obtain complexes which are quite smaller than the ones in [Mar17, Table 1], especially for large groups.

		0-cells	1-cells	2-cells	3-cells	4-cells	5-cells	6-cells
$B(G_{24})$	[Mar17]	1	14	38	25			
	Optimized	1	14	38	25			
$B(G_{27})$	[Mar17]	1	20	62	43			
	Optimized	1	20	60	41			
$B(G_{29})$	[Mar17]	1	25	127	207	108		
	Optimized	1	25	125	209	108		
$B(G_{33})$	[Mar17]	1	30	226	638	740	299	
	Optimized	1	30	223	616	705	283	
$B(G_{34})$	[Mar17]	1	56	711	3448	7520	7414	2686
	Optimized	1	56	646	2839	5691	5255	1812

Table A.1: Compared size of the Dehornoy-Lafont complexes

Sadly, although we obtain a smaller complex for $B(G_{34})$, it is not small enough to obtain the results that were missing in [CM14] regarding $H_*(B(G_{34}), \mathbb{Q}[t, t^{-1}])$. However, we were able to compute $H_*(B(G_{34}), k[t, t^{-1}])$ where k ranges among some finite fields. These computations give us a reasonable conjecture regarding $H_*(B(G_{34}), \mathbb{Q}[t, t^{-1}])$.

A.3.5 The complex braid group $B(G_{31})$

The last exceptional group to consider is $B(G_{31})$. Although this group does not appear (to our knowledge) as a group of fraction of some Gaussian monoid, it is equivalent to the enveloping groupoid of some Garside category. Indeed the complex reflection group G_{31} appears as the centralizer of some regular element (in the sense of Springer) in the Coxeter group E_8. Following the work of Bessis in [Bes15, Section 11], this description gives rise to a Garside category $\mathcal{C}_{31}$, whose enveloping groupoid $\mathcal{B}_{31}$ is equivalent to $B(G_{31})$. A detailed description of this category can be found in [Gar23b].

This category admits 88 objects, 660 atoms, and a total of 2603 simple morphisms (excluding the identities). The first possible approach used to study the homology of this category (see [CM14, Section 5.3]) was to construct the Charney-Meier-Whittlesey complex for this category (as defined in [Bes07, Section 7]). Sadly this complex is too large to be dealt with without a strong computational power.

We give in Table A.2 the size of the Charney-Meier-Whittlesey complex for $\mathcal{C}_{31}$ compared to the size of the Dehornoy-Lafont complex (note that in this case, the bounds given by Lemma A.2.14 are 1655 and 1845, respectively):

	0-cells	1-cells	2-cells	3-cells	4-cells
CMW	88	2603	11065	15300	6750
DL	88	660	1665	1735	642

Table A.2: Compared size of the complexes for $\mathcal{C}_{31}$

Let M be one of the $B(G_{31})$ modules we are considering. We first extend M to a $\mathcal{B}_{31}$ module, using the construction of Section A.1.3. We then restrict this module to the category $\mathcal{C}_{31}$. In the case of $k[t,t^{-1}]$, the matrix we obtain may contain coefficients in $k[t,t^{-1}] \setminus k[t]$, as opposed to the case of a monoid, in which the action gives rise to matrices in $k[t]$. In theory this is not a problem since $k[t,t^{-1}]$ is a principal ideal domain. But in practice, it is far simpler to look for the Smith normal form of a matrix in $k[t]$. To avoid this issue, we multiply our matrices by a big enough power of t (which is an invertible element in $k[t,t^{-1}]$) to obtain matrices in $k[t]$. We only then need to divide the elementary divisors we obtain by powers of t if need be.

A.3.6 Computational results

We use the notation $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$. The computations for the complexes and the differentials were made on the `CHEVIE` package for `GAP3` ([CHE]). The computations of the Smith normal forms were made using the softwares `Macaulay2` ([MAC]) and `MAGMA` ([MAG]).

For each row, we indicate the representative of the isodiscriminantality class of which we computed the homology.

The results in Table A.3 are not new. The case of complexified real reflection groups is already known from [Sal94]; the case of complex reflection groups which are not isodiscriminantal to groups in the infinite series is given in [CM14] and [Mar17]. We reproduce their results here for the convenience of the reader

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$A_2 \sim G_4$, G_8, G_{16}	$\mathbb{Z}$	$\mathbb{Z}$	0						
$I_2(4) \sim G_5$, G_{10}, G_{18}	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
$I_2(6) \sim G_6$, G_9, G_{17}	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
G_7, G_{11}, G_{19}	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$						
G_{12}	$\mathbb{Z}$	$\mathbb{Z}$	0						
G_{13}	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
$I_2(8) \sim G_{14}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
G_{15}	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$						
$I_2(5) \sim G_{20}$	$\mathbb{Z}$	$\mathbb{Z}$	0						
$I_2(10) \sim G_{21}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$						
G_{22}	$\mathbb{Z}$	$\mathbb{Z}$	0						
$G_{23} = H_3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$					
G_{24}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$					
$A_3 \sim G_{25}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0					
$B_3 \sim G_{26}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$					
G_{27}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_3 \times \mathbb{Z}$	$\mathbb{Z}$					
$G_{28} = F_4$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$				
G_{29}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}$				
$G_{30} = H_4$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$				
G_{31}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z}$	$\mathbb{Z}$				
$A_4 \sim G_{32}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$				
G_{33}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}$	$\mathbb{Z}$			
G_{34}	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_3^2 \times \mathbb{Z}_6$	$\mathbb{Z}_3^2 \times \mathbb{Z}$	$\mathbb{Z}$		
$G_{35} = E_6$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$	0		
$G_{36} = E_7$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6^2$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z}$	$\mathbb{Z}$	
$G_{37} = E_8$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}$	$\mathbb{Z}$

Table A.3: Homology of exceptional braid groups in $\mathbb{Z}$ (after Salvetti, Callegaro and Marin)

In the same vein, Table A.4 gives the homology of exceptional braid groups with coefficients in the sign representation. This homology has already been studied in [Sal94] for complexified real reflection groups and in [Mar17] for the exceptional groups $B(G_{12}), B(G_{13}), B(G_{22}), B(G_{24}), B(G_{27})$, $B(G_{29})$, $B(G_{33}), B(G_{34})$. The first homology group was studied for all complex reflection group in [CM14, Section 7.2]. We restate the results of [Sal94, Table 2] and [Mar17, Table 3] among the results for other exceptional groups (we frame the results which are new).

Lastly, we give in Table A.5 the homology with coefficients in the representation $k[t, t^{-1}]$. In the case $k = \mathbb{Q}$, this was already studied in [Sal94] for complexified real exceptional reflection groups; in [CPS01] for real reflection groups of type A; in [CPSS99] for real reflection groups of type B; and in [Mar17] for the exceptional groups $B(G_{12}), B(G_{13}), B(G_{22}), B(G_{24}), B(G_{27}), B(G_{29})$, $B(G_{33})$, $B(G_{34})$, although the results are incomplete for this last group. We let $\Phi_n \in \mathbb{Z}[t]$ denote the n-th cyclotomic polynomial. In the Table, for each $P \in \mathbb{Q}[t, t^{-1}]$, the presence of P in the Table symbolizes the $\mathbb{Q}[t, t^{-1}]$ module $\mathbb{Q}[t, t^{-1}]/(P)$, and $\mathbb{Q}$ is a shortcut for $\mathbb{Q}[t, t^{-1}]/(t - 1)$.

Note that our apparent results regarding the groups studied in [Mar17] differ from [Mar17, Table 4] because of a slight mistake in the latter: for all groups except $B(G_{13})$, the results should be "shifted to the left", for instance $\Phi_6 \oplus \Phi_{12}$ is not $H_2(B(G_{12}), \mathbb{Q}[t, t^{-1}])$, but rather $H_1(B(G_{12}), \mathbb{Q}[t, t^{-1}])$.

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$A_2 \sim G_4,\, G_8,\, G_{16}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	0						
$I_2(4) \sim G_5,\, G_{10},\, G_{18}$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0						
$I_2(6) \sim G_6,\, G_9,\, G_{17}$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	0						
$G_7,\, G_{11},\, G_{19}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\boxed{0}$						
G_{12}	$\mathbb{Z}_2$	$\mathbb{Z}_3$	0						
G_{13}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0						
$I_2(8) \sim G_{14}$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0						
G_{15}	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\boxed{0}$						
$I_2(5) \sim G_{20}$	$\mathbb{Z}_2$	$\mathbb{Z}_5$	0						
$I_2(10) \sim G_{21}$	$\mathbb{Z}_2$	$\mathbb{Z}_{10}$	0						
G_{22}	$\mathbb{Z}_2$	0	0						
$G_{23} = H_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0					
G_{24}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0					
$A_3 \sim G_{25}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0					
$B_3 \sim G_{26}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0					
G_{27}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0					
$G_{28} = F_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{24}$	0				
G_{29}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_{40}$	0				
$G_{30} = H_4$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	0				
G_{31}	$\mathbb{Z}_2$	0	$\boxed{\mathbb{Z}_6}$	$\boxed{\mathbb{Z}_{20}}$	$\boxed{0}$				
$A_4 \sim G_{32}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_5$	0				
G_{33}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0			
G_{34}	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{252}$	0		
$G_{35} = E_6$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_9$	0		
$G_{36} = E_7$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
$G_{37} = E_8$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{240}$	0

Table A.4: Homology of exceptional braid groups in $\mathbb{Z}_\varepsilon$ (framed results are new).

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$A_2 \sim G_4$, G_8, G_{16}	$\mathbb{Q}$	Φ_6	0						
$I_2(4) \sim G_5$, G_{10}, G_{18}	$\mathbb{Q}$	$\Phi_1\Phi_4$	0						
$I_2(6) \sim G_6$, G_9, G_{17}	$\mathbb{Q}$	$\frac{t^6-1}{t+1}$	0						
G_7, G_{11}, G_{19}	$\mathbb{Q}$	$\boxed{\mathbb{Q} \oplus (t^3-1)}$	$\boxed{0}$						
G_{12}	$\mathbb{Q}$	$\Phi_6\Phi_{12}$	0						
G_{13}	$\mathbb{Q}$	$\Phi_1\Phi_9$	0						
$I_2(8) \sim G_{14}$	$\mathbb{Q}$	$\frac{t^8-1}{t+1}$	0						
G_{15}	$\mathbb{Q}$	$\boxed{\mathbb{Q} \oplus t^5-1}$	$\boxed{0}$						
$I_2(5) \sim G_{20}$	$\mathbb{Q}$	Φ_{10}	0						
$I_2(10) \sim G_{21}$	$\mathbb{Q}$	$\frac{t^{10}-1}{t+1}$	0						
G_{22}	$\mathbb{Q}$	Φ_{15}	0						
$G_{23} = H_3$	$\mathbb{Q}$	0	$\frac{t^5-1}{t-1}\Phi_3$	0					
G_{24}	$\mathbb{Q}$	0	$\Phi_1\Phi_3\Phi_7$	0					
$A_3 \sim G_{25}$	$\mathbb{Q}$	Φ_6	Φ_4	0					
$B_3 \sim G_{26}$	$\mathbb{Q}$	$\mathbb{Q}$	t^3-1	0					
G_{27}	$\mathbb{Q}$	0	$(t^{15}-1) \oplus \Phi_3$	0					
$G_{28} = F_4$	$\mathbb{Q}$	$\mathbb{Q}$	$\frac{t^6-1}{t+1}$	$\frac{t^{12}-1}{t+1}\Phi_8$	0				
G_{29}	$\mathbb{Q}$	0	$\Phi_4 \oplus \Phi_4$	$\frac{t^{20}-1}{t+1} \oplus \Phi_4$	0				
$G_{30} = H_4$	$\mathbb{Q}$	0	0	$\frac{t^{30}-1}{t+1}\Phi_4\Phi_{12}\Phi_{20}$	0				
G_{31}	$\mathbb{Q}$	$\boxed{0}$	$\boxed{\Phi_6}$	$\boxed{\frac{t^{10}-1}{t+1}\Phi_{15}}$	$\boxed{0}$				
$A_4 \sim G_{32}$	$\mathbb{Q}$	0	Φ_4	Φ_{10}	0				
G_{33}	$\mathbb{Q}$	0	0	0	$(t^9-1)\Phi_5$	0			
G_{34}	$\mathbb{Q}$	0	Φ_6	?	?	?	?		
$G_{35} = E_6$	$\mathbb{Q}$	0	0	0	$\Phi_3\Phi_8$	$\frac{t^{12}-1}{t^4-1}\Phi_{18}$	0		
$G_{36} = E_7$	$\mathbb{Q}$	0	0	0	Φ_3	Φ_3	$(t^9-1)\Phi_7$	0	
$G_{37} = E_8$	$\mathbb{Q}$	0	0	0	Φ_4	0	$\Phi_8\Phi_{12}$	$\frac{t^{30}-1}{t+1}\frac{t^{24}-1}{t^6-1}\Phi_{20}$	0

Table A.5: Homology of exceptional braid groups in $\mathbb{Q}[t, t^{-1}]$ (framed results are new).

We finish with the case of a finite field k. Following [Mar17], we restrict our attention to the case $k = \mathbb{F}_p$ with $p \in \{2, 3, 5, 7\}$. We denote by $\phi_{n,k}$ the n-th cyclotomic polynomial with coefficients in k. As $\Phi_n := \phi_{n,\mathbb{Q}}$ lies in $\mathbb{Z}[X]$, we can consider its image in $\mathbb{F}_p[X]$ for some prime p. We also denote this polynomial by Φ_n. It is well known that $\Phi_n = \phi_{n,\mathbb{F}_p} \bmod p$ if p does not divide n. Furthermore, if p does not divide n, then we have

$$\forall r > 0,\ \Phi_{np^r} \equiv (\Phi_n)^{p^r - p^{r-1}} = (\phi_{n,\mathbb{F}_p})^{p^r - p^{r-1}} \bmod p,$$

as stated in [Gue68]. Most of the time, the homology $H_*(B(G_i), \mathbb{F}_p[t, t^{-1}])$ is given by the same polynomials as $H_*(B(G_i), \mathbb{Q}[t, t^{-1}])$, reduced mod p. For instance we have, in the case of $B(G_{12})$:

$$H_1(B(G_{12}), \mathbb{F}_2[t, t^{-1}]) = \mathbb{F}_2[t, t^{-1}]/(P),$$

where

$$P(X) = X^6 - X^5 + X^3 - X + 1 = (X^2 + X + 1)^3 = \Phi_6(X)\Phi_{12}(X) \equiv \phi_{3,\mathbb{F}_2}(X)^3 \bmod 2.$$

We list here the cases where $H_*(B(G_i), \mathbb{F}_p[t, t^{-1}])$ is not given by the reduction modulo p of the polynomials giving $H_*(B(G_i), \mathbb{Q}[t, t^{-1}])$. We first consider the case $W \neq G_{34}$. The results for $W \in \{G_{24}, G_{29}, G_{33}\}$ already appear in [Mar17].

- When $W = G_{29}$, we have
  - $H_3(B(G_{29}), \mathbb{F}_2[t, t^{-1}]) = (t + 1)^3 \oplus \Phi_4$.
  - $H_4(B(G_{29}), \mathbb{F}_2[t, t^{-1}]) = (t^{20} - 1) \oplus \Phi_4$.
- When $W = G_{30}$, we have $H_3(B(G_{30}), \mathbb{F}_2[t, t^{-1}]) = (t^{30} - 1)\Phi_4\Phi_{12}\Phi_{20}$.
- When $W = G_{31}$, we have
  - $H_2(B(G_{31}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1\Phi_6$ and $H_2(B(G_{31}), \mathbb{F}_3[t, t^{-1}]) = \Phi_1\Phi_6$.
  - $H_3(B(G_{31}), \mathbb{F}_2[t, t^{-1}]) = (t^{10} - 1)\Phi_{15}$ and $H_3(B(G_{31}), \mathbb{F}_3[t, t^{-1}]) = \frac{t-1}{t+1}(t^{10} - 1)\Phi_{15}$.
- When $W = G_{33}$, we have
  - $H_2(B(G_{33}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$ and $H_2(B(G_{33}), \mathbb{F}_3[t, t^{-1}]) = \Phi_1$.
  - $H_3(B(G_{33}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$ and $H_3(B(G_{33}), \mathbb{F}_3[t, t^{-1}]) = \Phi_1$.
- When $W = G_{35}$, we have $H_2(B(G_{35}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$ and $H_3(B(G_{35}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$.
- When $W = G_{36}$, we have
  - $H_2(B(G_{36}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$.
  - $H_3(B(G_{36}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1 \oplus \Phi_1$.
  - $H_4(B(G_{36}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1 \oplus (t^3 + 1)$ and $H_4(B(G_{36}), \mathbb{F}_3[t, t^{-1}]) = t^3 - 1$.
  - $H_5(B(G_{36}), \mathbb{F}_2[t, t^{-1}]) = t^3 + 1$, and $H_5(B(G_{36}), \mathbb{F}_3[t, t^{-1}]) = t^3 - 1$.
- When $W = G_{37}$, we have
  - $H_2(B(G_{37}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$.
  - $H_3(B(G_{37}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1$.
  - $H_4(B(G_{37}), \mathbb{F}_2[t, t^{-1}]) = \Phi_1\Phi_4$ and $H_4(B(G_{37}), \mathbb{F}_3[t, t^{-1}]) = \Phi_1\Phi_4$.

- $H_5(B(G_{37}), \mathbb{F}_2[t,t^{-1}]) = \Phi_1$ and $H_5(B(G_{37}), \mathbb{F}_3[t,t^{-1}]) = \Phi_1 \oplus \Phi_1$.
- $H_6(B(G_{37}), \mathbb{F}_2[t,t^{-1}]) = \Phi_1\Phi_8\Phi_{12}$ and $H_6(B(G_{37}), \mathbb{F}_3[t,t^{-1}]) = \Phi_1\Phi_8\Phi_{12}$.
- $H_7(B(G_{37}), \mathbb{F}_2[t,t^{-1}]) = \frac{(t^{30}-1)(t^{24}-1)}{(t^6-1)}\Phi_{20}$.

We notice that, in all of these cases, the homology $H_*(B(G_i), \mathbb{F}_p[t,t^{-1}])$ for $p \in \{5,7\}$ is given by the reduction modulo p of the polynomials giving $H_*(B(G_i), \mathbb{Q}[t,t^{-1}])$. This allows us to give a conjecture about the values of $H_*(B(G_{34}), \mathbb{Q}[t,t^{-1}])$ which we were not able to compute.

The homology $H_*(B(G_{34}), \mathbb{F}_p[t,t^{-1}])$ for $p \in \{2,3\}$ was computed in [Mar17]. The results are in Table A.6.

G_{34}	H_0	H_1	H_2	H_3	H_4	H_5	H_6
$\mathbb{F}_2[t,t^{-1}]$	$\mathbb{F}_2$	0	(t^3-1)	$\mathbb{F}_2$	$\Phi_3 \oplus \Phi_3 \oplus (t^3-1)$	$(t^{42}-1) \oplus \Phi_3 \oplus \Phi_3$	0
$\mathbb{F}_3[t,t^{-1}]$	$\mathbb{F}_3$	0	$\mathbb{F}_3 \oplus \Phi_6$	$\mathbb{F}_3$	$\Phi_3 \oplus \Phi_3 \oplus \Phi_3 \oplus \Phi_2$	$(t^{42}-1) \oplus \Phi_3 \oplus \Phi_3$	0

Table A.6: Homology of $B(G_{34})$ with coefficients in $\mathbb{F}_2[t,t^{-1}]$ and $\mathbb{F}_3[t,t^{-1}]$ (after Marin).

We were able to compute the homology $H_*(B(G_{34}), \mathbb{F}_p[t,t^{-1}])$ for all primes $5 \leqslant p \leqslant 97$. For each of these cases, we see in Table A.7 that the homology is given by the same polynomials:

G_{34}	H_0	H_1	H_2	H_3	H_4	H_5	H_6
$\mathbb{F}_p[t,t^{-1}]$	$\mathbb{F}_p$	0	Φ_6	0	$\Phi_3 \oplus \Phi_3 \oplus \Phi_3$	$\frac{(t^{42}-1)}{t+1} \oplus \Phi_3 \oplus \Phi_3$	0

Table A.7: Homology of $B(G_{34})$ with coefficients in $\mathbb{F}_p[t,t^{-1}]$ (for primes p between 5 and 97).

We obtain a conjecture about $H_*(B(G_{34}), \mathbb{Q}[t,t^{-1}])$, which we state in Table A.8.

G_{34}	H_3	H_4	H_5	H_6
$\mathbb{Q}[t,t^{-1}]$	0	$\Phi_3 \oplus \Phi_3 \oplus \Phi_3$	$\frac{t^{42}-1}{t+1} \oplus \Phi_3 \oplus \Phi_3$	0

Table A.8: Conjectural homology of $B(G_{34})$ with coefficients in $\mathbb{Q}[t,t^{-1}]$.

Appendix B

Circular and hosohedral-type Garside groups

In this chapter, we consider a particular class of Garside groups, which covers complex braid groups of rank 2, and which we call circular groups. We mainly prove that roots are unique in these groups, up to conjugacy. We also consider a generalization of circular groups, called hosohedral-type groups. These groups are defined using circular groups, and a procedure called the Δ-product, which we study in generality. This chapter is taken (except B.1.3) from my third paper [Gar24c].

Contents

B.1 Circular monoids

In this section we define circular monoids by their (monoid) presentations. These monoids already appeared in [DP99, Example 5], where the authors showed that they were Garside monoids. Here we propose an in-depth study of their Garside properties.

B.1.1 Definition, first properties

Let m, ℓ be two positive integers that we fix throughout this section. Let also $\{a_0, \ldots, a_{m-1}\}$ be an alphabet. For $i \in \mathbb{Z}$ and $p \in \mathbb{Z}_{\geqslant 1}$, we define $s(i,p)$ as the word

$$s(i,p) := \prod_{k=i'}^{p} a_k = a_{i'} a_{i'+1} \cdots a_{i'+p-1}$$

where i' is the remainder in the Euclidean division of i by m, and with the convention that, for $j \geqslant 1$, $a_j := a_{j'}$ where j' is the remainder in the Euclidean division of j by m. We also define $s(i,0)$ to be the empty word for all $i \in \mathbb{Z}$.

Definition B.1.1 (Circular groups). Let m, ℓ be two positive integers. The *circular monoid* $M(m,\ell)$ is defined by the monoid presentation

$$M(m,\ell) := \langle a_0, \ldots, a_{m-1} \mid \forall i [\![0, m-1]\!], s(i,\ell) = s(i+1,\ell) \rangle^+ .$$

The enveloping group $G(m,\ell)$ of $M(m,\ell)$ is called a *circular group*.

From now on, we assimilate the word $s(i,p)$ ($i \in \mathbb{Z}$, $p \geqslant 0$) with its image in $M(m,\ell)$.

Example B.1.2. The monoid $M(3,3)$ is given by $\langle a,b,c \mid abc = bca = cab\rangle^+$. The group $G(3,3)$ is the fundamental group of the complement of 3 lines going through the origin in $\mathbb{C}^2$ (cf. [DP99, Example 5]). The monoid $M(2,3)$ is given by $\langle s,t \mid sts = tst\rangle^+$. It is the Artin monoid of type A_2.

We note that the presentation of $M(m,\ell)$ is homogeneous. That is, the defining relations are equalities between words of the same length. The function sending an element of $M(m,\ell)$ to the length of any word representing it is then a length function, making $M(m,\ell)$ into a homogeneous monoid. As the only defining relations of $M(m,\ell)$ are between words of length ℓ, two distinct words of length less than ℓ cannot represent the same element of $M(m,\ell)$. In particular, for $0 < p < \ell$ and $i, i' \in \mathbb{Z}$, we have $s(i,p) = s(i',p)$ in $M(m,\ell)$ if and only if $i \equiv i'[m]$.

Lemma B.1.3. *[DP99, Example 5]*
The monoid $M(m,\ell)$ is a homogeneous Garside monoid with Garside element $\Delta = s(0,\ell)$. Its simple elements are the $s(i,p)$ for $i \in [\![0, m-1]\!]$ and $p \in [\![0,\ell]\!]$. The Garside automorphism of $M(m,\ell)$ is given by $\phi(s(i,p)) = s(i+\ell,p)$.

The only statement which is not showed in [DP99, Example 5] is the statement on the Garside automorphism, which comes from the fact that, for any simple element $s(i,p)$ of $M(m,\ell)$, we have

$$s(i,p)\Delta = s(i,p)s(i+p,\ell) = s(i,\ell+p) = s(i,\ell)s(i+\ell,p) = \Delta s(i+\ell,p).$$

Let $s(i,p)$ be a simple element of $M(m,\ell)$. We have $s(i,p)s(i+p,\ell-p) = s(i,\ell) = \Delta$, thus the right-complement (resp. left-complement) of $s(i,p)$ in Δ is given by

$$s(i,p)^* = s(i+p-\ell,\ell-p) \text{ (resp. } \overline{s(i,p)} = s(i+p,\ell-p)).$$

The fact that any simple element different from 1 and Δ admits a unique decomposition as a product of atoms has the following consequences:

Lemma B.1.4 (Gcd and lcm of two simple elements). *Let $s(i,p)$ and $s(i',p')$ be two simple elements of $M(m,\ell)$.*

(a) We have $s(i,p) \preccurlyeq s(i',p')$ if and only if $i = i'$ and $p \leqslant p'$, or if $p = 0$, or if $p' = \ell$.

(b) The left-gcd of $s(i,p)$ and $s(i',p')$ is given by

$$s(i,p) \wedge s(i',p') = \begin{cases} s(i,p) & \text{if } s(i,p) \preccurlyeq s(i',p'), \\ s(i',p') & \text{if } s(i',p') \preccurlyeq s(i,p) \\ 1 & \text{otherwise.} \end{cases}$$

(c) The right-lcm of $s(i,p)$ *and* $s(i',p')$ *is given by*

$$s(i,p) \vee s(i',p') = \begin{cases} s(i',p') & \text{if } s(i,p) \preccurlyeq s(i',p'), \\ s(i,p) & \text{if } s(i',p') \preccurlyeq s(i,p) \\ \Delta & \text{otherwise.} \end{cases}$$

Proof. (a) The cases $p = 0$ and $p' = \ell$ are immediate: they give $s(i,p) = 1$ and $s(i',p') = \Delta$, respectively. If $i = i'$ and $p \leqslant p'$, we have $s(i,p)s(i+p,p'-p) = s(i',p')$ and $s(i,p) \preccurlyeq s(i',p')$. Conversely, suppose that $s(i,p) \preccurlyeq s(i',p')$. Since we assume that $p' < \ell$ and $0 < p$, there is only one way to write both $s(i,p)$ and $s(i',p')$ as a product of atoms. Thus the assumption that $s(i,p)$ can be written as a prefix of some word expressing $s(i',p')$ implies that $s(i,p)$ is in fact the only prefix of length p of $s(i',p')$. We obtain that $p \leqslant p'$ and $s(i',p) = s(i,p)$, which gives $i = i'$.
(b) The first two cases are obvious. Suppose that we have $s(i,p) \not\preccurlyeq s(i',p')$ and $s(i',p') \not\preccurlyeq s(i,p)$. By the first point we have $p,p' \notin \{0,\ell\}$ and $i \neq i'$. Again by point (a), a nontrivial common left-divisor $s(j,q)$ of $s(i,p)$ and $s(i',p')$ should be such that $j = i = i'$, which is impossible. We apply similar reasoning to prove point (c). □

This very strict behavior of gcds and lcms allows for an easy description of greedy normal forms of a product of two simple elements in $M(m,\ell)$.

Lemma B.1.5 (Greedy normal form of a product of two nontrivial simples). *Let* $s(i,p)$ *and* $s(i',p')$ *be two simple elements of* $M(m,\ell)$ *with* $0 < p,p' < \ell$*. The greedy normal form of* $s(i,p)s(i',p')$ *is given by*

$$s(i,p)s(i',p') = \begin{cases} s(i,p)s(i',p') & \text{if } i+p \not\equiv i'[m], \\ s(i,p+p') & \text{if } i+p \equiv i'[m] \text{ and } p+p' < \ell, \\ \Delta & \text{if } i+p \equiv i'[m] \text{ and } p+p' = \ell, \\ \Delta s(i+\ell, p+p'-\ell) & \text{if } i+p \equiv i'[m] \text{ and } p+p' > \ell. \end{cases}$$

Proof. We will apply Lemma 2.1.13. Since $0 < p,p' < \ell$, Lemma B.1.4 gives

$$\begin{aligned} \overline{s(i,p)} \wedge s(i',p') &= s(i+p,\ell-p) \wedge s(i',p'), \\ &= \begin{cases} s(i+p,\ell-p) & \text{if } i+p \equiv i'[m] \text{ and } \ell-p \leqslant p, \\ s(i',p') & \text{if } i+p \equiv i'[m] \text{ and } p' \leqslant \ell-p \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

These three cases give the desired result. □

This particular description of greedy normal forms in circular monoids will induce a convenient description of super-summit sets in Section B.1.2.

We finish this section by the study of the particular case where $m = \ell$. In this case, the group $G(m,m)$ is the fundamental group of the complement of m lines through the origin in $\mathbb{C}^2$ (cf. [DP99, Example 5]). Its presentation can be reinterpreted as a direct product.

Lemma B.1.6. *Let* m *be a positive integer, and let* F_{m-1} *be a free group with* $m-1$ *generators. The group* $G(m,m)$ *is isomorphic to* $\mathbb{Z} \times F_{m-1}$*. This isomorphism identifies* Δ *with* $(z,1)$*, where* z *is a generator of* $\mathbb{Z}$*.*

Proof. The result is obvious if $m \in \{1,2\}$. We denote by $x_0, \cdots, x_{m-2}$ the generators of F_{m-1}, and by z a generator of $\mathbb{Z}$. We define $f : \mathbb{Z} \times F_{m-1} \to G(m,m)$ by

$$\begin{cases} f(z) = \Delta = s(0,m), \\ f(x_i) = a_i & \forall i \in [\![0, m-2]\!]. \end{cases}$$

This induces a well-defined morphism since $\Delta \in Z(G(m,m))$. Conversely, we define $g : G(m,m) \to \mathbb{Z} \times F_{m-1}$ by

$$\begin{cases} g(a_i) = x_i & \forall i \in [\![1, m-2]\!], \\ g(a_{m-1}) = (x_0 \cdots x_{m-2})^{-1} z. \end{cases}$$

This induces a well-defined morphism since, for all $i \in [\![0, m-1]\!]$, we have

$$\begin{aligned} g(s(i,m)) &= g(s(i, m-1-i) a_{m-1} s(0,i)) \\ &= x_i \cdots x_{m-2} (x_0 \cdots x_{m-2})^{-1} z x_0 \cdots x_{i-1} \\ &= x_i \cdots x_{m-2} (x_0 \cdots x_{m-2})^{-1} x_0 \cdots x_{i-1} z = z = g(s(0,m)). \end{aligned}$$

It is straightforward to check that f and g are inverses of one another. □

B.1.2 Conjugacy in circular groups

Again we fix two positive integers m, ℓ. In this section, we study separately the conjugacy of periodic and non-periodic elements in circular groups. Our first result shows that we can restrict our attention to the case of periodic or rigid elements.

Proposition B.1.7. *Let x be an element of $G(m,\ell)$. If x lies in its own super-summit set, then it is either rigid or periodic.*

Proof. First, if $\inf(x) = \sup(x)$, then we have $x = \Delta^k$ for some $k \in \mathbb{Z}$. In this case, x is obviously both rigid and $(1,k)$-periodic.

Suppose now that $\sup(x) = \inf(x) + 1$. We have $x = \Delta^k s(i,p)$ for some $i \in [\![0, m-1]\!]$ and $p \in [\![1, \ell-1]\!]$. The element x is rigid if and only if the word $s(i,p)\phi^{-k}(s(i,p)) = s(i,p)s(i-k\ell, p)$ is greedy. By Lemma B.1.5 (greedy normal form of a product of two nontrivial simples), this is equivalent to $i + p \equiv i - k\ell[m]$, i.e. $k\ell + p \equiv 0[m]$. If $k\ell + p \equiv 0[m]$, then there is some integer v with $-k\ell + mv = p$. In particular we have $\ell \wedge m | p$. We have

$$\begin{aligned} x^n &= \Delta^{kn} \phi^{k(n-1)}(s(i,p)) \cdots \phi^k(s(i,p)) s(i,p) \\ &= \Delta^{kn} s(i + k(n-1)\ell, p) \cdots s(i + k\ell, p) s(i,p) \\ &= \Delta^{kn} s(i + k(n-1)\ell, np) \\ &= \Delta^{kn+a} s(i + (k(n-1)+a)\ell, r), \end{aligned}$$

where $np = a\ell + r$ is the Euclidean division of np by ℓ. The remainder r is 0 if and only if n is a multiple of $\frac{\ell \vee p}{p} = \frac{\ell}{\ell \wedge p}$. We obtain that x is (a,b)-periodic, for

$$a = \frac{\ell}{\ell \wedge p} \text{ and } b = k\frac{\ell \vee p}{p} + \frac{\ell \vee p}{\ell} = \frac{k\ell + p}{\ell \wedge p} = \frac{mv}{\ell \wedge p}.$$

Furthermore, a and b are coprime.

Lastly, suppose that $\sup(x) > \inf(x) + 1$. The left-weighted factorization of x is given by $\Delta^k s_1 \cdots s_r$ with $r > 1$. We claim that x is rigid. Otherwise, the word $s_r\phi^{-k}(s_1)$ is not greedy. By Lemma B.1.5, we either have that $s_r\phi^{-k}(s_1)$ is a simple element, or a product of the form Δs where s is a simple element different from 1 and Δ. In the first case, we have $\sup(\mathrm{cyc}(x)) < \sup(x)$. In the second case, we have $\inf(\mathrm{cyc}(x)) > \inf(x)$. In both cases, we have $x \notin \mathrm{SSS}(x)$. $\square$

Periodic elements

The proof of Proposition B.1.7 also gives the following result:

Lemma B.1.8. *Let m, ℓ be positive integers, and let $\Delta^k s(i,p)$ be in $G(m,\ell)$ with $0 < p < \ell$. The element $\Delta^k s(i,p)$ is periodic if and only if $p + k\ell \equiv 0[m]$, in which case it is a $(\frac{\ell}{p\wedge\ell}, \frac{mv}{p\wedge\ell})$-periodic element (where $p + k\ell = mv$).*

Now that we have a characterization of the elements of the super-summit sets of periodic elements, we can compute conjugacy graphs and centralizers. We distinguish two cases.

Lemma B.1.9. *Let $x = \Delta^k$ for some nonzero integer k. We have $\mathrm{SSS}(\Delta^k) = \{\Delta^k\}$. The centralizer of Δ^k in $G(m,\ell)$ is either $G(m,\ell)$ if $k\ell$ is a multiple of m, or cyclic and generated by Δ otherwise.*

Proof. We have that $y \in G(m,\ell)$ lies in $\mathrm{SSS}(\Delta^k)$ only if $\inf(y) = k = \sup(y)$. The only element satisfying this is Δ^k, which is conjugate to itself. Thus we have $\mathrm{SSS}(\Delta^k) = \{\Delta^k\}$. Now, let $s(j,q)$ be a simple element in $M(m,\ell)$. Since both 1 and Δ conjugate Δ^k to itself, we can assume that $q \in [\![1, m-1]\!]$. We have

$$\begin{aligned} s(j,q)^{-1}\Delta^k s(j,q) &= \overline{s(j,q)}\Delta^{k-1}s(j,q) \\ &= s(j+q, \ell-q)\Delta^{k-1}s(j,q) \\ &= \Delta^{k-1}s(j+q+(k-1)\ell, \ell-q)s(j,q). \end{aligned}$$

In order for this element to lie in $\mathrm{SSS}(\Delta^k)$, the word $s(j+q+(k-1)\ell, \ell-q)s(j,q)$ must not be greedy. This is equivalent to $j + k\ell \equiv j[m]$. If $k\ell$ is a multiple of m, this is true for all $j \in [\![0, m-1]\!]$, and we obtain $s(j,q)^{-1}\Delta^k s(j,q) = \Delta^k$: the arrows from Δ^k to itself in $\mathrm{CG}(\Delta^k)$ are given by all the simple elements. Otherwise, $j + k\ell \equiv j[m]$ is never true for $j \in [\![1, m-1]\!]$ and the only arrows from Δ^k to itself in $\mathrm{CG}(\Delta^k)$ are given by 1 and Δ. $\square$

Lemma B.1.10. *Let $x = \Delta^k s(i,p)$ be a periodic element in $M(m,\ell)$ with $p \in [\![1, m-1]\!]$. We have $\mathrm{SSS}(x) = \{\Delta^k s(n,p) \mid n \in [\![0, m-1]\!]\}$. The centralizer of $\Delta^k s(0,p)$ in $G(m,\ell)$ is cyclic and generated by $s(p,m)$.*

Proof. The assumption that x is periodic is equivalent to $k\ell + p \equiv 0[m]$ by Lemma B.1.8. Let $s(j,q)$ be a simple element in $M(m,\ell)$. We have

$$\begin{aligned} x^{s(j,q)} &= s(j,q)^{-1}\Delta^k s(i,p)s(j,q) \\ &= \Delta^{k-1}s(j+q+(k-1)\ell, \ell-q)s(i,p)s(j,q). \end{aligned}$$

Again, in order for this to lie in $\mathrm{SSS}(x)$, we must have either $j + k\ell \equiv i[m]$ or $i + p \equiv j[m]$. Since $k\ell + p \equiv 0[m]$, those two assertions are equivalent. If they are satisfied, then we have

$$x^{s(j,q)} = \Delta^k s(j+q-p, p) = \Delta^k s(i+q, p).$$

In particular, $s(p,n)$ gives a conjugating element from $\Delta^k s(0,p)$ to $\Delta^k s(n,p)$ for $n \in [\![0, m-1]\!]$. Moreover, for $\Delta^k s(n,p) \in \mathrm{SSS}(x)$, the simples s such that $(\Delta^k s(n,p))^s \in \mathrm{SSS}(x)$ are all divisible by $s(n+p,1)$. The conjugacy graph of x is then given by

$$\begin{tikzcd} \Delta^k s(0,p) \arrow[r, "s(p,1)"] & \Delta^k s(1,p) \arrow[r, "s(p+1,1)"] & \cdots \arrow[r, "s(p+m-2,1)"] & \Delta^k s(m-1,p) \arrow[lll, bend left, "s(p+m-1,1)"] \end{tikzcd},$$

and the centralizer of $\Delta^k s(0,p)$ is cyclic and generated by

$$s(p,1)s(p+1,1)\cdots s(p+m-1,1) = s(p,m).$$

$\square$

We can use these two lemmas to determine the center of circular groups. Recall that the Garside automorphism ϕ, corresponding to conjugacy by Δ on the right, sends a simple element $s(i,p)$ to $s(i+\ell,p)$. If $m \neq 1 \neq \ell$, then the smallest trivial power of ϕ is $\phi^{\frac{m}{m\wedge\ell}}$, and $\Delta^{\frac{m}{m\wedge\ell}}$ is the smallest central power of Δ in $G(m,\ell)$.

Corollary B.1.11 (Center of circular groups). *Let m,ℓ be two positive integers. If $m=1$ or $\ell=1$, then $G(m,\ell) \simeq \mathbb{Z}$ is abelian. If $m=\ell=2$, then $G(m,\ell) = \mathbb{Z}^2$ is abelian. Otherwise $Z(G(m,\ell))$ is infinite cyclic and generated by $\Delta^{\frac{m}{m\wedge\ell}}$.*

Proof. If $m=1$, then $M(1,\ell) = \langle a_0\rangle^+ \simeq \mathbb{Z}_{\geqslant 0}$ (with Garside element a_0^ℓ). If $\ell=1$, we have $M(m,1) = \langle a_0\rangle^+ \simeq \mathbb{Z}_{\geqslant 0}$ (with Garside element a_0). If $m=\ell=2$, then $G(m,\ell) = \langle a_0, a_1 \mid a_0a_1 = a_1a_0\rangle = \mathbb{Z}^2$ (with Garside element a_0a_1).

If $m>1$, we distinguish several cases. First, we assume that m does not divide ℓ. By Lemma B.1.9, the centralizer of Δ in $G(m,\ell)$ is cyclic and generated by Δ. As the center $Z(G(m,\ell))$ is included in $C_{G(m,\ell)}(\Delta)$, we obtain that $Z(G(m,\ell))$ is cyclic and generated by the smallest central power of Δ, which is $\Delta^{\frac{m}{m\wedge\ell}}$ since $\ell \neq 1$.

Now, if $m=\ell$, then Lemma B.1.6 gives an isomorphism $G(m,\ell) \simeq \mathbb{Z}\times F_{m-1}$. The cases $m=\ell\in\{1,2\}$ have already been studied. If $m \geqslant 3$, then the center of $\mathbb{Z}\times F_{m-1}$ is $\mathbb{Z}\times\{1\}$, which is identified with $\langle\Delta\rangle = \langle\Delta^{\frac{m}{m\wedge\ell}}\rangle$.

Lastly, we assume that $mk=\ell$ for some integer $k>1$. The element $s(0,m)$ is $(k,1)$-periodic in $G(m,\ell)$ and, by Lemma B.1.10, the centralizer of $s(0,m)$ in $G(m,\ell)$ is cyclic and generated by $s(0,m)$. Since $mk=\ell$, we have that $s(0,m)^k = \Delta = \Delta^{\frac{m}{m\wedge\ell}}$ is a central element. It remains to show that $s(0,m)$ admits no central power inferior to k. Let $1 \leqslant r \leqslant k-1$. We have $s(0,m)^r = s(0,mr)$ and

$$\begin{aligned} s(0,1)^{s(0,m)^r} &= s(0,1)^{s(0,mr)} \\ &= \Delta^{-1}s(mr-\ell,\ell-mr)s(0,1)s(0,mr) \\ &= \Delta^{-1}s(0,m(k-r))s(0,1)s(0,mr). \end{aligned}$$

As $m \neq 1$, this is a left-weighted factorization, in particular it is not equal to $s(0,1)$. We then obtain that the smallest central power of $s(0,m)$ is $s(0,m)^k = \Delta$, thus $Z(G(m,\ell)) = \langle\Delta\rangle$ as claimed. $\square$

Proposition B.1.12. *Let m, ℓ be positive integers, and let p, q be integers. Any two (p,q)-periodic elements in $G(m,\ell)$ are conjugate.*

Proof. First, by Theorem 3.4.4 (conjugacy of periodic elements), we can assume that p, q are coprime integers. If $p = 1$, then x and y are both conjugate to Δ^q. If $p > 1$, we can assume (up to conjugacy) that x, y are of the form $\Delta^k s(i,a)$ and $\Delta^{k'} s(i',a')$ with $0 < a, a' < \ell$, respectively. By Lemma B.1.8, there are two integers v, v' with $mv = k\ell + a$ and $mv' = k'\ell + a'$. Since $k\ell + a$ (resp. $k'\ell + a'$) is the length of x (resp. y) in $G(m,\ell)$, we have $v = v'$. By reducing modulo ℓ, we obtain $a \equiv a'[\ell]$. Since we assume that $a, a' \in [\![0, \ell-1]\!]$, $a \equiv a'[\ell]$ implies $a = a'$. The equality $k\ell + a = k'\ell + a'$ then gives $k = k'$. By Lemma B.1.10, we get that x and y are conjugate. ☐

Proposition B.1.13 (Rootless periodic elements). *Let m, ℓ be positive integers. Any periodic element in $G(m,\ell)$ is conjugate to a power of either $s(0,m)$ or Δ. Moreover the irreducible periodic elements of $G(m,\ell)$ are given (up to conjugacy) by*

$$\begin{cases} \{s(0,m)^{\pm 1}\} & \textit{if } m|\ell, \\ \{\Delta^{\pm 1}\} & \textit{if } \ell|m, \\ \{s(0,m)^{\pm 1}, \Delta^{\pm 1}\} & \textit{otherwise.} \end{cases}$$

Proof. Let $\rho \in G(m,\ell)$ be a periodic element. By Lemma B.1.10, we can assume up to conjugacy that $\rho = \Delta^k s(k\ell, p)$. If $p = 0$, then $\rho = \Delta^k$ is a power of Δ. If $p \neq 0$, then Lemma B.1.8 gives an integer v with $mv = k\ell + p$, we then have $\rho = s(0,m)^v$. If ρ is an irreducible periodic element of $G(m,\ell)$, then we have $\rho \in \{s(0,m)^{\pm 1}, \Delta^{\pm 1}\}$ (up to conjugacy). It only remains to check whether or not $s(0,m)$ and Δ are indeed irreducible. If $m|\ell$ (resp. $\ell|m$), then we have $s(0,m)^{\frac{\ell}{m}} = \Delta$ (resp. $\Delta^{\frac{m}{\ell}} = s(0,m)$). Since there must be at least one conjugacy class of irreducible periodic elements in $G(m,\ell)$, we get the desired result if $m|\ell$ or $\ell|m$.

Assume now that neither $m|\ell$ nor $\ell|m$. A proper root of $s(0,m)$ in $G(m,\ell)$ must have the form $s(0,n)$ with $0 < n < m$. By Lemma B.1.8, such an element cannot be periodic, thus it cannot be a root of $s(0,m)$, which is then irreducible. The same reasoning applies to Δ. ☐

Non-periodic elements

We now turn our attention to non-periodic elements. By Proposition B.1.7, such elements are exactly the conjugate of rigid elements in $G(m,\ell)$.

Proposition B.1.14. *Let $x \in G(m,\ell)$ be a non-periodic element. The super-summit set of x is made of rigid elements. Furthermore, the only arrows starting from an object y of* $\mathrm{CG}(x)$ *are labeled by* $\mathrm{init}(y)$ *and* $\overline{\mathrm{fin}(y)}$.

Proof. Let $y \in \mathrm{SSS}(x)$. Since x is not periodic, y is not periodic. It is then rigid by Proposition B.1.7. We then have $\sup(y) > \inf(y)$ and we can assume that the left-weighted factorization of y is $\Delta^k s(i_1,p_1)\cdots s(i_r,p_r)$ with $r > 0$. Since y is rigid, we have $i_r + p_r \not\equiv i_1 - k\ell[m]$ by Lemma B.1.5 (greedy normal form of a product of two nontrivial simples). Let $s(j,q)$ be a simple element with $q \in [\![1, m-1]\!]$. We have

$$y^{s(j,q)} = \Delta^{k-1} s(j+q+(k-1)\ell, \ell - q) s(i_1,p_1)\cdots s(i_r,p_r) s(j,q).$$

In order for this to lie in $\mathrm{SSS}(x)$, we must have either $j + k\ell \equiv i_1[m]$ or $i_r + p_r \equiv j[m]$. Since $i_r + p_r + kl \not\equiv i_1[m]$, these cases are mutually exclusive.

- Assume that $j + k\ell \equiv i_1[m]$. By Lemma B.1.5, the left-weighted factorization of $y^{s(j,q)}$ is given by

$$\begin{cases} \Delta^{k-1}s(j+q+(k-1)\ell, \ell-q+p_1)\cdots s(i_r,p_r)s(j,q) & \text{if } p_1 < q, \\ \Delta^k s(i_2,p_2)\cdots s(i_r,p_r)s(j,q) & \text{if } p_1 = q, \\ \Delta^k s(j+q+k\ell, p_1-q)s(i_2,p_2)\cdots s(i_r,p_r)s(j,q) & \text{if } p_1 > q. \end{cases}$$

Thus, $y^{s(j,q)} \in \mathrm{SSS}(x)$ in this case if and only if $p_1 = q$. We then have that $s(j,q) = s(i_1 - k\ell, p)$ is the initial factor of y.

- Assume that $i_r + p_r \equiv j[m]$. By Lemma B.1.5, the left-weighted factorization of of $y^{s(j,q)}$ is given by

$$\begin{cases} \Delta^{k-1}s(j+q+k\ell-\ell, \ell-q)s(i_1,p_1)\cdots s(i_r,p_r+q) & \text{if } p_r+q < \ell, \\ \Delta^k \phi(s(j+q+k\ell-\ell,\ell-q)s(i_1,p_1)\cdots s(i_{r-1},p_{r-1})) & \text{if } p_r+q = \ell, \\ \Delta^k \phi(s(j+q+k\ell-\ell,\ell-q)s(i_1,p_1)\cdots s(i_{r-1},p_{r-1}))s(i_r+\ell,p_r+q-\ell) & \text{if } p_r+q > \ell. \end{cases}$$

Thus, $y^{s(j,q)} \in \mathrm{SSS}(x)$ in this case if and only if $p_r + q = \ell$. We then have that $s(j,q) = s(i_r + p_r, \ell - p_r) = \overline{s(i_r,p_r)}$.

□

Let $x \in G(m,\ell)$ be a rigid element. The conjugation of x by $\overline{\mathrm{fin}(x)}$ is equal to $\phi(\mathrm{dec}(x))$. The last proposition then gives

Corollary B.1.15. *Let $x \in G(m,\ell)$ be a non-periodic element. One can go from any element of $\mathrm{SSS}(x)$ to any other by a finite sequence of cyclings, decyclings and applications of the Garside automorphism.*

We can now state our main result on uniqueness of roots up to conjugacy.

Theorem B.1.16 (Uniqueness of roots up to conjugacy in circular groups). *Let m,ℓ be two positive integers. If $\alpha,\beta \in G(m,\ell)$ are such that $\alpha^n = \beta^n$ for some nonzero integer n, then α and β are conjugate.*

Proof. First, if α is (p,q)-periodic for some integers p and q. We have that α^n is (p,nq)-periodic and that β is also (np,nq)-periodic. The elements α and β are then conjugate by Theorem 3.4.4 and Proposition B.1.12.

Up to replacing α and β with α^{-1} and β^{-1}, we can assume that $n > 0$. Assume now that α is not periodic, we also have that $x := \alpha^n$ and β are non periodic. Up to conjugacy, we can assume that $\alpha \in \mathrm{SSS}(\alpha)$. By Proposition B.1.7, we have that α is rigid. The element x is then rigid as a power of the rigid element α. Let now $c \in G(m,\ell)$ be so that $\beta^c \in \mathrm{SSS}(\beta)$. Since β is not periodic, β^c is rigid as well as $x^c = (\beta^c)^n$. We have $x, x^c \in \mathrm{SSS}(x)$. By Corollary B.1.15, there is a finite sequence of cyclings, decyclings, and applications of the Garside automorphism sending x to x^c. By Lemma 3.2.13, applying the same transformations to α gives a rigid element α' whose n-th power is x^c. Again by Lemma 3.2.13, we have $\alpha' = \beta^c$ and thus α and β are conjugate. □

B.1.3 Parabolic subgroups in circular groups

We fix two positive integers m, ℓ. In this section, we study parabolic subgroups of circular groups. As we will see, the situation is not very rich, as we will show that nontrivial parabolic subgroups of circular groups are all cyclic. Nonetheless, this can be seen as an easy example of how to study support-preservingness in a Garside group.

Lemma B.1.17. *Let m, ℓ be two integers.*

(a) If $m = 1$, then the parabolic Garside elements of $G(m, \ell)$ are $1, \Delta$.

(b) If $m \neq 1$, then the parabolic Garside elements of $G(m, \ell)$ are $\{1, \Delta\} \cup \{a_i \mid i \in [\![0, m-1]\!]\}$.

Proof. First, if $m = 1$, then $G(m, \ell) = \langle a \rangle$ and $\Delta = a^\ell$. The simple elements are $\{1, a, a^2, \ldots, a^\ell\}$ and the result is obvious. Now, if $m \neq 1$, then powers of atoms are never simple elements, thus atoms are parabolic Garside elements by Example 5.1.3. Conversely, if $\ell = 1$, then all the atoms are equal, and equal to Δ. The result is then obvious. If $\ell > 1$, then the atoms are the only simple elements different from $1, \Delta$ which are balanced, whence the result. ☐

In particular, if $G(m, \ell)$ is nonabelian, then the nontrivial parabolic subgroups of $G(m, \ell)$ are exactly the conjugates of the subgroups generated by the atoms of $M(m, \ell)$.

In order to study support-preservingness of the shoal of all standard parabolic subgroups of $G(m, \ell)$, we compute the minimal positive conjugators of elements of $M(m, \ell)$.

Lemma B.1.18 (Minimal positive conjugators in circular groups). *Let $x \in M(m, \ell)$ have greedy normal form $x = s_1 \cdots s_r$ with $s_k := s(i_k, p_k)$ for $k \in [\![1, r]\!]$.*

(a) If $s_1 = \Delta$, then the minimal positive conjugators of x are the atoms of $M(m, \ell)$.

(b) If $s_1 \neq \Delta$, then we have

$$\forall j \in [\![0, m-1]\!],\ \rho_{a_j}(x) = \begin{cases} a_j & \text{if } j = i_1, \\ \overline{s_r} & \text{if } j = i_r + p_r, \\ \Delta & \text{otherwise.} \end{cases}$$

In particular, the minimal positive conjugators of x are a_{i_1} and $\overline{s_r}$.

Proof. First, if $s_1 = \Delta$, then any simple element s is a positive conjugator for x, since $s^{-1}x \in M(m, \ell)$. The minimal positive conjugators of x are then simply the atoms of $M(m, \ell)$.

Now, assume that $s_1 \neq \Delta$. Since $s_1 \cdots s_r$ is a greedy word, we have $s_k \neq \Delta$ for all $k \in [\![1, r]\!]$. Let $j \in [\![0, m-1]\!]$, and let us compute $\rho_{a_j}(x)$. First, we set $c_0 = a_j$, and we compute the converging prefix $c_1 = c_0 \vee x/c_0 = a_j \vee x/a_j$ (see Lemma 5.2.21).

If $j = i_1$, then $a_i \preccurlyeq s_1 \preccurlyeq x$ and $x \vee c_0 = x$. We then have $c_1 = a_j \vee 1 = a_j = c_0$. Since $c_1 = c_0$, we obtain that $\rho_{a_j}(x) = c_1 = a_j$. If $j \neq i_1$, then we have $a_j \vee s_1 = \Delta$, and $s_1/a_j = \overline{s_1} \neq 1$ since $s_1 \neq \Delta$. We then deduce the following diagram of lcms:

$$\begin{tikzcd}
{} \arrow[r, "s_1"] \arrow[d, "a_j"'] & {} \arrow[r, "s_2"] \arrow[d, "\overline{s_1}"] & {} \arrow[r, dotted] \arrow[d, "\overline{s_2}"] & {} \arrow[r, "s_r"] \arrow[d, "\overline{s_{r-1}}"] & {} \arrow[d, "\overline{s_r}"] \\
{} \arrow[r, "\overline{a_i}"'] & {} \arrow[r, "\phi(s_1)"'] & {} \arrow[r, dotted] & {} \arrow[r, "\phi(s_{r-1})"'] & {}
\end{tikzcd}$$

We have $x/a_j = \overline{s_r}$. Now, we either have $j = i_r + p_r$ and $a_j \preccurlyeq \overline{s_r}$, or $j \neq i_r + p_r$ and $a_j \vee \overline{s_r} = \Delta$. In both cases, we obtain that c_1 is a positive conjugator of x, and thus $c_1 = \rho_{a_j}(x)$, whence the result. $\square$

Corollary B.1.19. *The shoal of all parabolic subgroups of $G(m,\ell)$ is support-preserving.*

Proof. If $G(m,\ell)$ is abelian, the conjugacy is trivial in $G(m,\ell)$ and the result is trivial. Now, assume that $G(m,\ell)$ is not abelian, and let $x \in M(m,\ell)$ be written in greedy normal form $s_1 \cdots s_r$. If all the s_i are equal to the same atom of $M(m,\ell)$, say a_j, then $\mathrm{SPC}(x) = G(m,\ell)_{a_j}$, otherwise, $\mathrm{SPC}(x) = G(m,\ell)$.

Let ρ be a minimal positive conjugator for x. If $s_1 = \Delta$, then $\mathrm{SPC}(x) = G(m,\ell)$, and we can write $\rho = a_j$ for some $j \in [\![0, m-1]\!]$ by Lemma B.1.18. We then have $s(j+1,\ell-1) = a_j^{-1}\Delta \preccurlyeq x^\rho$, and $x^\rho \succcurlyeq a_j$. The standard parabolic closure $G(m,\ell)_s$ of x^ρ is then such that $a_j, s(j+1,\ell-1) \preccurlyeq s$, whence $s = \Delta$ and $\mathrm{SPC}(x^\rho) = G(m,\ell) = \mathrm{SPC}(x)^\rho$.

Now, assume that $s_1 \neq \Delta$, then we have either $\rho = a_i$ where $s_1 = s(i,p)$, or $\rho = \overline{s_r}$. If x is a power of an atom, then $x = a_i^r$, in this case $x^\rho = x$ of $\rho = a_i$, or $x^\rho = \phi(x)$ if $\rho = \overline{s_r} = \overline{a_i}$. In both case, we have $\mathrm{SPC}(x^\rho) = \mathrm{SPC}(x)^\rho$. If x is not a power of an atom, then $\mathrm{SPC}(x) = G(m,\ell)$, and we have to say that x^ρ is also not a power of an atom. If $\rho = a_i$, then we can apply the same reasoning as in the first part to get $\mathrm{SPC}(x^\rho) = \mathrm{SPC}(x)^\rho$. In the second case, we have

$$x^\rho = \overline{s_r}^{-1} s_1 \cdots s_r \overline{s_r} = \overline{s_r}^{-1} \Delta \phi(s_1) \cdots \phi(s_{r-1}) = \phi(s_r)\phi(s_1)\cdots\phi(s_{r-1}) = \mathrm{cyc}(\phi(x)).$$

In particular, x^ρ is not a power of an atom, and $\mathrm{SPC}(x^\rho) = G(m,\ell)$. $\square$

Since circular groups are homogeneous, we obtain that intersection of parabolic subgroups of circular groups are again parabolic subgroups by Theorem 5.2.33 (intersection of parabolic subgroups). Moreover, we can apply Proposition 5.2.12 to obtain that, if $U \subset G(m,\ell)$ has finite index, then $Z(U) \subset Z(G(m,\ell))$.

B.1.4 Some group theoretic properties

Homology of circular groups

The homology of a Garside group can be studied using a particular complex introduced by Dehornoy and Lafont in [DL03, Section 4]. This complex is built using atoms and lcms in the underlying Garside monoid. The particular behavior of circular monoids with regards to lcms induces strong results on the associated complex.

Let us start by quickly recalling the definition of the Dehornoy-Lafont complex. We start by considering a homogeneous Garside monoid (M,Δ), with set of simples S and set of atoms A. We fix an arbitrary strict linear ordering $<$ on A. For any $x \in M$, we define $\mathrm{md}(x)$ to be the $<$-minimal element of A which right-divides x.

An n-cell is defined as an n-tuple $[\alpha_1, \ldots, \alpha_n]$ of atoms of M such that $\alpha_1 < \ldots < \alpha_n$, and $\alpha_i = \mathrm{md}(\alpha_i \vee_L \alpha_{i+1} \vee_L \cdots \vee_L \alpha_n)$ for all $i \in [\![1,n]\!]$. We denote by $\mathcal{X}_n$ the set of n-cells. The set C_n of n-chains is then defined as the free $\mathbb{Z}G(M)$-module with basis the set of n-cells. Endowed with a convenient differential ∂_n (which we will not define here), the complex $(C_n,\partial_n)_{n\in\mathbb{N}}$ is an exact resolution of the trivial $\mathbb{Z}G(M)$-module $\mathbb{Z}$. Since both S and A are finite, we have $\mathcal{X}_n = \varnothing$ for $n \geqslant |A|$, thus $(C_n,\partial_n)_{n\geqslant 0}$ is bounded above and below.

Lemma B.1.20. *Let m, ℓ be two positive integers, and let $M := M(m,\ell)$. We have $\mathcal{X}_0 = \{[\varnothing]\}$, $\mathcal{X}_1 = \{[a_0],\ldots,[a_{m-1}]\}$, $\mathcal{X}_2 = \{[a_0, a_i] \mid i \in [\![1, m-1]\!]\}$ and $\mathcal{X}_n = \varnothing$ for $n \geqslant 3$.*

Proof. The statements on $\mathcal{X}_0$ and $\mathcal{X}_1$ are straightforward. By definition, an n-tuple $[\alpha_1,\ldots,\alpha_n]$ is an n-cell if and only if $[\alpha_2,\ldots,\alpha_n]$ is an $n-1$-cell and $\alpha_1 = \mathrm{md}(\alpha_2 \vee_L \cdots \vee_L \alpha_n)$.

If $n = 2$, then we get that a couple $[a_i, a_j]$ is a 2-cell if and only if $a_i = \mathrm{md}(a_i \vee_L a_j)$. Since $a_i \neq a_j$ by assumption, we have $\mathrm{md}(a_i \vee_L a_j) = \mathrm{md}(\Delta) = a_0$. Thus we get the result on $\mathcal{X}_2$. Lastly, if $[\alpha_1, \cdots, \alpha_n]$ is an n-cell for $n \geqslant 2$, then $[\alpha_{n-1}, \alpha_n]$ is a 2-cell. Thus $\alpha_{n-1} = a_0$, and $\alpha_{n-2} < a_0$ is impossible if $n > 2$. We obtain that $\mathcal{X}_n = \varnothing$ if $n \geqslant 3$. □

Let $M = M(m,\ell)$ be a circular monoid. We know that $H_0(G(m,\ell),\mathbb{Z}) = \mathbb{Z}$ and that $H_1(G(m,\ell),\mathbb{Z})$ is the abelianization $G(m,\ell)^{\mathrm{ab}}$ of $G(m,\ell)$. Since $C_n = \{0\}$ for $n \geqslant 3$, we have $H_n(G(m,\ell),\mathbb{Z}) = \{0\}$ for $n \geqslant 3$. Furthermore, the group $H_2(G(m,\ell),\mathbb{Z})$ is the kernel of the map $\partial_2 : C_2 \otimes \mathbb{Z} \to C_1 \otimes \mathbb{Z}$. In particular, it is a free abelian group. Since the Euler characteristic of the complex (C_n, ∂_n) is 0, we get that $H_2(G(m,\ell),\mathbb{Z}) \simeq \mathbb{Z}^{r-1}$ where r is the rank of the free part of $H_1(G(m,\ell),\mathbb{Z})$. Thus the integral homology of $G(m,\ell)$ can be computed by only computing $G(m,\ell)^{\mathrm{ab}}$.

Lemma B.1.21 (Integral homology of circular groups). *Let m,ℓ be positive integers. We have $G(m,\ell)^{\mathrm{ab}} \simeq \mathbb{Z}^{m\wedge\ell}$. The integral homology of $G(m,\ell)$ is then given by*

$$H_n(G(m,\ell),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{m\wedge\ell} & \text{if } n = 1, \\ \mathbb{Z}^{m\wedge\ell-1} & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

Proof. Let $a_0,\ldots,a_{m-1}$ denote the atoms of $M(m,\ell)$. In $G(m,\ell)^{\mathrm{ab}}$, for all $i \in [\![0, m-1]\!]$, we have $a_{i+\ell} = s(i,\ell)^{-1} a_i s(i,\ell) = a_i$. Conversely, in the group $\mathbb{Z}^m$ quotiented by the relations $a_i = a_{i+\ell}$ for all $i \in [\![0, m-1]\!]$, we have

$$s(i+1,\ell) = s(i+1,\ell-1)a_{i+\ell} = s(i+1,\ell-1)a_i = a_i s(i+1,\ell-1) = s(i,\ell)$$

for all $i \in [\![0, m-1]\!]$. Thus we have

$$G(m,\ell)^{\mathrm{ab}} = \left\langle a_0,\ldots,a_{m-1} \;\middle|\; \begin{cases} a_i = a_{i+\ell} & \forall i \in [\![0, m-1]\!] \\ a_i a_j = a_j a_i & \forall i,j \in [\![0, m-1]\!] \end{cases} \right\rangle.$$

This group is free abelian, with rank the cardinality of $(\mathbb{Z}/m\mathbb{Z})/(\ell\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/(m\wedge\ell)\mathbb{Z}$. □

In particular, for $m = 2$, we recover the result of [Sal94, Table 1] on the homology of spherical Artin groups of rank 2.

Remarkable isomorphisms.

In this section we give the classification of circular groups up to group isomorphism. First, Corollary B.1.11 (Center of circular groups) gives that, for any positive integers m,ℓ, the group $G(m,\ell)$ is abelian if and only if $m = 1$ or $\ell = 1$ or $m = \ell = 2$, in which case $G(m,\ell) \simeq \mathbb{Z}$, $G(m,\ell) \simeq \mathbb{Z}$ and $G(m,\ell) \simeq \mathbb{Z}^2$, respectively.

Proposition B.1.22. *Let m, ℓ, m', ℓ' be four positive integers. If the groups $G(m,\ell)$ and $G(m',\ell')$ are isomorphic and nonabelian, then $(m',\ell') \in \{(m,\ell),(\ell,m)\}$.*

Proof. Let $d := m \wedge \ell$ and $d' := m' \wedge \ell'$. If $G(m,\ell)$ and $G(m',\ell')$ are isomorphic, then Lemma B.1.21 gives that

$$\mathbb{Z}^d \simeq H_1(G(m,\ell),\mathbb{Z}) \simeq H_1(G(m',\ell'),\mathbb{Z}) \simeq \mathbb{Z}^{d'}.$$

In particular we have $d' = d$.

Since $G(m,\ell)$ and $G(m',\ell')$ are nonabelian, both the centers of $G(m,\ell)$ and $G(m',\ell')$ are cyclic and generated by some power of Δ. An isomorphism $f : G(m,\ell) \to G(m',\ell')$ then induces a bijection between irreducible periodic elements of $G(m,\ell)$ and of $G(m',\ell')$. By Proposition B.1.13 (rootless periodic elements), we have

- If $m|\ell$, then we either have $m'|\ell'$, in which case we have $\frac{\ell}{m} = \frac{\ell'}{m'}$, $d = m = m' = d'$ and $\ell = \ell'$, or $\ell'|m'$, in which case we have $\frac{\ell}{m} = \frac{m'}{\ell'}$, $d = m = \ell' = d'$ and $\ell = m'$.
- If $\ell|m$, the same reasoning gives $(m,\ell) = (m',\ell')$ or $(m,\ell) = (\ell',m')$.
- Lastly, if neither $m|\ell$ nor $\ell|m$, then we have neither $m'|\ell'$ nor $\ell'|m'$. We then have either

$$\frac{\ell}{d} = \frac{\ell'}{d'} \text{ and } \frac{m}{d} = \frac{m'}{d'} \text{ or } \frac{\ell}{d} = \frac{m'}{d'} \text{ and } \frac{m}{d} = \frac{\ell'}{d'}.$$

  Since $d' = d$, we obtain $(m',\ell') \in \{(m,\ell),(\ell,m)\}$.

□

This proposition strongly restricts the possible isomorphisms between circular groups. We can then show that all the remaining possible isomorphisms actually occur:

Proposition B.1.23. *Let m, ℓ be two positive integers. There is an isomorphism of groups between $G(m,\ell)$ and $G(\ell,m)$, which sends atoms of $M(m,\ell)$ to conjugates of atoms in $M(\ell,m)$.*

Proof. The result is immediate if $m = \ell$. Up to exchanging m and ℓ, we can assume that $m < \ell$. Let $\ell = mp + r$ be the Euclidean division of ℓ by m.

We denote by $\{a_0,\ldots,a_{m-1}\}$ the atoms of $M(m,\ell)$ and by $\{b_0,\ldots,b_{\ell-1}\}$ the atoms of $M(\ell,m)$. We also consider F_m to be the free group generated by $\{a_0,\ldots,a_{m-1}\}$. Exceptionally, we denote the simple elements of $M(\ell,m)$ by $t(i,p)$ instead of $s(i,p)$ to avoid confusions with the simple elements of $M(m,\ell)$. We also denote by $\widetilde{s}(i,p)$ the product $a_i \cdots a_{i+p}$ in F_m.

Let $\widetilde{f} : F_m \to G(\ell,m)$ be the morphism defined by

$$\begin{cases} f(a_0) := b_{m-1}, \\ f(a_i) := (b_{m-i-1})^{f(\widetilde{s}(0,i))} \quad \forall i \in [\![1, m-1]\!]. \end{cases}$$

By an immediate induction, we get that $\widetilde{f}(\widetilde{s}(0,k)) = t(m-k,k)$ for all $k \in [\![0, m-1]\!]$. We show that $\widetilde{f}$ induces a well defined group morphism $f : G(m,\ell) \to G(\ell,m)$. Let $i \in [\![1, m-1]\!]$, we have

$$\begin{aligned}
\widetilde{f}(\widetilde{s}(i,\ell)) &= \widetilde{f}(\widetilde{s}(0,i)^{-1}\widetilde{s}(0,\ell+i)) \\
&= \widetilde{f}(\widetilde{s}(0,i)^{-1})\widetilde{f}(\widetilde{s}(0,\ell))\widetilde{f}(\widetilde{s}(0,r+i)) \\
&= t(m-i,i)^{-1}\widetilde{f}(\widetilde{s}(0,m)^q))t(m-r-i,r+i) \\
&= t(m-i,i)^{-1}t(0,m)^q t(m-r-i,r+i) \\
&= t(0,m)^q t(m-i+qm,i)^{-1}t(m-r-i,r+i) \\
&= t(0,m)^q t(m-i-r,i)^{-1}t(m-r-i,r+i) \\
&= t(0,m)^q t(0,r) = \widetilde{f}(\widetilde{s}(0,\ell)).
\end{aligned}$$

We show that f is an isomorphism by constructing its inverse. First, by definition of f, we have

$$\forall i \in [\![0, m-1]\!],\ b_{m-i-1} = f({}^{s(0,i)}a_i),$$

and we define $g(b_j) = a_{m-j-1}^{s(0,m-j-1)^{-1}}$ for $j \in [\![0, m-1]\!]$. We also have $\Delta = t(0,m) = f(s(0,m))$ and we define $g(t(0,m)) = s(0,m)$. Let $j \in [\![0, \ell-1]\!]$ and let $j = mp + j'$ be the Euclidean division of j by m. We have $b_j = \phi^p(b_{j'}) = \Delta^{-p}b_{j'}\Delta^p$ and we define

$$g(b_j) = g(\Delta^p b_{j'}\Delta^p) := s(0,m)^{-p}g(b_{j'})s(0,m)^p = a_{m-j'-1}^{s(m-j'-1,j'+1)s(0,m)^{p-1}}.$$

In order to show that g does define a group morphism $G(\ell,m) \to G(m,\ell)$, we have to show that $g(t(i,m))$ does not depend on i. We have $g(t(0,m)) = s(0,m)$ by definition. Then, let $i \in [\![1, \ell-1]\!]$ be such that $g(t(i-1,m)) = s(0,m)$. Let $i-1 = mp+k$ be the Euclidean division of $i-1$ by m. We have

$$\begin{aligned}
g(t(i,m)) &= g(b_{i-1}^{-1})g(t(i-1,m))g(b_{i+m-1}) = s(0,m) \\
&= a_{m-k-1}^{-1\ s(m-k-1,k+1)s(0,m)^{p-1}} s(0,m) a_{m-k-1}^{s(m-k-1,k+1)s(0,m)^p} \\
&= s(0,m).
\end{aligned}$$

We obtain that $g(t(i,m)) = g(t(0,m)) = s(0,m)$ by induction. It is an immediate check to see that f and g are inverse to each other. □

If we combine Proposition B.1.22 and Proposition B.1.23, we get a complete classification of circular groups up to group isomorphism.

Corollary B.1.24 (Classification of circular groups). *Let m, ℓ, m', ℓ' be four positive integers. The groups $G(m,\ell)$ and $G(m',\ell')$ are isomorphic if and only if one of the following holds:*

- *$1 \in \{m,\ell\}$ and $1 \in \{m',\ell'\}$. In this case, $G(m,\ell) \simeq G(m',\ell') \simeq \mathbb{Z}$.*
- *$(m',\ell') \in \{(m,\ell),(\ell,m)\}$.*

Example B.1.25. If $m = 2$, then $M(2,\ell)$ is the Artin monoid for the Artin group of type $I_2(\ell)$, while $M(\ell,2)$ is the dual braid monoid for the same Artin group. The isomorphism $G(2,\ell) \to G(\ell,2)$ constructed in the above proof is already known: it sends a_0 to b_1 and a_1 to $b_0^{b_1} = b_1^{-1}\Delta = b_2$.

B.1.5 Application to complex braid groups of rank 2

We refer the reader to Chapter 6 for general results on complex reflection groups. They are finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ generated by complex (pseudo)-reflections.

Fact. *Let $W \subset \mathrm{GL}_2(\mathbb{C})$ be a complex reflection group of rank 2. The braid group $B(W)$ is isomorphic to a circular group.*

This is mostly a rephrasing of [Ban76, Theorem 1 and Theorem 2]:

- The only non-irreducible cases are groups of the form $W = \mathbb{Z}/d\mathbb{Z}/ \times \mathbb{Z}/d'\mathbb{Z}$ (with $d, d' \geqslant 1$). In this case we have $B(W) \simeq \mathbb{Z} \times \mathbb{Z} \simeq G(2,2)$.
- If $W = G(de, e, 2)$ for e odd and $d \geqslant 2$ or $W \in \{G_5, G_{10}, G_{18}\}$, then $B(W) \simeq G(2,4)$.
- If $W = G(de, e, 2)$ for e even and $d \geqslant 2$ or $W \in \{G_7, G_{11}, G_{15}, G_{19}\}$, then $B(W) \simeq G(3,3)$.
- If $W = G(e, e, 2)$ for $e \geqslant 3$, then $B(W) \simeq G(2, e)$.
- If $W \in \{G_4, G_8, G_{16}\}$, then $B(W) \simeq G(2,3)$.
- If $W \in \{G_6, G_9, G_{13}, G_{17}\}$, then $B(W) \simeq G(2,6)$.
- If $W = G_{14}$, then $B(W) \simeq G(2,8)$.
- If $W = G_{20}$, then $B(W) \simeq G(2,5)$.
- If $W = G_{21}$, then $B(W) \simeq G(2,10)$.
- If $W = G_{12}$, then $B(W) \simeq G(3,4)$.
- If $W = G_{22}$, then $B(W) \simeq G(3,5)$.

A direct application of Theorem B.1.16 (uniqueness of roots up to conjugacy) then gives

Theorem B.1.26. *Let W be a complex reflection group of rank 2, and let $B(W)$ be its braid group. If $\alpha, \beta \in B(W)$ are such that $\alpha^n = \beta^n$ for some nonzero integer n, then α and β are conjugate in $B(W)$.*

Note also that, as we saw in Section B.1.3, the center of a finite index subgroup of a circular group is included in the center of the group. In particular, this is also true for complex braid groups of rank 2.

Remark B.1.27. Our approach only covers complex reflection groups of rank 2. Indeed, by Lemma B.1.20, circular groups have homological dimension at most 2, and a complex braid group of rank r has homological dimension r by [CM14, Proposition 1.1]. Thus only rank 2 complex reflection groups can have a braid group isomorphic to a circular group.

B.2 Δ-product and hosohedral-type groups

In this section we present hosohedral-type Garside groups as a generalization of circular groups. These groups are enveloping groups of so-called hosohedral-type monoids. These monoids were first introduced by Picantin in his PhD thesis under the name "monoïdes de type fuseau" ([Pic00, Définition 1.3]). In [Pic00, Proposition 2.4], Picantin shows that these monoids are exactly the Garside monoids whose lattice of simples has the shape of a hosohedron ("fuseau" in french). The name "hosohedral-type monoid" was suggested to us by Picantin.

More recently, hosohedral-type groups were identified by Mireille Soergel in [Soe23, Theorem 4.6] as the Garside groups satisfying a particular nonpositive curvature property (namely, the systolicity of the flag complex associated to the “Garside presentation” as in [Soe23, Lemma 4.3]).

Here we introduce these groups as a particular case of a general construction, already present in [DP99], which we call the Δ-product of Garside monoids.

B.2.1 Δ-product of Garside monoids

Let $(M_1, \Delta_1), \dots, (M_h, \Delta_h)$ be a finite family of homogeneous Garside monoids that we fix throughout this section. We denote by $\ell_1, \dots, \ell_h$ the associated length functions, and by $S_1, \dots, S_h$ the associated set of simples. We also set $A_1, \dots, A_h$ the set of atoms of $M_1, \dots, M_h$, respectively.

The free product $M_1 * \cdots * M_h$ is not a Garside monoid because two atoms coming from different factors M_i and M_j do not have any common multiple (in particular, no lcms). We can fix this by forcing the Δ_i to all be equal.

Definition B.2.1 (Δ-product). Let $(M_1, \Delta_1), \dots, (M_h, \Delta_h)$ be a family of homogeneous Garside monoids. The Δ-*product* of the M_i is defined by

$$M_1 *_\Delta M_2 *_\Delta \cdots *_\Delta M_h := *_{i=1}^{h} M_i \Big/ (\Delta_i = \Delta_j \ \forall i, j \in [\![1, h]\!]) \cdot$$

Likewise, we define the Δ-product of the enveloping groups $G(M_i)$ by

$$G(M_1) *_\Delta G(M_2) *_\Delta \cdots *_\Delta G(M_h) = *_{i=1}^{h} G(M_i) \Big/ (\Delta_i = \Delta_j \ \forall i, j \in [\![1, h]\!]) \cdot$$

Remark B.2.2. The definition of the Δ-product of Garside monoids really depends on the Garside element and not only on the monoids themselves. For instance, we have $M(1, \ell) \simeq \mathbb{Z}_{\geqslant 0}$ as a monoid for every $\ell \geqslant 1$. However, we have

$$M(1, p) *_\Delta M(1, q) \simeq \langle a, b \mid a^p = b^q \rangle^+.$$

If $p, q \geqslant 2$, then this monoid has two atoms and cannot be isomorphic to $\mathbb{Z}_{\geqslant 0} \simeq M(1,1) *_\Delta M(1,1)$. Furthermore, the enveloping group of this monoid is not always isomorphic to $\mathbb{Z} = G(M(1,1) *_\Delta M(1,1))$.

Remark B.2.3. Note that, if (M, Δ) is a Garside monoid, the monoid $M *_\Delta M(1,1)$ is naturally isomorphic to M. Thus we can assume that all the (M_i, Δ_i) are distinct from $M(1,1)$.

We first show that the enveloping group of the Δ-product of the (M_i, Δ_i) identifies with the Δ-product of the enveloping groups $G(M_i)$.

Lemma B.2.4. *Let $(M_1, \Delta_1), \dots, (M_h, \Delta_h)$ be a family of homogeneous Garside monoids. Let also M be the Δ-product $M_1 *_\Delta \cdots *_\Delta M_h$. The groups $G(M)$ and $G(M_1) *_\Delta \cdots *_\Delta G(M_h)$ are naturally isomorphic.*

Proof. If M, M' are two monoids, we denote by $\mathrm{Hom}(M, M')$ the set of monoid morphisms from M to M'. If G and G' are two groups, $\mathrm{Hom}(G, G')$ is in fact the set of group morphisms from G

to G'. Let H be a group. By definition of the enveloping group and of the Δ-product, we have natural bijections

$$\begin{aligned}
\mathrm{Hom}(G(M),H) &\simeq \mathrm{Hom}(M_1 *_\Delta \cdots *_\Delta M_h, H)\\
&\simeq \{f \in \mathrm{Hom}(M_1 * \cdots * M_h, H) \mid \forall i,j \in [\![1,h]\!], f(\Delta_i) = f(\Delta_j)\}\\
&\simeq \left\{(f_i) \in \prod_{i=1}^n \mathrm{Hom}(M_i,H) \ \mid \forall i,j, f_i(\Delta_i) = f_j(\Delta_j)\right\}\\
&\simeq \left\{(f_i) \in \prod_{i=1}^n \mathrm{Hom}(G(M_i),H) \ \mid \forall i,j, f_i(\Delta_i) = f_j(\Delta_j)\right\}\\
&\simeq \{f \in \mathrm{Hom}(G(M_1) * \cdots * G(M_h), H) \mid \forall i,j \in [\![1,h]\!], f(\Delta_i) = f(\Delta_j)\}\\
&\simeq \mathrm{Hom}(G(M_1) *_\Delta \cdots *_\Delta G(M_h), H).
\end{aligned}$$

Applying this to $H := G(M)$ gives a bijection $\mathrm{Hom}(G(M),G(M)) \simeq \mathrm{Hom}(G(M_1) *_\Delta \cdots *_\Delta G(M_h), G(M))$. The image of the identity morphism of $G(M)$ under this bijection gives the desired result. □

From now on, we will identify $G(M_1 *_\Delta \cdots *_\Delta M_h)$ with $G(M_1) *_\Delta \cdots *_\Delta G(M_h)$. We now fix a family $(M_1,\Delta_1),\ldots,(M_h,\Delta_h)$ of Garside monoids, all distinct from $M(1,1)$. We denote by M the associated Δ-product, and by A its set of atoms. Let $i \in [\![1,h]\!]$. By definition of the Δ-product as a quotient of the free product, there is a natural morphism $\varphi_i : G(M_i) \to G(M)$. We also denote by φ_i the restriction from M_i to M.

Proposition B.2.5. *[DP99, Proposition 5.3]*
The atoms of the monoid M are given by $A := \bigsqcup_{i=1}^h \varphi_i(A_i)$. Furthermore, (M,Δ) is a homogeneous Garside monoid with simple elements

$$S = \bigcup_{i=1}^h \varphi_i(S_i).$$

Where $\Delta = \varphi_i(\Delta_i)$ for any $i \in [\![1,h]\!]$. The Garside automorphism ϕ of (M,Δ) is given on a simple $\varphi_i(s)$ by $\phi(\varphi_i(s)) = \varphi_i(\phi_i(s))$, where ϕ_i is the Garside automorphism of (M_i,Δ_i).

The assertion on the atoms and the assertion that (M,Δ) is a homogeneous Garside monoid are in the original statement of [DP99, Proposition 5.3]. For $i \in [\![1,h]\!]$, we identify A_i with the subset $\varphi_i(A_i)$ of A from now on. The proof of [DP99, Proposition 5.3] can also be used to show the assertion on the simple elements:

First, let $i \in [\![1,h]\!]$ and let $s \in S_i$ be a simple element of (M_i,Δ_i). We have $s\overline{s} = \Delta_i$ and $\varphi_i(s)\varphi_i(\overline{s}) = \Delta$. Thus $\varphi_i(s)$ is simple and $\overline{\varphi_i(s)} = \varphi_i(\overline{s})$. We also obtain that the Garside automorphism is given by $\phi(\varphi_i(s)) = \overline{\overline{\varphi_i(s)}} = \varphi_i(\phi_i(s))$. Conversely, we have to show that, if $s \in S$ is a simple element of (M,Δ), then there is some $i \in [\![1,h]\!]$ and some simple $\widetilde{s} \in S_i$ with $s = \varphi_i(\widetilde{s})$. This is a direct consequence of the following lemma:

Lemma B.2.6. *Let w be a word in A which expresses a simple element $s \in S$. There is some $i \in [\![1,h]\!]$ such that all the letters of w lie inside $\varphi_i(A_i)$. The word w then also represents some $\widetilde{s} \in S_i$ with $\varphi_i(\widetilde{s}) = s$. Furthermore, if $s \notin \{1,\Delta\}$, then the integer i and the simple $\widetilde{s} \in S_i$ are unique.*

Proof. Let $i \in [\![1, h]\!]$. For each pair $a, b \in A_i$, we choose two words $f_i(a, b)$ and $f_i(b, a)$ such that $af_i(a, b)$ and $bf_i(b, a)$ are two words in A_i expressing $a \vee b$ in M_i. By [DP99, Theorem 4.1], the monoid M_i admits the following presentation:

$$M_i = \langle af_i(a, b) = bf_i(b, a) \mid a, b \in A_i \rangle^+.$$

Now, for $a \in A_i$, we choose a word $c(a)$ in A_i representing $\overline{a}$ in M_i. The proof of [DP99, Proposition 5.3] gives that M admits the presentation

$$M \simeq \left\langle af(a, b) = bf(b, a) \;\middle|\; f(a, b) = \begin{cases} f_i(a, b) & \text{if } a, b \in A_i \\ c(a) & \text{if } a \in A_i, b \in A_j, i \neq j \end{cases} \right\rangle^+. \tag{B.2.1}$$

By definition, if $w_1 = w_2$ is a relation in this presentation, then we either have

- There is an $i \in [\![1, h]\!]$ such that all the letters of both w_1 and w_2 lie in A_i. In this case, $w_1 = w_2$ also holds in M_i.
- There are two distinct integers $i, j \in [\![1, h]\!]$ and two atoms $a \in A_i, b \in A_j$ such that $w_1 = ac(a)$ and $w_2 = bc(b)$. In this case, w_1 (resp. w_2) represents Δ_i in M_i (resp. Δ_j in M_j).

Let w be a word in A which represents Δ in M, and let $a \in A$ be an atom of M. By assumption, there is a sequence of words $w_1, \ldots, w_m$ in A such that $w_1 = w$, $w_m = ac(a)$ and each w_k is equivalent to w_{k+1} by the use of one relation of presentation (B.2.1) for $k \in [\![1, m-1]\!]$. Up to changing the atom a, we can assume that m is the first integer such that w_m is equal to a word of the form $bc(b)$ for some atom b.

Let $i \in [\![1, h]\!]$ be such that $a \in A_i$. We claim that for all $j \in [\![1, m]\!]$, the word w_j contains letters only in A_i. If this is not the case, then let k_0 be the last integer in $[\![1, m]\!]$ such that w_{k_0} contains letters not lying in $\varphi_i(A_i)$ (we have $k_0 < m$ by assumption). The defining relations of (B.2.1) giving $w_{k_0} = w_{k_0+1}$ in M then have the form $bc(b) = ac(a)$. Since w_{k_0} and $bc(b)$ both express Δ in M, we have $w_{k_0} = bc(b)$, which contradicts the minimality assumption on m.

Since all the letters of w_j lie inside A_i for $j \in [\![1, m]\!]$, we have that the relations of (B.2.1) giving $w_j = w_{j+1}$ for $j \in [\![1, m-1]\!]$ also hold in M_i. Thus w_1 is a word in A_i, which expresses the element Δ_i in M_i.

Let now $s \in S$ be a simple element of M. By definition, there is a word $w_1 = ww_2$ in A expressing Δ such that w expresses s in M. The first part of the proof gives that there is some $i \in [\![1, h]\!]$ such that w_1, w and w_2 are actually words in A_i. The word w (resp. w_2) then expresses an element $\widetilde{s}$ (resp. $\widetilde{s}'$) in M_i, with $\widetilde{s}\widetilde{s}' = \Delta_i$. We then have $\widetilde{s} \in S_i$ and $\varphi_i(\widetilde{s}) = s$.

Suppose now that $s \notin \{1, \Delta\}$, and let w' be another word in A expressing s in M. By definition, there is a sequence of words $w_1, \ldots, w_n$ of words in A, such that $w_1 = w$, $w_n = w'$ and each w_k is equivalent to w_{k+1} by the use of one relation of presentation (B.2.1). Since $s \prec \Delta$, none of the w_k contain a subword expressing Δ in M. We deduce that all the relations giving $w_k = w_{k+1}$ in M for $k \in [\![1, n-1]\!]$ are between words in A_i and also hold in M_i. Thus w_k also expresses $\widetilde{s}$ for all $k \in [\![1, n]\!]$. □

Proposition B.2.7. *Let $i \in [\![1, n]\!]$ and let $s \in S_i$. The morphism φ_i induces an injective morphism of lattices from $\{t \in S_i \mid t \preccurlyeq s\}$ to $\{t \in S \mid t \preccurlyeq \varphi_i(s)\}$. Furthermore, if $s \neq \Delta_i$, then this morphism is also surjective.*

Proof. We show the injectivity and surjectivity assumptions before considering lcms and gcds. Let $s \in S_i$ and let X and Y denote the two considered sets. Since φ_i is a morphism of monoids, it restricts to a poset morphism $X \to Y$. First, we show that $\varphi_i : X \to Y$ is always injective. Let $t, t' \in X$ be such that $\varphi_i(t) = \varphi_i(t')$. Let w and w' be two words in A_i expressing t and t', respectively. The words w and w' express the same element $\varphi_i(t)$ in M. If $\varphi_i(t) = \Delta$, then w and w' are two words in A_i representing Δ in M. The proof of Lemma B.2.6 gives that w and w' represent the same element in M_i, which is Δ_i. If $\varphi_i(t) \neq \Delta$, then w and w' express the same element in S_i by Lemma B.2.6, thus $t = t'$.

Now, suppose that $s \neq \Delta$. We show that $\varphi_i : X \to Y$ is surjective. Let $t \preccurlyeq \varphi_i(s)$ in M. By definition, there is a word $w = w_1 w_2$ in A expressing $\varphi_i(s)$ such that w_1 expresses t. By Lemma B.2.6, w and w_1 are words in A_i. The word w_1 then expresses some element $\widetilde{t} \in S_i$ such that $\varphi_i(\widetilde{t}) = t$ and $\varphi_i : X \to Y$ is surjective.

We now show that $\varphi_i : S_i \to S$ is a morphism of lattices. That is, φ_i preserves right-lcms and left-gcds. Let $s, t \in S_i$ be two simples of (M_i, Δ_i), and let $u = s \wedge t$. Let $x := \varphi_i(s) \wedge \varphi_i(t)$. Since the simple $\varphi_i(u)$ is obviously a left-divisor of $\varphi_i(s)$ and $\varphi_i(t)$, we have $\varphi_i(u) \preccurlyeq x$. If $s = \Delta_i$, then $u = t$ and $x = \varphi_i(t) = \varphi_i(u)$. Likewise the result is clear if $t = \Delta_i$. We assume from now on that $t, s \neq \Delta_i$. Since $s \neq \Delta_i$ (resp. $t \neq \Delta_i$), the first part of the proof gives the existence of a unique $\widetilde{x}$ (resp. $\widetilde{x}'$) in S_i such that $\widetilde{x} \preccurlyeq s$ (resp. $\widetilde{x}' \preccurlyeq t$) and $\varphi_i(\widetilde{x}) = x = \varphi_i(\widetilde{x}')$. Since φ_i is injective on S_i, we get that $\widetilde{x} = \widetilde{x}'$ is a common divisor of s and t in M_i. We obtain $\widetilde{x} \preccurlyeq u$, $x \preccurlyeq \varphi_i(u)$ and $x = \varphi_i(u)$.

Let now $v = s \vee t$ and $y = \varphi_i(s) \vee \varphi_i(t)$. Again, since $\varphi_i(v)$ is an obvious right-multiple of both s and t, we have $y \preccurlyeq \varphi_i(v)$. By the first part of the proof, there is a unique $\widetilde{y} \in S_i$ such that $\widetilde{y} \preccurlyeq v$ and $\varphi_i(\widetilde{y}) = y$. The first part of the proof also gives that $s, t \preccurlyeq \widetilde{y}$. Thus $v \preccurlyeq \widetilde{y}$, $\varphi_i(v) \preccurlyeq \varphi_i(\widetilde{y}) = y$ and $\varphi_i(v) = y$. □

Of course, one can show by similar arguments that φ_i is an injective morphism of lattices from $(S_i, \succcurlyeq)$ to $(S, \succcurlyeq)$.

Corollary B.2.8. *Let s, t be two simple elements in M, both different from Δ and 1. Assume that $s = \varphi_i(\widetilde{s})$ and $t = \varphi_j(\widetilde{t})$ for $i \neq j$, $\widetilde{s} \in S_i$ and $\widetilde{t} \in S_j$. We have $s \wedge t = 1$ and $s \vee t = \Delta$ in M. Furthermore, the word st is greedy in (M, Δ).*

Proof. Let $u := s \wedge t$. By Proposition B.2.7, there is a unique $\widetilde{u} \in S_i$ (resp. $\widetilde{u}' \in S_j$) such that $\varphi_i(\widetilde{u}) = \varphi_j(\widetilde{u}') = u$. Since $i \neq j$, Lemma B.2.6 gives that $u = 1$. We apply similar reasoning for lcms. This gives in particular that $\overline{s} \wedge t = 1$, thus the path st is greedy. □

Proposition B.2.9. *Let $i \in [\![1, h]\!]$. The morphism $\varphi_i : G(M_i) \to G(M)$ preserves left-weighted factorizations. In particular it is injective.*

Proof. We first show that $\varphi_i : M_i \to M$ preserves greedy normal forms. By definition, we only have to show that, if st is a greedy word in M_i, then $\varphi_i(s)\varphi_i(t)$ is a greedy word in M. If st is a greedy word of length 2 in M_i, then Proposition B.2.7 gives

$$\overline{\varphi_i(s)} \wedge \varphi_i(t) = \varphi_i(\overline{s}) \wedge \varphi_i(t) = \varphi_i(\overline{s} \wedge t) = 1.$$

The word $\varphi_i(s)\varphi_i(t)$ is then greedy by Lemma 2.1.13.

Let now $x = \Delta_i^k s_1 \cdots s_r$ be the left-weighted factorization of some $x \in M_i$. Since $\Delta_i \not\preccurlyeq s_1 \cdots s_r$, we have $\Delta \not\preccurlyeq \varphi_i(s_1 \cdots s_r)$. Furthermore, the word $\varphi_i(s_1) \cdots \varphi_i(s_r)$ is greedy because $\varphi_i : M_i \to M$

preserves greediness. The word $\Delta^k\varphi_i(s_1)\cdots\varphi_i(s_r)$ is then the left-weighted factorization of $\varphi_i(x)$ by definition. $\square$

Proposition B.2.10. *Let $i \in [\![1,h]\!]$, and let $x \in G(M_i)$. We have $\varphi_i(\mathrm{SSS}(x)) = \mathrm{SSS}(\varphi_i(x))$. Furthermore, if x is not conjugate to a power of Δ_i in M_i, then the centralizer of $\varphi_i(x)$ in $G(M)$ is the image under φ_i of the centralizer of x in $G(M_i)$.*

Proof. Let x be an element of $G(M_i)$. By Proposition B.2.9, we have $\varphi_i(\mathrm{init}(x)) = \mathrm{init}(\varphi_i(x))$ and $\varphi_i(\mathrm{fin}(x)) = \mathrm{fin}(\varphi_i(x))$. Thus, we also have $\varphi_i(\mathrm{cyc}(x)) = \mathrm{cyc}(\varphi_i(x))$ and $\varphi_i(\mathrm{dec}(x)) = \mathrm{dec}(\varphi_i(x))$.

If x is conjugate to some Δ_i^k with $k \in \mathbb{Z}$, then we have $\mathrm{SSS}(x) = \{\Delta^k\} = \mathrm{SSS}(\varphi_i(x))$.

From now on, we suppose that x is not conjugate to any power of Δ. We show that $\inf(\mathrm{SSS}(x)) = \inf(\mathrm{SSS}(\varphi_i(x))$ and $\sup(\mathrm{SSS}(x)) = \sup(\mathrm{SSS}(\varphi_i(x))$. First, let $x' \in \mathrm{SSS}(x)$, Proposition B.2.9 gives that

$$\inf(\mathrm{SSS}(\varphi_i(x))) \geqslant \inf(\varphi_i(x')) = \inf(x') = \inf(\mathrm{SSS}(x)),$$

$$\sup(\mathrm{SSS}(\varphi_i(x))) \leqslant \sup(\varphi_i(x')) = \sup(x') = \sup(\mathrm{SSS}(x)).$$

Conversely, one can reach an element y of $\mathrm{SSS}(\varphi_i(x))$ by applying a sequence of cyclings and decyclings to $\varphi_i(x)$. Applying the same operations to x gives a conjugate $\widetilde{y}$ of x in $G(M_i)$ such that $\varphi_i(\widetilde{y}) = y$. We then have

$$\inf(\mathrm{SSS}(x)) \geqslant \inf(\widetilde{y}) = \inf(y) = \inf(\mathrm{SSS}(\varphi_i(x))),$$

$$\sup(\mathrm{SSS}(x)) \leqslant \sup(\widetilde{y}) = \sup(y) = \sup(\mathrm{SSS}(\varphi_i(y))).$$

Thus, $\inf(\mathrm{SSS}(x)) = \inf(\mathrm{SSS}(\varphi_i(x)))$ and $\sup(\mathrm{SSS}(x)) = \sup(\mathrm{SSS}(\varphi_i(x)))$ as claimed.

Now, we show that $\varphi_i(\mathrm{SSS}(x)) \subset \mathrm{SSS}(\varphi_i(x))$. Let $x' \in \mathrm{SSS}(x)$, we have $\inf(\varphi_i(x')) = \inf(x') = \inf(\mathrm{SSS}(x)) = \inf(\mathrm{SSS}(\varphi_i(x)))$, and likewise, $\sup(\varphi_i(x')) = \sup(\mathrm{SSS}(\varphi_i(x)))$. Since $\varphi_i(x')$ is conjugate to $\varphi_i(x)$, we have $\varphi_i(x') \in \mathrm{SSS}(\varphi_i(x))$.

Let $x' \in \mathrm{SSS}(x)$, and let $s \in S$ be a simple element of M such that $\varphi_i(x')^s \in \mathrm{SSS}(\varphi_i(x))$. We show that $s \in \varphi_i(S_i)$. Let $x' = \Delta_i^k s_1 \cdots s_r$ be the left-weighted factorization of x' in M_i. The left-weighted factorization of $\varphi_i(x')$ is given by $\Delta^k\varphi_i(s_1)\cdots\varphi_i(s_r)$. We have

$$\varphi_i(x')^s = \Delta^{k-1}\phi^{k-1}(\overline{s})\varphi_i(s_1)\cdots\varphi_i(s_r)s.$$

If $s \notin \varphi_i(S_i)$, then the words $\phi^{k-1}(\overline{s})\varphi_i(s_1)$ and $\varphi_i(s_r)s$ are in greedy normal form by Corollary B.2.8. The above expression is then the left-weighted factorization of $\varphi_i(x')^s$ in M. Thus $\inf(\varphi_i(x')^s) = k-1$ and $\varphi_i(x')^s \notin \mathrm{SSS}(\varphi_i(x'))$. The same reasoning shows that if $\varphi_i(x')^{s^{-1}} \in \mathrm{SSS}(\varphi_i(x'))$, then $s \in \varphi_i(S_i)$. An immediate induction then shows that the connected component of $\varphi_i(x')$ in $\mathrm{CG}(\varphi_i(x))$ is contained in $\varphi_i(\mathrm{SSS}(x))$. Since $\mathrm{CG}(\varphi_i(x))$ is connected, we have $\mathrm{SSS}(\varphi_i(x)) \subset \varphi_i(\mathrm{SSS}(x))$.

This also shows that the conjugacy graph $\mathrm{CG}(\varphi_i(x))$ is the image under of $\mathrm{CG}(x)$ under φ_i, whence the result on centralizers. $\square$

Proposition B.2.11 (Periodic elements in a Δ-product). *Let p, q be nonzero integers, and let $\rho \in G(M)$ be a (p,q)-periodic element. There is some $i \in [\![1,h]\!]$ and some (p,q)-periodic element $\sigma \in G(M_i)$ such that ρ is conjugate to $\varphi_i(\sigma)$ in $G(M)$.*

Proof. Let $\rho \in G(M)$ be a (p,q)-periodic element. If ρ is conjugate to a power of Δ, then the result is obvious. Otherwise, Theorem 3.4.4 gives that ρ is conjugate to an element of the form $\Delta^k s$ for some $s \in S$. By Proposition B.2.5, there is some $i \in [\![1,h]\!]$ and some $\widetilde{s} \in S$ such that $\varphi_i(\widetilde{s}) = s$. We then have $\Delta^k s = \varphi_i(\Delta_i^k \widetilde{s})$. The element $\Delta_i^k \widetilde{s}$ is (p,q)-periodic in M_i by Proposition B.2.9. □

Let $i \in [\![1,h]\!]$ and let k_i be the smallest integer such that $\Delta_i^{k_i}$ is central in $G(M_i)$. By Proposition B.2.5, the smallest central power of Δ in $G(M)$ is given by the lcm of the k_i for $i \in [\![1,h]\!]$.

Proposition B.2.12 (Center of a nontrivial Δ-product). *Assume that $h \geqslant 2$. The intersection in $G(M)$ of all the $\varphi_i(G(M_i))$ for $i \in [\![1,h]\!]$ is the subgroup generated by Δ. The center of $G(M)$ is cyclic and generated by the smallest central power of Δ in $G(M)$.*

Proof. Since $h \geqslant 2$, we can choose $i \neq j$ in $[\![1,h]\!]$. Let $x_i \in G(M_i)$ and $x_j \in G(M_j)$ be such that $x := \varphi_i(x_i) = \varphi_j(x_j) \in G(M)$. If $x_i = \Delta_i^{k_i} s_1 \dots s_r$ (resp. $x_j = \Delta_j^{k_j} t_1 \dots t_u$) is the left-weighted factorization of x_i in $G(M_i)$ (resp. of x_j in $G(M_j)$), then by Proposition B.2.9, the left-weighted factorization of x in $G(M)$ is then given by

$$x = \Delta^{k_i} \varphi_i(s_1) \cdots \varphi_i(s_r) = \Delta^{k_j} \varphi_j(t_1) \cdots \varphi_j(t_u).$$

We deduce that $k_i = k_j$,$r = u$ and $\varphi_i(s_k) = \varphi_j(t_k)$ for all $k \in [\![1,r]\!]$. Since $i \neq j$, Lemma B.2.6 gives that $r = 0$ and x is a power of Δ.

Let now x be an element of $Z(G(M))$, and let $i \in [\![1,h]\!]$. By definition, x lies in the centralizer of $\varphi_i(a)$ for all $a \in A_i$. Since all the M_i are distinct from $M(1,1)$, the elements of A_i are not conjugate to a power of Δ. Thus Proposition B.2.10 gives that x actually lies in the image under φ_i of the common centralizer of all elements of A_i in M_i. In other words we have $x \in \varphi_i(Z(G(M_i))$. We then have that $x \in \bigcap_{i=1}^h \varphi_i(Z(G(M_i)))$ is a power of Δ, whence the result. □

B.2.2 Hosohedral-type Garside groups

Definition B.2.13 (Hosohedral-type Garside groups). A Garside monoid (M, Δ) is a *hosohedral-type monoid* if it is a Δ-product of circular monoids. The enveloping group of a hosohedral-type monoid is a *hosohedral-type group*.

For instance, torus knot groups are hosohedral-type groups. Indeed, for p, q two positive coprime integers, we have

$$\langle a, b \mid a^p = b^q \rangle \simeq G(M(1,p) *_\Delta M(1,q)).$$

In his PhD thesis, Picantin introduced hosohedral-type monoids as the Garside monoids whose lattice of simple elements satisfy a strong property regarding gcds and lcms.

Theorem B.2.14. *[Pic00, Proposition 2.5]*
A finite lattice $(S, \wedge, \vee, 0, 1)$ has hosohedral-type *if any couple (s,t) in S satisfies*

$$(a \wedge b, a \vee b) \in \{(a,b), (b,a), (0,1)\}.$$

A Garside monoid (M, Δ) is a hosohedral-type monoid if and only if the lattice $(S, \preccurlyeq)$ of its simple elements has hosohedral-type.

Note that this theorem also covers the case of non-homogeneous Garside monoids, which we do not consider here.

In his proof, Picantin directly classifies all the Garside monoids whose lattice of simples is a fixed hosohedral-type lattice, in terms of the length of maximal chains. Under the notation of [Pic00, Proposition VI.2.5 and Definition VI.2.7], the monoid

$$\mathrm{fus}^+[\![h_1^{u_{1,1}}\cdot\ldots\cdot h_1^{u_{1,k_1}}\cdot h_2^{u_{2,1}}\cdot\ldots\cdot h_n^{u_{n,k_n}}]\!]$$

is isomorphic to the Δ-product

$$M(u_{1,1},h_1) *_\Delta \cdots *_\Delta M(u_{1,k_1},h_1) *_\Delta M(u_{2,1},h_2) *_\Delta \cdots *_\Delta M(u_{n,k_n},h_n).$$

This property regarding lcms and gcds of simple elements also induces strong geometric properties for the associated Garside group, as pointed out by Mireille Soergel in [Soe23, Theorem 4.6]. By [DDGKM, Proposition VI.1.11], every Garside monoid (M,Δ) with set of simples S admits a presentation

$$M \simeq \langle S \mid s\cdot t = st\ \forall s,t\in S \text{ such that } st\in S\rangle^+ \tag{B.2.2}$$

Theorem B.2.15. *[Soe23, Theorem 4.6]*
Let (M,Δ) be a Garside monoid. The flag complex of the Cayley graph of $G(M)$ associated to presentation (B.2.2) is systolic (in the sense of [Soe23, Section 2]) if and only if (M,Δ) is a hosohedral-type monoid.

This theorem also covers the case of non-homogeneous Garside monoids.

Example B.2.16. In [Soe23, Question after Remark 4.8], Soergel considers the Garside group G_k, defined for an integer $k\geqslant 2$ by the presentation $G_k := \langle a,b \mid aba = b^k\rangle$. She asks whether or not this group is isomorphic to a hosohedral-type group. We have the following isomorphisms of groups, given by Tietze transformations:

$$\begin{aligned}\langle a,b \mid aba=b^k\rangle &= \langle a,b,x \mid x=b^k a^{-1}, aba=b^k\rangle\\ &= \langle a,b,x \mid a=x^{-1}b^k, aba=b^k\rangle\\ &= \langle b,x \mid x^{-1}b^k b x^{-1} b^k = b^k\rangle\\ &= \langle b,x \mid b^{k+1}=x^2\rangle.\end{aligned}$$

The last group is a hosohedral-type group. More generally, a conjecture of Picantin ([Pic00, Conjecture 1]) states that, if M is a Garside monoid with two atoms, then the enveloping group $G(M)$ is isomorphic to either a group of the form $\langle a,b \mid a^p=b^q\rangle$ for positive integers p,q, or to an Artin group of dihedral type. In either case this would mean that the enveloping group of a Garside monoid with two generators is always a hosohedral-type group.

From now on, let $M = M(m_1,\ell_1) *_\Delta \cdots M(m_h,\ell_h)$ be a hosohedral-type monoid. Using Remark B.2.3, we assume that $(m_j,\ell_j)\neq(1,1)$ for all $j\in[\![1,h]\!]$. We also assume that $h\geqslant 2$, otherwise we recover results from Section B.1. By Proposition B.2.9, we can identify the factors $M(m_j,\ell_j)$ (resp. $G(m_i,\ell_i)$) with the associated subgroup of M (resp. of $G(M)$).

To avoid confusion, the simple $s(i,p)$ of the factor $M(m_j,\ell_j)$ will be denoted by $s_j(i,p)$.

Conjugacy

Like in the case of circular groups, an element in a super-summit set of a hosohedral-type monoid is either rigid or periodic.

Proposition B.2.17. *Let x be an element of $G(M)$. If x lies in its own super-summit set, then it is either rigid or periodic.*

Proof. The proof imitates the case of circular groups. First, if $\inf(x) = \sup(x)$, then we have $x = \Delta^k$ for some $k \in \mathbb{Z}$. In this case, x is obviously both rigid and $(1, k)$-periodic.

Suppose now that $\sup(x) = \inf(x) + 1$. We have $x = \Delta^k s_j(i,p)$ for some $j \in [\![1, h]\!]$, $i \in [\![0, m_j - 1]\!]$ and $0 < p < \ell_j$. The element x is periodic if and only if it is periodic as an element of the factor $G(m_j, \ell_j)$. By Lemma B.1.8, this is equivalent to $p + k\ell_j \equiv 0[m_j]$. If this is not the case, the word $s_j(i,p)\phi^{-k}(s_j(i,p))$ is greedy in $M(m_i, \ell_i)$. It is then also greedy in M by Proposition B.2.9.

Lastly, suppose that $\sup(x) > \inf(x) + 1$. The left-weighted factorization of x is given by $\Delta^k s_1 \cdots s_r$ with $r > 1$. We claim that x is rigid. Otherwise, the word $s_r\phi^{-k}(s_1)$ is not greedy. By Corollary B.2.8, this implies that s_r and s_1 lie in the same factor $M(m_i, \ell_i)$. We can then apply the last part of the proof of Proposition B.1.7 to show that we either have $\sup(\mathrm{cyc}(x)) < \sup(x)$ or $\inf(\mathrm{cyc}(x)) > \inf(x)$. In both cases, we have $x \notin \mathrm{SSS}(x)$. □

By Proposition B.2.11, there are (p, q)-periodic elements in $G(M)$ if and only if there are (p, q)-periodic elements in some factor $G(m_j, \ell_j)$. However, two (p, q)-periodic elements coming from two different factors are not conjugate in general.

Proposition B.2.18. *Let p, q be integers. Two (p, q)-periodic elements of $G(M)$ are conjugate if and only if they both admit a conjugate lying the the same factor $G(m_j, \ell_j)$.*

Proof. Let ρ, σ be two (p, q)-periodic elements in $G(M)$. If ρ and σ are conjugate, then $\mathrm{SSS}(\rho) = \mathrm{SSS}(\sigma)$. Let $x \in \mathrm{SSS}(\rho)$. We have either $x = \Delta^k s$ for some simple s and some integer k or $x = \Delta^k$. In both cases, x is a conjugate of both ρ and σ lying in a fixed factor $M(m_i, \ell_i)$ by Proposition B.2.5.

Conversely, if x and y are respective conjugate of ρ and σ in $M(m_i, \ell_i)$, then x and y are (p, q)-periodic elements of $M(m_i, \ell_i)$. They are conjugate in $G(m_i, \ell_i)$ (in particular in $G(M)$) by Proposition B.1.12. □

Example B.2.19. Let n be a positive integer, and consider $M = M(1, n) *_\Delta M(1, n) = \langle a, b \mid a^n = b^n \rangle^+$. By Proposition B.2.10, we have $\mathrm{SSS}(a) = \{a\}$ and $\mathrm{SSS}(b) = \{b\}$, thus a and b are two $(n, 1)$-periodic elements that are not conjugate in $G(M)$.

On the other hand, the conjugacy of non-periodic elements in a hosohedral-type group behaves in the same way as in a circular group.

Proposition B.2.20. *Let $x \in G(M)$ be a non-periodic element. The super-summit set of x is made of rigid elements. The only arrows starting from an object y of $\mathrm{CG}(x)$ are labeled by $\mathrm{init}(y)$ and $\overline{\mathrm{fin}(y)}$. In particular, one can go from an element of $\mathrm{SSS}(x)$ to any other by a finite sequence of cyclings, decyclings and applications of the Garside automorphism.*

Proof. We mimic the proof of Proposition B.1.14. Let $y \in \mathrm{SSS}(x)$. Since x is not periodic, y is not periodic. It is then rigid by Proposition B.2.17, and we have $\sup(y) > \inf(y)$. We assume that the left-weighted factorization of y is $\Delta^k s_{j_1}(i_1, p_1) \cdots s_{j_r}(i_r, p_r)$. Let s be a simple element of M. We have

$$y^s = \Delta^{k-1}\phi^{k-1}(\overline{s})s_{j_1}(i_1, p_1) \cdots s_{j_r}(i_r, p_r)s.$$

If $s \in \{1, \Delta\}$, then $y^s \in \mathrm{SSS}(x)$ is obvious. Otherwise, let j be such that s lies in the factor $M(m_j, \ell_j)$. If $j \notin \{j_1, j_r\}$, we have $\inf(y^s) = k - 1$ and $y \notin \mathrm{SSS}(x)$.

- If $j_r = j_1$, then the proof of Proposition B.1.14 gives that $y^s \in \mathrm{SSS}(x)$ if and only if $s \in \{\mathrm{init}(y), \overline{\mathrm{fin}(y)}\}$.
- If $j_r \neq j_1$ and $j = j_1$, then the word $s_{j_r}(i_r, p_r)s$ is greedy in M by Corollary B.2.8. We then have $y^s \in \mathrm{SSS}(x)$ if and only if $\phi^{k-1}(\overline{s})s_{j_1}(i_1, p_r) = \Delta$, i.e. if $s = \mathrm{init}(y)$.
- If $j_r \neq j_1$ and $j = j_r$, then the word $\phi^{k-1}(\overline{s})s_{j_1}(i_1, p_1)$ is greedy in M by Corollary B.2.8. We then have $y^s \in \mathrm{SSS}(x)$ if and only if $s_{j_r}(i_r, p_r)s = \Delta$, i.e. if $s = \overline{\mathrm{fin}(y)}$.

□

Theorem B.2.21. *Let M be a hosohedral-type monoid. If $\alpha, \beta \in G(M)$ are such that $\alpha^n = \beta^n$ for some nonzero integer n, then we either have*

- *α and β are conjugate.*
- *α and β are nonconjugate periodic elements of $G(M)$.*

Proof. Again, α is periodic if and only if its power α^n is periodic, if and only if β is periodic. Suppose that α and β are not nonconjugate periodic elements of $G(M)$. Then α and β are either conjugate periodic elements (in which case they are conjugate) or non-periodic elements. Up to replacing α, β with α^{-1}, β^{-1}, we now assume that $n > 0$.

We assume from now on that a and b are non-periodic elements of $G(M)$. Up to conjugacy, we can assume that $a \in \mathrm{SSS}(a)$. By Proposition B.2.17, we have that a is rigid. The element x is then rigid as a power of the rigid element a. Let now $c \in G(M)$ be so that $b^c \in \mathrm{SSS}(b)$. Since b is not periodic, b^c is rigid as well as $x^c = (b^c)^n$. We have $x, x^c \in \mathrm{SSS}(x)$. By Proposition B.2.20, there is an finite sequence of cyclings, decyclings, and applications of the Garside automorphism sending x to x^c. By Lemma 3.2.13, applying the same transformations to a gives a rigid element a' whose n-th power is x^c. Again by Lemma 3.2.13, we have $a' = b^c$ and thus a and b are conjugate. □

Partial computations of homology.

Let $M = M(m_1, \ell_1) *_\Delta \cdots *_\Delta M(m_h, \ell_h)$ be a hosohedral-type monoid. We can construct the Dehornoy-Lafont complex (cf. Section B.1.4) associated to M to try and compute the homology of $G(M)$. Since the lcm of two distinct atoms of a hosohedral-type group is Δ by Corollary B.2.8 and Lemma B.1.4 (gcd and lcm of two simple elements), we can mimic the proof of Lemma B.1.20 to get

Lemma B.2.22. *If M is a hosohedral-type monoid with m atoms, we order its atoms by $a_0 < a_1 < \ldots < a_{m-1}$. We have $\mathcal{X}_0 = \{[\varnothing]\}$, $\mathcal{X}_1 = \{[a_0], \ldots, [a_{m-1}]\}$, $\mathcal{X}_2 = \{[a_0, a_i] \mid i \in [\![1, m-1]\!]\}$ and $\mathcal{X}_n = \varnothing$ for $n \geqslant 3$.*

Like in the case of a circular monoid, this implies that a hosohedral-type group has homological dimension at most 2. Furthermore, as $H_2(G(M), \mathbb{Z})$ is free, the computation of $G(M)^{\mathrm{ab}}$ is sufficient to determine all the integral homology of $G(M)$. Unfortunately, we are only able to compute the free part of $G(M)^{\mathrm{ab}}$.

Proposition B.2.23. *Let $M = M(m_1, \ell_1) *_\Delta \cdots *_\Delta M(m_h, \ell_h)$ be a hosohedral-type monoid. For $i \in [\![1, h]\!]$, denote by d_i the gcd of $m_i \wedge \ell_i$. The free part of $G(M)^{\mathrm{ab}}$ has rank $1 + \sum_{i=1}^{h}(d_i - 1)$.*

Proof. First, let $j \in [\![1, h]\!]$. Let also $e_0, \ldots, e_{d-1}$ denote the canonical basis of $\mathbb{Z}^{d_j}$. By Lemma B.1.21 (integral homology of circular groups), the morphism $s_j(i, 1) \mapsto e_{i'}$, where i' is the remainder in the Euclidean division of i by d_j induces an isomorphism between $G(m_j, \ell_j)$ and $G(m, \ell)^{\mathrm{ab}} \simeq \mathbb{Z}^{d_j}$. Under this isomorphism, we have

$$\Delta_j = s_j(0, m) \mapsto \left(\sum_{i=0}^{d_j - 1} e_i \right)^{\frac{m_j}{d_j}} = \sum_{i=0}^{d_j - 1} \frac{m_j}{d_j} e_i.$$

Let $A \simeq \mathbb{Z}^{\sum_{i=1}^{n} d_i}$ be the direct product of the $G(m_i, \ell_i)^{\mathrm{ab}}$ for $i \in [\![1, h]\!]$. The isomorphisms $G(m_j, \ell_j)^{\mathrm{ab}} \simeq \mathbb{Z}^{d_j}$ described above induce a morphism p from the free product of the $G(m_i, \ell_i)$ to A, and we have $A = (G(m_1, \ell_1) * \cdots * G(m_h, \ell_h))^{\mathrm{ab}}$. We then have

$$G(M)^{\mathrm{ab}} = A \Big/ \langle p(s_1(0, m_1)) - p(s_i(0, m_i)),\ \forall i \in [\![2, h]\!] \rangle \cdot$$

As the vectors $p(s_1(0, m_1)) - p(s_i(0, m_i))$ are linearly independent, they span a submodule of A of rank $h - 1$. The free part of $G(M)^{\mathrm{ab}}$ then has rank

$$\sum_{i=1}^{h} d_i - h + 1 = 1 + \sum_{i=1}^{h} (d_i - 1).$$

□

Example B.2.24. Let $p, q \geqslant 2$ be integers, and consider the group

$$G := \langle a, b \mid a^p = b^q \rangle \simeq G(M(1, p) *_\Delta M(1, q)).$$

The proof of Proposition B.2.23 gives that G^{ab} is isomorphic to the quotient of $\mathbb{Z}^2$ by the submodule generated by the vector $p(s_1(0, p)) - p(s_2(0, q)) = (-p, q)$. We obtain $H_1(G, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, where $d = p \wedge q$.

Bibliography

[AA08] P. Achar and A. Aubert. "On rank 2 complex reflection groups". In: *Comm. Algebra* 36.6 (2008), pp. 2092–2132. DOI: 10.1080/00927870801949559.

[Adj66] S. Adjan. *Defining relations and algorithmic problems for groups and semigroups.* Vol. No. 85 (1966). Proceedings of the Steklov Institute of Mathematics. Translated from the Russian by M. Greendlinger. American Mathematical Society, Providence, RI, 1966, pp. iii+152.

[Ale28] J. Alexander. "Topological invariants of knots and links". In: *Trans. Amer. Math. Soc.* 30.2 (1928), pp. 275–306. URL: https://doi.org/10.2307/1989123.

[Ari95] S. Ariki. "Representation theory of a Hecke algebra of $G(r,p,n)$". In: *J. Algebra* 177.1 (1995), pp. 164–185. DOI: 10.1006/jabr.1995.1292.

[AK94] S. Ariki and K Koike. "A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations". In: *Adv. Math.* 106.2 (1994), pp. 216–243. DOI: 10.1006/aima.1994.1057.

[Arn70] V. I. Arnold. "Certain topological invariants of algebraic functions". In: *Trudy Moskov. Mat. Obšč.* 21 (1970), pp. 27–46.

[Art47] E. Artin. "Theory of braids". In: *Ann. of Math. (2)* 48 (1947), pp. 101–126. DOI: 10.2307/1969218.

[Art25] Emil Artin. "Theorie der Zöpfe". In: *Abh. Math. Sem. Univ. Hamburg* 4.1 (1925), pp. 47–72. DOI: 10.1007/BF02950718.

[ASS06] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1.* Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458. ISBN: 978-0-521-58423-4; 978-0-521-58631-3; 0-521-58631-3. DOI: 10.1017/CBO9780511614309.

[Ban76] E. Bannai. "Fundamental groups of the spaces of regular orbits of the finite unitary groups of rank 2". In: *J. Math. Soc. Japan* 28 (1976), pp. 447–454. DOI: 10.2969/jmsj/02830447.

[Bes16] D. Bessis. *Complex reflection groups are somehow real (talk at Lausanne).* 2016. URL: https://fr.slideshare.net/slideshow/complex-reflection-groups-are-somehow-real/66336346.

[Bes15] D. Bessis. "Finite complex reflection arrangements are $K(\pi,1)$". In: *Ann. of Math. (2)* 181.3 (2015), pp. 809–904. DOI: 10.4007/annals.2015.181.3.1.

[Bes07] D. Bessis. *Garside categories, periodic loops and cyclic sets.* 2007. DOI: 10.48550/ARXIV.MATH/0610778.

[Bes00] D. Bessis. "Groupes des tresses et éléments réguliers". In: *J. Reine Angew. Math.* 518 (2000), pp. 1–40. DOI: 10.1515/crll.2000.005.

[Bes03] D. Bessis. "The dual braid monoid". In: *Ann. Sci. École Norm. Sup. (4)* 36.5 (2003), pp. 647–683. DOI: 10.1016/j.ansens.2003.01.001.

[Bes01] D. Bessis. "Zariski theorems and diagrams for braid groups". In: *Invent. Math.* 145.3 (2001), pp. 487–507. DOI: 10.1007/s002220100155.

[BC06] D. Bessis and R. Corran. "Non-crossing partitions of type (e, e, r)". In: *Adv. Math.* 202.1 (2006), pp. 1–49. ISSN: 0001-8708. DOI: 10.1016/j.aim.2005.03.004.

[BDM02] D. Bessis, F. Digne, and J. Michel. "Springer theory in braid groups and the Birman-Ko-Lee monoid". In: *Pacific J. Math.* 205.2 (2002), pp. 287–309. ISSN: 0030-8730. DOI: 10.2140/pjm.2002.205.287.

[BM04] D. Bessis and J. Michel. "Explicit Presentations for Exceptional Braid Groups". In: *Experiment. Math.* 13.3 (2004), pp. 257–266. ISSN: 1058-6458.

[Bes99] M. Bestvina. "Non-positively curved aspects of Artin groups of finite type". In: *Geom. Topol.* 3 (1999), pp. 269–302. ISSN: 1465-3060. DOI: 10.2140/gt.1999.3.269.

[Big01] S. Bigelow. "Braid groups are linear". In: *J. Amer. Math. Soc.* 14.2 (2001), pp. 471–486. DOI: 10.1090/S0894-0347-00-00361-1.

[Big99] S. Bigelow. "The Burau representation is not faithful for $n = 5$". In: *Geom. Topol.* 3 (1999), pp. 397–404. DOI: 10.2140/gt.1999.3.397.

[BGG07a] J S. Birman, V. Gebhardt, and J. González-Meneses. "Conjugacy in Garside groups. I. Cyclings, powers and rigidity". In: *Groups Geom. Dyn.* 1.3 (2007), pp. 221–279. DOI: 10.4171/GGD/12.

[Bir74] J. Birman. *Braids, links, and mapping class groups.* Vol. No. 82. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.

[BGG08] J. Birman, V. Gebhardt, and J. González-Meneses. "Conjugacy in Garside groups. II. Structure of the ultra summit set". In: *Groups Geom. Dyn.* 2.1 (2008), pp. 13–61. DOI: 10.4171/GGD/30.

[BGG07b] J. Birman, V. Gebhardt, and J. González-Meneses. "Conjugacy in Garside groups. III. Periodic braids". In: *J. Algebra* 316.2 (2007), pp. 746–776. DOI: 10.1016/j.jalgebra.2007.02.002.

[BKL98] J. Birman, K-H. Ko, and S-J. Lee. "A new approach to the word and conjugacy problems in the braid groups". In: *Adv. Math.* 139.2 (1998), pp. 322–353. DOI: 10.1006/aima.1998.1761.

[Bor94] F. Borceux. *Handbook of categorical algebra : Volume 1, Basic Category Theory.* Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994, pp. xvi+345. ISBN: 0-521-44178-1.

[MAG] Wieb Bosma, John Cannon, and Catherine Playoust. "The Magma algebra system. I. The user language". In: *J. Symbolic Comput.* 24.3-4 (1997). Computational algebra and number theory (London, 1993), pp. 235–265.

[BT18] S. Bouc and J. Thévenaz. "Correspondence functors and finiteness conditions". In: *J. Algebra* 495 (2018), pp. 150–198. DOI: 10.1016/j.jalgebra.2017.11.010.

[Bou81] N. Bourbaki. *Groupes et algèbres de Lie, chapitres 4, 5, 6.* Masson, 1981.

[BW02] T. Brady and C. Watt. “$K(\pi, 1)$’s for Artin groups of finite type”. In: *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000).* Vol. 94. 2002, pp. 225–250. DOI: 10.1023/A:1020902610809.

[Bri71] E. Brieskorn. “Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe.” In: *Invent. Math.* 12 (1971), pp. 57–61. DOI: 10.1007/BF01389827.

[Bri73] E. Brieskorn. “Sur les groupes de tresses [d’après V. I. Arnold]”. In: *Séminaire Bourbaki, 24ème année (1971/1972).* Vol. Vol. 317. Lecture Notes in Math. Springer, Berlin-New York, 1973, Exp. No. 401, pp. 21–44.

[BS72] E. Brieskorn and K. Saito. “Artin-Gruppen und Coxeter-Gruppen”. In: *Invent. Math.* 17 (1972), pp. 245–271. DOI: 10.1007/BF01406235.

[Bro10] M. Broué. *Introduction to complex reflection groups and their braid groups.* Vol. 1988. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2010, pp. xii+138. ISBN: 978-3-642-11174-7. DOI: 10.1007/978-3-642-11175-4.

[Bro01] M. Broué. “Reflection groups, braid groups, Hecke algebras, finite reductive groups”. In: *Current developments in mathematics, 2000.* Int. Press, Somerville, MA, 2001, pp. 1–107.

[BM93] M. Broué and G. Malle. “Zyklotomische Heckealgebren”. In: 212. Représentations unipotentes génériques et blocs des groupes réductifs finis. 1993, pp. 119–189.

[BMM93] M. Broué, G. Malle, and J. Michel. “Generic blocks of finite reductive groups”. In: 212. Représentations unipotentes génériques et blocs des groupes réductifs finis. 1993, pp. 7–92.

[BMR98] M. Broué, G. Malle, and R. Rouquier. “Complex reflection groups, braid groups, Hecke algebras”. In: *J. Reine Angew. Math.* 500 (1998), pp. 127–190.

[BM97] M. Broué and J. Michel. “Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées”. In: *Finite reductive groups (Luminy, 1994).* Vol. 141. Progr. Math. Birkhäuser Boston, Boston, MA, 1997, pp. 73–139.

[Bur35] W. Burau. “Über Zopfgruppen und gleichsinnig verdrillte Verkettungen”. In: *Abh. Math. Sem. Univ. Hamburg* 11.1 (1935), pp. 179–186. DOI: 10.1007/BF02940722.

[Cal05] F. Callegaro. “On the cohomology of Artin groups in local systems and the associated Milnor fiber”. In: *J. Pure Appl. Algebra* 197.1-3 (2005), pp. 323–332. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2004.10.002.

[CM14] F. Callegaro and I. Marin. “Homology computations for complex braid groups”. In: *J. Eur. Math. Soc. (JEMS)* 16.1 (2014), pp. 103–164. DOI: 10.4171/JEMS/429.

[CCC20] M. Calvez, B. Cisneros de la Cruz, and M. Cumplido. “Conjugacy stability of parabolic subgroups of Artin-Tits groups of spherical type”. In: *J. Algebra* 556 (2020), pp. 621–633. DOI: 10.1016/j.jalgebra.2020.03.017.

[CMW04] R. Charney, J. Meier, and K. Whittlesey. “Bestvina’s Normal Form Complex and the Homology of Garside Groups”. In: *Geom. Dedicata* 105 (2004), pp. 171–188. ISSN: 0046-5755. DOI: 10.1023/B:GEOM.0000024696.69357.73.

[Cha18] E. Chavli. “The BMR freeness conjecture for the tetrahedral and octahedral families”. In: *Comm. Algebra* 46.1 (2018), pp. 386–464. DOI: 10.1080/00927872.2017.1324872.

[Che55] C. Chevalley. “Invariants of finite groups generated by reflections”. In: *Amer. J. Math.* 77 (1955), pp. 778–782. DOI: 10.2307/2372597.

[CW02] A. Cohen and D. Wales. “Linearity of Artin groups of finite type”. In: *Israel J. Math.* 131 (2002), pp. 101–123. DOI: 10.1007/BF02785852.

[Coh73a] F. Cohen. “Cohomology of braid spaces”. In: *Bull. Amer. Math. Soc.* 79 (1973), pp. 763–766. DOI: 10.1090/S0002-9904-1973-13306-3.

[Coh73b] F. Cohen. “Homology of $\Omega^{(n+1)}\Sigma^{(n+1)}X$ and $C_{(n+1)}X$, $n > 0$”. In: *Bull. Amer. Math. Soc.* 79 (1973), 1236–1241 (1974). DOI: 10.1090/S0002-9904-1973-13394-4.

[CLM76] F. Cohen, T. Lada, and P. May. *The homology of iterated loop spaces.* Vol. Vol. 533. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1976.

[CPS01] C. De Concini, C. Procesi, and M. Salvetti. “Arithmetic properties of the cohomology of braid groups”. In: *Topology* 40.4 (2001), pp. 739–751. DOI: 10.1016/S0040-9383(99)00081-6.

[CPSS99] C. De Concini, C. Procesi, M. Salvetti, and F. Stumbo. “Arithmetic properties of the cohomology of Artin groups”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 28.4 (1999), pp. 695–717.

[CLL15] R. Corran, E-K. Lee, and S-J. Lee. “Braid groups of imprimitive complex reflection groups”. In: *J. Algebra* 427 (2015), pp. 387–425. DOI: 10.1016/j.jalgebra.2015.01.004.

[CP11] R. Corran and M. Picantin. “A new Garside structure for the braid groups of type (e, e, r)”. In: *J. Lond. Math. Soc. (2)* 84.3 (2011), pp. 689–711. DOI: 10.1112/jlms/jdr030.

[Cri00] J. Crisp. “Symmetrical subgroups of Artin groups”. In: *Adv. Math.* 152.1 (2000), pp. 159–177. DOI: 10.1006/aima.1999.1895.

[CGGW19] M. Cumplido, V. Gebhardt, J. González-Meneses, and B. Wiest. “On parabolic subgroups of Artin-Tits groups of spherical type”. In: *Adv. Math.* 352 (2019), pp. 572–610. DOI: 10.1016/j.aim.2019.06.010.

[CMV23] M. Cumplido, A. Martin, and N. Vaskou. “Parabolic subgroups of large-type Artin groups”. In: *Math. Proc. Cambridge Philos. Soc.* 174.2 (2023), pp. 393–414. DOI: 10.1017/S0305004122000342.

[DH21] M. Davis and J. Huang. “Bordifications of hyperplane arrangements and their curve complexes”. In: *J. Topol.* 14.2 (2021), pp. 419–459. ISSN: 1753-8416,1753-8424. DOI: 10.1112/topo.12184.

[Deh94] P. Dehornoy. “Braid groups and left distributive operations”. In: *Trans. Amer. Math. Soc.* 345.1 (1994), pp. 115–150. DOI: 10.2307/2154598.

[Deh02] P. Dehornoy. “Groupes de Garside”. In: *Ann. Sci. École Norm. Sup. (4)* 35.2 (2002), pp. 267–306. DOI: 10.1016/S0012-9593(02)01090-X.

[Deh19] P. Dehornoy. *La calcul des tresses—une introduction, et au-delà.* Calvage et Mounet, Paris, [2019] ©2019.

[Deh92] P. Dehornoy. “Preuve de la conjecture d’irréflexivité pour les structures distributives libres”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 314.5 (1992), pp. 333–336.

[Deh00] P. Dehornoy. “The fine structure of LD-equivalence”. In: *Adv. Math.* 155.2 (2000), pp. 264–316. DOI: 10.1006/aima.2000.1938.

[DDGKM] P. Dehornoy, F. Digne, E. Godelle, D. Krammer, and J. Michel. *Foundations of Garside Theory*. Vol. 22. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2015, pp. xviii+691. ISBN: 978-3-03719-139-2. DOI: 10.4171/139.

[DL03] P. Dehornoy and Y. Lafont. "Homology of gaussian groups". In: *Ann. Inst. Fourier (Grenoble)* 53.2 (2003), pp. 489–540.

[DP99] P. Dehornoy and L. Paris. "Gaussian groups and Garside groups, two generalisations of Artin groups". In: *Proc. London Math. Soc. (3)* 79.3 (1999), pp. 569–604. DOI: 10.1112/S0024611599012071.

[Del72] P. Deligne. "Les immeubles des groupes de tresses généralisés". In: *Invent. Math.* 17 (1972), pp. 273–302. DOI: 10.1007/BF01406236.

[DL76] P. Deligne and G. Lusztig. "Representations of reductive groups over finite fields". In: *Ann. of Math. (2)* 103.1 (1976), pp. 103–161. DOI: 10.2307/1971021.

[DPS22] E. Delucchi, G. Paolini, and M. Salvetti. *Dual structures on Coxeter and Artin groups of rank three*. 2022. arXiv: 2206.14518v1.

[Dig03] F. Digne. "On the linearity of Artin braid groups". In: *J. Algebra* 268.1 (2003), pp. 39–57. DOI: 10.1016/S0021-8693(03)00327-2.

[DMM11] F. Digne, I. Marin, and J. Michel. "The center of pure complex braid groups". In: *J. Algebra* 347 (2011), pp. 206–213. DOI: 10.1016/j.jalgebra.2011.06.034.

[DM06] F. Digne and J. Michel. *Garside and locally Garside categories*. 2006. arXiv: 0612652 [math.AT].

[Dou17] T. Douvropoulos. "Applications of geometric techniques in Coxeter-Catalan combinatorics". PhD thesis. University of Minnesota, 2017.

[Eli+20] B. Elias, S. Makisumi, U. Thiel, and G. Williamson. *Introduction to Soergel bimodules*. Vol. 5. RSME Springer Series. Springer, Cham, [2020] ©2020. DOI: 10.1007/978-3-030-48826-0.

[FN62a] E. Fadell and L. Neuwirth. "Configuration spaces". In: *Math. Scand.* 10 (1962), pp. 111–118. DOI: 10.7146/math.scand.a-10517.

[FN62b] R. Fox and L. Neuwirth. "The braid groups". In: *Math. Scand.* 10 (1962), pp. 119–126. DOI: 10.7146/math.scand.a-10518.

[FG03a] N. Franco and J. González-Meneses. "Computation of centralizers in braid groups and Garside groups". In: *Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001)*. Vol. 19. 2. 2003, pp. 367–384. DOI: 10.4171/RMI/352.

[FG03b] N. Franco and J. González-Meneses. "Conjugacy problem for braid groups and Garside groups". In: *J. Algebra* 266.1 (2003), pp. 112–132. DOI: 10.1016/S0021-8693(03)00292-8.

[Fuk70] D. Fuks. "Cohomology of the braid group mod 2". In: *Funkcional. Anal. i Priložen.* 4.2 (1970), pp. 62–73.

[Gar24a] O. Garnier. "Generalization of the Dehornoy–Lafont Order Complex to Categories: Application to Exceptional Braid Groups". In: *Appl. Categ. Structures* 32.1 (2024), Paper No. 1, 28. DOI: 10.1007/s10485-023-09757-6.

[Gar24b] O. Garnier. *Parabolic subgroups of complex braid groups: the remaining case.* 2024. arXiv: 2403.02209v1.

[Gar23a] O. Garnier. "Regular theory in complex braid groups". In: *J. Algebra* 620 (2023), pp. 534–557. DOI: 10.1016/j.jalgebra.2022.12.031.

[Gar23b] O. Garnier. *Springer categories for regular centralizers in well-generated complex braid groups.* 2023. arXiv: 2306.13022v1.

[Gar24c] O. Garnier. "Uniqueness of roots up to conjugacy in circular and hosohedral-type Garside groups". In: *J. Group Theory* 27.5 (2024), pp. 1091–1128. DOI: 10.1515/jgth-2023-0268.

[Gar69] F. A. Garside. "The braid group and other groups". In: *Quart. J. Math. Oxford Ser. (2)* 20 (1969), pp. 235–254. DOI: 10.1093/qmath/20.1.235.

[Geb05] V. Gebhardt. "A new approach to the conjugacy problem in Garside groups". In: *J. Algebra* 292.1 (2005), pp. 282–302. DOI: 10.1016/j.jalgebra.2005.02.002.

[GG10a] V. Gebhardt and J. González-Meneses. "Solving the conjugacy problem in Garside groups by cyclic sliding". In: *J. Symbolic Comput.* 45.6 (2010), pp. 629–656. DOI: 10.1016/j.jsc.2010.01.013.

[GG10b] V. Gebhardt and J. González-Meneses. "The cyclic sliding operation in Garside groups". In: *Math. Z.* 265.1 (2010), pp. 85–114. DOI: 10.1007/s00209-009-0502-2.

[GP00] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras.* Vol. 21. London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 2000, pp. xvi+446.

[GWY23] Joel Gibson, Geordie Williamson, and Oded Yacobi. *4-Strand Burau is Unfaithful Modulo 5.* 2023. arXiv: 2310.02403v1. URL: https://arxiv.org/abs/2310.02403.

[Gob24] T. Gobet. "Toric Reflection Groups". In: *J. Aust. Math. Soc.* 116.2 (2024), pp. 171–199. DOI: 10.1017/S1446788723000101.

[God01] E. Godelle. "Normalisateurs et centralisateurs des sous-groupes paraboliques dans les groupes d'Artin-Tits". PhD thesis. Université de Picardie Jules Verne, 2001.

[God07] E. Godelle. "Parabolic subgroups of Garside groups". In: *J. Algebra* 317.1 (2007), pp. 1–16. DOI: 10.1016/j.jalgebra.2007.05.024.

[God10] E. Godelle. "Parabolic subgroups of Garside groups II: ribbons". In: *J. Pure Appl. Algebra* 214.11 (2010), pp. 2044–2062. DOI: 10.1016/j.jpaa.2010.02.010.

[Gon03] J. González-Meneses. "The nth root of a braid is unique up to conjugacy". In: *Algebr. Geom. Topol.* 3 (2003), pp. 1103–1118. DOI: 10.2140/agt.2003.3.1103.

[GM22] J. González-Meneses and I. Marin. *Parabolic subgroups of complex braid groups.* 2022. arXiv: 2208.11938v2.

[GW04] J. González-Meneses and B. Wiest. "On the structure of the centralizer of a braid". In: *Ann. Sci. École Norm. Sup. (4)* 37.5 (2004), pp. 729–757. DOI: 10.1016/j.ansens.2004.04.002.

[MAC] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry.* Available at http://www.math.uiuc.edu/Macaulay2/.

[Gue68] W. J. Guerrier. "The Factorization of the Cyclotomic Polynomials mod p". In: *Amer. Math. Monthly* 75 (1968), p. 46. DOI: 10.2307/2315109.

[HH23] T. Haettel and J. Huang. *New Garside structures and applications to Artin groups.* 2023. arXiv: 2305.11622v1 [math.GR].

[HH24] Thomas Haettel and Jingyin Huang. "Lattices, Garside structures and weakly modular graphs". In: *J. Algebra* 656 (2024), pp. 226–258. DOI: 10.1016/j.jalgebra.2023.08.034.

[Hal24] I. Haladjian. *Braid groups of J-reflection groups and associated classical and dual Garside structures.* 2024. arXiv: 2410.18828v1 [math.GR].

[Kra08] D. Krammer. "A class of Garside groupoid structures on the pure braid group". In: *Trans. Amer. Math. Soc.* 360.8 (2008), pp. 4029–4061. DOI: 10.1090/S0002-9947-08-04313-4.

[Kra02] D. Krammer. "Braid groups are linear". In: *Ann. of Math. (2)* 155.1 (2002), pp. 131–156. DOI: 10.2307/3062152.

[Kra00] D. Krammer. "The braid group B_4 is linear". In: *Invent. Math.* 142.3 (2000), pp. 451–486. DOI: 10.1007/s002220000088.

[Kre72] G. Kreweras. "Sur les partitions non croisées d'un cycle". In: *Discrete Math.* 1.4 (1972), pp. 333–350. DOI: 10.1016/0012-365X(72)90041-6.

[LL11] E-K Lee and S-J Lee. "Periodic elements in Garside groups". In: *J. Pure Appl. Algebra* 215.10 (2011), pp. 2295–2314. DOI: 10.1016/j.jpaa.2010.12.011.

[LL10] E-K. Lee and S-J. Lee. "Uniqueness of roots up to conjugacy for some affine and finite type Artin groups". In: *Math. Z.* 265.3 (2010), pp. 571–587. DOI: 10.1007/s00209-009-0530-y.

[LT09] G. I. Lehrer and D. E. Taylor. *Unitary reflection groups.* Vol. 20. Australian Mathematical Society Lecture Series. Cambridge University Press, Cambridge, 2009, pp. viii+294. ISBN: 978-0-521-74989-3.

[Lek83] H. van der Lek. "The homotopy type of complex hyperplane complements." PhD thesis. Katholieke Universiteit Nijmegen, 1983.

[LP93] D. Long and M. Paton. "The Burau representation is not faithful for $n \geq 6$". In: *Topology* 32.2 (1993), pp. 439–447. DOI: 10.1016/0040-9383(93)90030-Y.

[LS01] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory.* Classics in Mathematics. Reprint of the 1977 edition. Springer-Verlag, Berlin, 2001. ISBN: 3-540-41158-5. DOI: 10.1007/978-3-642-61896-3.

[Mag34] W. Magnus. "Über Automorphismen von Fundamentalgruppen berandeter Flächen". In: *Math. Ann.* 109.1 (1934), pp. 617–646. DOI: 10.1007/BF01449158.

[Mak71] G. Makanin. "The normalizers of a braid group". In: *Mat. Sb. (N.S.)* 86(128) (1971), pp. 171–179.

[MM10] G. Malle and J. Michel. "Constructing representations of Hecke algebras for complex reflection groups". In: *LMS J. Comput. Math.* 13 (2010), pp. 426–450. DOI: 10.1112/S1461157009000412.

[Mar21] I. Marin. *Groupes de tresses des groupes de réflexions complexes (Masters course).* 2021.

[Mar17] I. Marin. “Homology computations for complex braid groups II”. In: *J. Algebra* 477 (2017), pp. 540–547. DOI: 10.1016/j.jalgebra.2017.01.044.

[Mar12a] I. Marin. “Krammer representations for complex braid groups”. In: *J. Algebra* 371 (2012), pp. 175–206. DOI: 10.1016/j.jalgebra.2012.07.046.

[Mar19] I. Marin. “Proof of the BMR conjecture for G_{20} and G_{21}”. In: *J. Symbolic Comput.* 92 (2019), pp. 1–14. DOI: 10.1016/j.jsc.2017.11.015.

[Mar09] I. Marin. “Reflection groups acting on their hyperplanes”. In: *J. Algebra* 322.8 (2009), pp. 2848–2860. DOI: 10.1016/j.jalgebra.2009.08.005.

[Mar07] I. Marin. “Sur les représentations de Krammer génériques”. In: *Ann. Inst. Fourier (Grenoble)* 57.6 (2007), pp. 1883–1925. DOI: 10.5802/aif.2317.

[Mar12b] I. Marin. “The cubic Hecke algebra on at most 5 strands”. In: *J. Pure Appl. Algebra* 216.12 (2012), pp. 2754–2782. DOI: 10.1016/j.jpaa.2012.04.013.

[Mar14] I. Marin. “The freeness conjecture for Hecke algebras of complex reflection groups, and the case of the Hessian group G_{26}”. In: *J. Pure Appl. Algebra* 218.4 (2014), pp. 704–720. DOI: 10.1016/j.jpaa.2013.08.009.

[MP17] I. Marin and G. Pfeiffer. “The BMR freeness conjecture for the 2-reflection groups”. In: *Math. Comp.* 86.306 (2017), pp. 2005–2023. DOI: 10.1090/mcom/3142.

[Mar45] A. Markov. “Foundations of the algebraic theory of tresses”. In: *Trav. Inst. Math. Stekloff* 16 (1945), p. 53.

[MM99] H. Masur and Y. Minsky. “Geometry of the complex of curves. I. Hyperbolicity”. In: *Invent. Math.* 138.1 (1999), pp. 103–149. DOI: 10.1007/s002220050343.

[McD79] D. McDuff. “On the classifying spaces of discrete monoids”. In: *Topology* 18.4 (1979), pp. 313–320. DOI: 10.1016/0040-9383(79)90022-3.

[Mic04] J. Michel. *Groupes de réflexions complexes*. 2004. URL: https://webusers.imj-prg.fr/~jean.michel/papiers/cours2004.pdf.

[CHE] J. Michel. “The development version of the CHEVIE package of GAP3”. In: *J. Algebra* 435 (2015), pp. 308–336. DOI: 10.1016/j.jalgebra.2015.03.031.

[Mit72] B. Mitchell. “Rings with several objects”. In: *Advances in Math.* 8 (1972), pp. 1–161. DOI: 10.1016/0001-8708(72)90002-3.

[Moo91] J. Moody. “The Burau representation of the braid group B_n is unfaithful for large n”. In: *Bull. Amer. Math. Soc. (N.S.)* 25.2 (1991), pp. 379–384. DOI: 10.1090/S0273-0979-1991-16080-5.

[Mor21] R. Morris-Wright. “Parabolic subgroups in FC-type Artin groups”. In: *J. Pure Appl. Algebra* 225.1 (2021), Paper No. 106468, 13. DOI: 10.1016/j.jpaa.2020.106468.

[Nak83] T. Nakamura. “A note on the $K(\pi, 1)$ property of the orbit space of the unitary reflection group $G(m, l, n)$”. In: *Sci. Papers College Arts Sci. Univ. Tokyo* 33.1 (1983), pp. 1–6.

[Nea18] G. Neaime. “Interval Structures, Hecke Algebras, and Krammer’s Representations for the Complex Braid Groups B(e,e,n)”. PhD thesis. 2018. DOI: 10.48550/ARXIV.1812.03714.

[OS88a] P. Orlik and L. Solomon. “Discriminants in the invariant theory of reflection groups”. In: *Nagoya Math. J.* 109 (1988), pp. 23–45. DOI: 10.1017/S0027763000002749.

[OS88b] P. Orlik and L. Solomon. “The Hessian map in the invariant theory of reflection groups”. In: *Nagoya Math. J.* 109 (1988), pp. 1–21. DOI: 10.1017/S0027763000002737.

[OT92] P. Orlik and H. Terao. *Arrangements of Hyperplanes*. Vol. 300. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992, pp. xviii+325. DOI: 10.1007/978-3-662-02772-1.

[PS21] G. Paolini and M. Salvetti. “Proof of the $K(\pi, 1)$ conjecture for affine Artin groups”. In: *Invent. Math.* 224.2 (2021), pp. 487–572. DOI: 10.1007/s00222-020-01016-y.

[Pic00] M. Picantin. “Petits groupes gaussiens”. PhD thesis. Université de Caen, 2000.

[Pic01a] M. Picantin. “The center of thin Gaussian groups”. In: *J. Algebra* 245.1 (2001), pp. 92–122. DOI: 10.1006/jabr.2001.8894.

[Pic01b] M. Picantin. “The conjugacy problem in small Gaussian groups”. In: *Comm. Algebra* 29.3 (2001), pp. 1021–1039. DOI: 10.1081/AGB-100001665.

[Rei97] V. Reiner. “Non-crossing partitions for classical reflection groups”. In: *Discrete Math.* 177.1-3 (1997), pp. 195–222. DOI: 10.1016/S0012-365X(96)00365-2.

[EM94] E. A. El-Rifai and H. R. Morton. “Algorithms for positive braids”. In: *Quart. J. Math. Oxford Ser. (2)* 45.180 (1994), pp. 479–497. DOI: 10.1093/qmath/45.4.479.

[Rip10] V. Ripoll. “Orbites d’Hurwitz des factorisations primitives d’un élément de Coxeter”. In: *J. Algebra* 323.5 (2010), pp. 1432–1453. DOI: 10.1016/j.jalgebra.2009.12.010.

[Sal94] M. Salvetti. “The homotopy type of Artin groups”. In: *Math. Res. Lett.* 1.5 (1994), pp. 565–577. DOI: 10.4310/MRL.1994.v1.n5.a5.

[She52] G. Shephard. “Regular complex polytopes”. In: *Proc. London Math. Soc. (3)* 2 (1952), pp. 82–97. DOI: 10.1112/plms/s3-2.1.82.

[She53] G. Shephard. “Unitary groups generated by reflections”. In: *Canad. J. Math.* 5 (1953), pp. 364–383. DOI: 10.4153/cjm-1953-042-7.

[ST54] G. C. Shephard and J. A. Todd. “Finite unitary reflection groups”. In: *Canad. J. Math.* 6 (1954), pp. 274–304. ISSN: 0008-414X. DOI: 10.4153/cjm-1954-028-3.

[Shi07] J. Shi. “Formula for the reflection length of elements in the group $G(m, p, n)$”. In: *J. Algebra* 316.1 (2007), pp. 284–296. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2007.06.031.

[Shv96] O.V. Shvartsman. “Torsion in the quotient group of the Artin-Brieskorn braid group and regular springer numbers”. In: *Funktsional. Anal. i Prilozhen.* 30.1 (1996), pp. 39–46, 96. DOI: 10.1007/BF02383399.

[Sib02] H. Sibert. “Extraction of roots in Garside groups”. In: *Comm. Algebra* 30.6 (2002), pp. 2915–2927. DOI: 10.1081/AGB-120003997.

[Soe23] Mireille Soergel. “Systolic complexes and group presentations”. In: *Groups Geom. Dyn.* 17.3 (2023), pp. 899–921. DOI: 10.4171/ggd/717.

[Spr74] T.A. Springer. “Regular elements of finite reflection groups”. In: *Invent. Math.* 25 (1974), pp. 159–198. DOI: 10.1007/BF01390173.

[Squ94] C. Squier. “The homological algebra of Artin groups”. In: *Math. Scand.* 75.1 (1994), pp. 5–43. DOI: 10.7146/math.scand.a-12500.

[Thu88] W. Thurston. *Finite state algorithms for the braid group (Circulated notes)*. 1988.

[Tit66] J. Tits. “Normalisateurs de tores. I. Groupes de Coxeter étendus”. In: *J. Algebra* 4 (1966), pp. 96–116. DOI: 10.1016/0021-8693(66)90053-6.

[Tsu20] S. Tsuchioka. “BMR freeness for icosahedral family”. In: *Exp. Math.* 29.2 (2020), pp. 234–245. DOI: 10.1080/10586458.2018.1455072.

[Ver98] Vladimir V. Vershinin. “Homology of braid groups and their generalizations”. In: *Knot theory (Warsaw, 1995)*. Vol. 42. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 1998, pp. 421–446.

[Web07] P. Webb. “An introduction to the representations and cohomology of categories”. In: *Group representation theory*. EPFL Press, Lausanne, 2007, pp. 149–173.

[Wei94] C. A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: 10.1017/CBO9781139644136.

[Zhe06] H. Zheng. “A new approach to extracting roots in Garside groups”. In: *Comm. Algebra* 34.5 (2006), pp. 1793–1802. DOI: 10.1080/00927870500542762.

Index

Groupoïdes de Garside et groupes de tresses complexes

Résumé.

Ce travail est séparé en deux parties principales. La première partie est centrée autour de thèmes généraux en théorie de Garside, tandis que la seconde étudie des application de cette théorie dans l'étude des groupes de tresses complexes et de leurs sous-groupes paraboliques.
Les sous-groupes paraboliques des groupes de tresses complexes ont été introduits récemment par González-Meneses et Marin, dans le but de généraliser des résultats connus pour le groupe de tresses usuel à cette plus grande famille de groupes. En particulier, ils ont obtenu des résultats généraux sur ces sous-groupes paraboliques dans tous les cas sauf celui du groupe de tresses exceptionnel $B(G_{31})$. La particularité de ce groupe étant de n'avoir pas de structure de groupe de Garside connue, mais seulement une structure de groupoïde de Garside.
Dans la première partie, nous introduisons en particulier le concept de banc de sous-groupoïdes paraboliques standards. Ce concept nous permet de montrer -sous des hypothèses convenables- que les sous-groupes paraboliques dans un groupoïde de Garside sont stables par intersection. Ceci généralise l'approche de González-Meneses et Marin dans le cas des groupes de Garside.
Dans la seconde partie, nous étudions les sous-groupes paraboliques des groupes de tresses complexes en utilisant exclusivement les monoïdes duaux et les structures qui en dérivent, ce qui inclut le groupoïde de Garside défini par Bessis afin d'étudier le groupe de tresses complexe $B(G_{31})$. En utilisant ces structures, nous obtenons en particulier de nouvelles preuves de certains des résultats généraux de González-Meneses et Marin concernant les sous-groupes paraboliques des groupes bien-engendrés. Nous en déduisons une description complète des sous-groupes paraboliques d'un centralisateur régulier dans un groupe de tresses complexe arbitraire.
Ces résultats s'appliquent en particulier au groupe de tresses complexe $B(G_{31})$. Nous étendons les résultats principaux de González-Meneses et Marin à ce cas, complétant ainsi la preuve de ces résultats pour tous les groupes de tresses complexes.
Par ailleurs, nous introduisons une méthode générale pour calculer des présentations d'un groupe partant d'une présentation d'un groupoïde auquel il est équivalent. Nous appliquons cette méthode au groupoïde de Garside associé à $B(G_{31})$ pour obtenir plusieurs présentations de ce groupe.

Mots clés : Présentations, Groupes de Garside, Groupoïdes de Garside, Sous-groupes paraboliques, Groupes de réflexions complexes, Groupes de tresses complexes, Groupes d'Artin, Monoïde dual, Partitions non-croisées Tresses régulières.

Garside groupoids and complex braid groups

Abstract.

This work is split in two main parts. The first part is centered around general themes in Garside theory, while the seconds studies applications of Garside theory in the study of complex braid groups and their parabolic subgroups.
Parabolic subgroups of complex braid groups were recently introduced by González-Meneses and Marin, with the goal of generalizing known results on the usual braid group to complex braid groups. In particular, they proved general results on these parabolic subgroups in every case except that of the exceptional complex braid group $B(G_{31})$. This last group is particular among complex braid groups, as it is not known to admit a Garside group structure, but rather a Garside groupoid structure.
In the first part, we introduce in particular the concept of shoal of standard parabolic subgroupoids. This concept allows us to show -under suitable assumptions- that parabolic subgroups in a Garside groupoid are stable under intersection. This generalizes the approach of González-Meneses and Marin in the case of Garside groups.
In the second part, we study parabolic subgroups in complex braid groups using only dual braid monoids and other structures deduced from it, including the Garside groupoid defined by Bessis in order to study the complex braid group $B(G_{31})$. Using these structures, we obtain in particular new proofs of general results obtained by González-Meneses and Marin on parabolic subgroups of well-generated groups. We deduce a complete description of the parabolic subgroups of the centralizer of a regular braid in an arbitrary complex braid group.
These results apply in particular to the complex braid group $B(G_{31})$ and we extend the main results of González-Meneses and Marin to this case, thus completing the proof of these results for all complex braid groups.
Furthermore, we also introduce a general method suitable for computing presentations of a group starting from a presentation of a groupoid to which it is equivalent. We then use this method on the Garside groupoid attached to $B(G_{31})$ to obtain several presentations of this group.

Keywords: Presentations, Garside groups, Garside groupoids, Parabolic subgroups, Complex reflection groups, Complex braid groups, Artin groups, Dual braid monoid, Noncrossing partitions, Regular braids.