NORMALISERS OF PARABOLIC SUBGROUPS OF ARTIN–TITS GROUPS AND TITS CONE INTERSECTIONS

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Abstract. Let Γ be a Coxeter diagram and let $J \subseteq \Gamma$. Motivated by 3-fold flops, Iyama and Wemyss study the hyperplane arrangement in the Tits cone intersection of J, which is a J-relative generalisation of the classical Coxeter arrangement. For Γ of finite-type, we show that its complexified hyperplane complement is a $K(\pi, 1)$ space for the normaliser (quotient) of the standard parabolic subgroup of the Artin–Tits group attached to J. For general Γ we show that Brink–Howlett's groupoid, which describes normalisers of parabolic subgroups of Coxeter groups, has its universal cover described by the wall-and-chamber structure of the Tits cone intersection. We use this to show that wall crossing sequences satisfy an "atomic Matsumoto relation", generalising a theorem of Ko and answering questions raised by Iyama and Wemyss.

1. Introduction

Let V be the reflection representation of a Coxeter group W. The associated real and complex hyperplane arrangements arising from the W action on V are basic objects of study in Coxeter theory, and they play a crucial role in connecting structures in the Coxeter group W to structures in the associated Artin–Tits group A. In this paper, we develop a relative version of this connection for parabolic subdiagrams of Γ. In our setting, the algebraic objects are (quotients of) normalisers of parabolic subgroups (in both W and A), and the hyperplane arrangements are intersection arrangements that appear in the study of 3-fold flops [26, 29], further studied in the work of Iyama–Wemyss [20][1].

To explain in more detail, we first recall a few classical statements. Let $\Theta = V^*$ be the contragradient (i.e. dual) representation of W and consider the (locally-finite) hyperplane arrangement $(\mathrm{Cone}^\circ, \mathcal{H})$, where $\mathrm{Cone} \subseteq \Theta$ is the Tits cone of W and Cone° is its interior. Consider also the complexified hyperplane complement $\mathcal{X}^{\mathbb{C}}$ (see (2.1)) associated to $(\mathrm{Cone}^\circ, \mathcal{H})$, which inherits an action of W.

The space $\mathcal{X}^{\mathbb{C}}$ is a normal cover $\mathcal{X}^{\mathbb{C}} \to \mathcal{X}^{\mathbb{C}}/W$, and the covering map gives rise to the short exact sequence:

$$1 \to \pi_1(\mathcal{X}^{\mathbb{C}}, x) \to \pi_1(\mathcal{X}^{\mathbb{C}}/W, x) \to W \to 1. \tag{1.1}$$

This fundamental sequence relates the Coxeter group W, the Artin–Tits group $A \cong \pi_1(\mathcal{X}^{\mathbb{C}}/W, x)$ and the pure (or coloured) Artin–Tits group $P \coloneqq \pi_1(\mathcal{X}^{\mathbb{C}}, x)$. An important conjecture about Artin–Tits groups is that $\mathcal{X}^{\mathbb{C}}/W$ is a $K(\pi, 1)$ space for A. This is known for many families (including finite and affine types) [5, 7, 8, 15, 19, 23], but remains largely open.

Now, fix a subdiagram $J \subseteq \Gamma$. Iyama and Wemyss study an intersection arrangement known as the Tits cone intersection associated to J [20]. To this end, they consider a subspace $\mathrm{Cone}(J) \subseteq \mathrm{Cone}$, and a set of hyperplanes $\mathcal{H}_J$ obtained from $\mathcal{H}$ by restriction; see Section 3.1 for the precise definitions. As in the classical case, the hyperplane arrangement in the interior $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is locally finite (see Proposition 3.5). Moreover, if J is empty, then $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is the arrangement $(\mathrm{Cone}^\circ, \mathcal{H})$.

[1]Our citations to [20] refer to a version we received via private communication. The currently available version `https://www.maths.gla.ac.uk/~mwemyss/MainFile_for_web.pdf` has essentially all the results we cite, though sometimes in slightly less generality than in [20].

The complexified hyperplane complement $\mathcal{X}_J^{\mathbb{C}}$ (see (3.21)) associated to $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ plays an important role in the (homological) minimal model program and the theory of 3-fold flops, as its fundamental group acts on derived categories that appear in the various algebraic and geometric settings [9, 16, 17, 20]. Motivated by this, Iyama and Wemyss classify these arrangements in low ranks [20], noting that they are not in general Coxeter arrangements.

In this paper, we establish explicit relations between these arrangements and subquotients of Coxeter and Artin–Tits groups. To explain this, consider the standard parabolic subgroups $W_J \leq W$ and $A_J \leq A$. A key observation is that $\mathrm{Cone}(J)$ carries a natural action by the normaliser quotient of W_J (see Section 3.3):

$$N(W, J) \coloneqq \mathrm{Norm}_W(W_J)/W_J. \tag{1.2}$$

In [3], Brink and Howlett introduce a groupoid $\mathscr{BH}$ (see Section 4) whose vertex group is isomorphic to $N(W, J)$, and use it to obtain a presentation of this group. Our first main result realises $\mathscr{BH}$ geometrically by relating it to the wall-and-chamber structures of the Tits cone intersection.

Theorem 1.3. *The universal cover $\widetilde{\mathscr{BH}}$ of $\mathscr{BH}$ has objects given by the chambers in the Tits cone intersection and morphisms between adjacent chambers given by simple wall crossings. Moreover, the $N(W, J)$-action on $\mathrm{Cone}(J)$ induces an action on $\widetilde{\mathscr{BH}}$, which realises $N(W, J)$ as the deck transformation group of this covering.*

For the precise statement of Theorem 1.3 see Theorem 4.8. As a consequence of this theorem, using the groupoid presentation of $\mathscr{BH}$ obtained by Brink and Howlett in [3], we deduce in Corollary 4.15 that sequences of wall crossings satisfy an "atomic Matsumoto theorem". This generalises [21, Theorem 1.8] by Ko beyond the finitary case, and answers a question raised by Iyama–Wemyss [20, Remark 1.62].

Now, the action of $N(W, J)$ on $\mathrm{Cone}(J)$ also induces an action on the complexified hyperplane complement $\mathcal{X}_J^{\mathbb{C}}$. We once again obtain a normal covering $\mathcal{X}_J^{\mathbb{C}} \to \mathcal{X}_J^{\mathbb{C}}/N(W, J)$, which induces the following short exact sequence that generalises (1.1):

$$1 \to \pi_1(\mathcal{X}_J^{\mathbb{C}}, x) \to \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W, J), x) \to N(W, J) \to 1. \tag{1.4}$$

Our goal is to describe the fundamental groups in (1.4) directly as subquotients of the ambient Artin–Tits group A. To that end, we set

$$N(A, J) \coloneqq \mathrm{Norm}_A(A_J)/A_J, \tag{1.5}$$

and let $N(P, J)$ to be the image of $P \cap \mathrm{Norm}_A(A_J)$ in $N(A, J)$:

$$N(P, J) \coloneqq \{\beta A_J \in N(A, J) \mid P \cap \beta A_J \neq \varnothing\}. \tag{1.6}$$

It is not hard to show directly that these groups fit into a short exact sequence (Lemma 2.4)

$$1 \to N(P, J) \to N(A, J) \to N(W, J) \to 1. \tag{1.7}$$

Our second main result is that when W is finite, this short exact sequence recovers (1.4):

Theorem 1.8. *Assume that W is a finite Coxeter group. The sequences* (1.4) *and* (1.7) *are isomorphic. In particular, we have the following isomorphisms:*

$$N(A, J) \cong \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W, J), x),$$
$$N(P, J) \cong \pi_1(\mathcal{X}_J^{\mathbb{C}}, x).$$

Moreover, $\mathcal{X}_J^{\mathbb{C}}/N(W, J)$ is a $K(\pi, 1)$ space for $N(A, J)$.

On the one hand, this theorem provides an algebraic description of the fundamental group $\pi_1(\mathcal{X}_J^{\mathbb{C}}, x)$ in terms of the pure Artin–Tits group P. On the other hand, it shows that the normaliser quotient $N(A, J)$ also has (the quotient of) a complexified hyperplane complement space as its

$K(\pi, 1)$. One obtains from this, for example, that the cohomology of $N(A, J)$ has an explicit presentation in terms of the Orlik–Solomon algebra associated to the hyperplane arrangement.

The proof of Theorem 1.8 appears in Section 6 (Theorem 6.5 and Corollary 6.10). The essential idea is to relate two groupoids:

- the Deligne groupoid $\mathscr{D}$ [7] associated to $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$, and
- the reduced ribbon groupoid $\mathscr{R}$ of Godelle [13].

The Deligne groupoid of a finite arrangement in a real vector space is a combinatorially defined groupoid. In the setting of Theorem 1.8, we obtain the groupoid $\mathscr{D}$ which is equivalent to the fundamental groupoid $\pi_1(\mathcal{X}_J^{\mathbb{C}})$ of $\mathcal{X}_J^{\mathbb{C}}$. Moreover, $\mathscr{D}$ inherits an action of $N(W, J)$, and this induces a groupoid covering $\mathcal{Q} : \mathscr{D} \to \mathscr{D}/N(W, J)$ (see Section 3.5).

On the other hand, in the study of normalisers of finite-type Artin–Tits groups (and more generally Garside groups), Godelle introduces the reduced ribbon groupoid $\mathscr{R}$, whose key feature is that its vertex groups are isomorphic to $N(A, J)$.

We define a functor $\mathcal{G} : \mathscr{D} \to \mathscr{R}$ and prove in Proposition 6.3 and Corollary 6.4 that it is a groupoid covering isomorphic to the natural quotient $\mathcal{Q} : \mathscr{D} \to \mathscr{D}/N(W, J)$. As a result, we obtain the identification of groupoids:

$$\mathscr{R} \cong \mathscr{D}/N(W, J) \cong \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W, J)),$$

from which Theorem 1.8 follows.

Finally, we conclude with a $K(\pi, 1)$-conjecture for normaliser quotients of Artin–Tits groups:

Conjecture 1.9. *Theorem 1.8 holds for any Coxeter group W.*

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2. Background

2.1. **Hyperplane arrangements.** Let V be a real vector space. Throughout this paper, a hyperplane arrangement will consist of a convex subset $T \subseteq V$ and a (possibly infinite) set $\mathcal{H}$ of hyperplanes in V such that $H \cap T \neq \varnothing$ for all $H \in \mathcal{H}$. We will denote a hyperplane arrangement by the pair $(T, \mathcal{H})$. Given a hyperplane arrangement $(T, \mathcal{H})$ and a convex subset $T' \subseteq T$, we can define a restricted hyperplane arrangement in T' by removing from $\mathcal{H}$ all the hyperplanes that do not intersect with T'. For notation simplicity, we will simply denote this arrangement by $(T', \mathcal{H})$, with the understanding that only hyperplanes in $\mathcal{H}$ that intersect T' are considered. Finally, we say that a hyperplane arrangement is locally finite if for all $x \in T$, there exists an open neighbourhood U_x of x in T such that U_x only intersects finitely many hyperplanes in $\mathcal{H}$.

Given a hyperplane arrangement $(T, \mathcal{H})$, the connected components of $T \setminus \bigcup_{H \in \mathcal{H}} H$ are called chambers. Following [20], a wall of a chamber C is the intersection of $\overline{C}$ with a hyperplane $H \in \mathcal{H}$ that has codimension 1. (Note that the support of a face is called a wall in [2, 7, 25], which we have reserved for a different usage above.)

2.2. **Coxeter and Artin–Tits groups.** Let Γ be a Coxeter diagram with vertex set Γ_0. Let W denote the associated Coxeter group, with simple generators s_i, $i \in \Gamma_0$. We say that Γ is spherical-type (or finite-type), if W is finite. Note that we only impose the finite-type assumption from Section 5 onwards.

We denote the simple roots of Γ by $\{\alpha_i \mid i \in \Gamma_0\}$, and let $V = \bigoplus_{i \in \Gamma_0} \mathbb{R}\alpha_i$ be the reflection representation of W. The set of roots $\Phi = \{w \cdot \alpha_i \mid w \in W, i \in \Gamma_0\}$ decomposes into positive and negative roots, which we denote by Φ^+ and Φ^- respectively. For $x \in W$, we let $\ell(x)$ denote the Coxeter length of x. That is $\ell(x) = |\{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}|$.

Following [20], we use $\Theta = V^*$ to denote the contragradient representation of W over $\mathbb{R}$. Recall that Θ realises W as a reflection group in the sense of Vinberg [28]. For $\alpha \in \Phi$, consider the hyperplanes:

$$H_\alpha = \{\varphi \in \Theta \mid \varphi(\alpha) = 0\},$$

and let $\mathcal{H} = \{H_\alpha \mid \alpha \in \Phi\}$ denote the set of (real) hyperplanes.

We will be particularly interested in hyperplane arrangements and their associated complexified complements. To define these, first consider the open fundamental chamber given by

$$C = \{\varphi \in \Theta \mid \varphi(\alpha_i) > 0 \text{ for all } i \in \Gamma_0\}.$$

The Tits cone is defined as

$$\mathrm{Cone} \coloneqq \bigcup_{w \in W} w \cdot \overline{C}.$$

Let Cone° denote the interior of $\mathrm{Cone} \subseteq \Theta$. The pair $(\mathrm{Cone}^\circ, \mathcal{H})$ is a locally-finite hyperplane arrangement. Note that W is finite if and only if $\mathrm{Cone} = \Theta$.

The complexified hyperplane complement is defined as

$$\mathcal{X}^{\mathbb{C}} \coloneqq \Big(\mathrm{Cone}^\circ \times \mathrm{Cone}^\circ\Big) \setminus \bigcup_{H \in \mathcal{H}} H \times H. \tag{2.1}$$

When W is finite, this space is described more simply as follows:

$$\mathcal{X}^{\mathbb{C}} \cong \Theta \otimes_{\mathbb{R}} \mathbb{C} \setminus \bigcup_{H \in \mathcal{H}} H \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.2}$$

In general, the action of W on $\mathcal{X}^{\mathbb{C}}$ is free and properly discontinuous, and the normal covering map $\mathcal{X}^{\mathbb{C}} \to \mathcal{X}^{\mathbb{C}}/W$ induces the following short exact sequence (where $Z \in \mathcal{X}^{\mathbb{C}}$):

$$1 \to \pi_1(\mathcal{X}^{\mathbb{C}}, Z) \to \pi_1(\mathcal{X}^{\mathbb{C}}/W, Z) \xrightarrow{\pi} W \to 1. \tag{2.3}$$

We define the Artin–Tits group $A \coloneqq \pi_1(\mathcal{X}^{\mathbb{C}}/W, Z)$, and the pure Artin–Tits group is $P \coloneqq \pi_1(\mathcal{X}^{\mathbb{C}}, Z)$. Note that the surjection $\pi : A \to W$ admits a set-theoretic section $w \mapsto \sigma_w$, the "positive lift" of w. We note that A also has a presentation with Artin generators denoted $\sigma_i := \sigma_{s_i},\ i \in \Gamma_0$. We won't be explicitly using the presentation here, and refer the reader to [27] for more details.

2.3. **Normalisers of parabolic subgroups.** For $J \subseteq \Gamma_0$, let $W_J \coloneqq \langle s_j \mid j \in J\rangle$ (respectively $A_J \coloneqq \langle \sigma_j \mid j \in J\rangle$) denote the associated standard parabolic subgroup of W (respectively of A). For $J \subseteq \Gamma_0$ such that W_J is finite, we denote by $w_J \in W_J$ the longest element, and by $\Delta_J \in A_J$ its positive lift to the Artin-Tits group. The element w_J induces a bijection $\iota_J : J \to J$ defined by the formula $w_J \cdot \alpha_j = -\alpha_{\iota(j)}$. An associate of J is by definition a subset $I \in \Gamma_0$ such that W_I and W_J are conjugate in W.

The normalisers of parabolic subgroups and their quotients will be of primary interest to us, and we recall the definitions of $N(W, J), N(A, J)$ and $N(P, J)$ from (1.2), (1.5) and (1.6). There is an extensive literature on the structure of the normaliser of parabolic subgroups. For Coxeter groups, the papers [1, 18, 22] are some of the first to describe their generators from various perspectives. Most important for us is Brink and Howlett's groupoid approach, which also provides the defining relations of $N(W, J)$ [3].

The normalisers of parabolic subgroups of Artin–Tits groups have also been extensively studied; the works [10, 24] for spherical types are particularly relevant, and even more so Godelle's Garside groupoid approach in [13], which provides a version of Brink and Howlett's presentation for spherical type Artin–Tits groups. We note that [3] works for general Coxeter type, whereas Godelle's results rely on Garside structures and apply only to the spherical types. A related (weaker) conjectural description of normalisers in terms of quasi-centralisers (i.e. ribbons instead of reduced ribbons; see Section 5) for Artin–Tits groups in general can be found in [12, Conjecture 1, Property $\star$], which is proven for some families beyond spherical types e.g. FC type [11], two-dimensional type [12], and affine type $\widetilde{A}$ and $\widetilde{C}$ [14].

For later use, we record here an easy lemma relating the groups above.

Lemma 2.4. *The map $\pi : A \to W$ induces a short exact sequence*

$$1 \to N(P, J) \to N(A, J) \to N(W, J) \to 1. \tag{2.5}$$

Proof. Write $\overline{\pi}$ for the map $N(A, J) \to N(W, J)$ induced by π. Let $\beta A_J \in N(A, J)$ and suppose $\beta A_J \in \ker(\overline{\pi})$, i.e. $\overline{\pi}(\beta A_J) = W_J$. Then $w \coloneqq \pi(\beta) \in W_J$, and hence $\beta A_J = \beta\sigma_w^{-1} A_J$. Moreover, $\beta\sigma_w^{-1} \in P$. This shows that $\beta A_J \in N(P, J)$, and hence $\ker(\overline{\pi}) \subseteq N(P, J)$. The other inclusion is clear, proving the short exact sequence. $\square$

2.4. **Groupoid coverings.** The theory of groupoid coverings is a direct translation of the classical theory of covering spaces (see for instance [4, Section 10.2]). This is important for our key arguments, and we recall the basics here.

A groupoid $\mathscr{G}$ is a category in which every morphism is invertible. For us, all groupoids are assumed to be connected and small. For $x \in \mathscr{G}$, we let $\mathscr{G}_x \coloneqq \mathrm{End}_{\mathscr{G}}(x)$ denote the vertex group of $\mathscr{G}$ at x. Up to isomorphism, $\mathscr{G}_x$ is independent of the choice of x.

A functor of groupoids $\mathcal{F} : \widetilde{\mathscr{G}} \to \mathscr{G}$ is a groupoid covering if any morphism in $\mathscr{G}$ has a unique lift in the following sense: given $\alpha \in \mathrm{Hom}_{\mathscr{G}}(x, y)$ and $\widetilde{x} \in \mathcal{F}^{-1}(x)$, there exist unique $\widetilde{y} \in \widetilde{\mathscr{G}}$ and $\widetilde{\alpha} \in \mathrm{Hom}_{\widetilde{\mathscr{G}}}(\widetilde{x}, \widetilde{y})$ such that $\mathcal{F}(\widetilde{\alpha}) = \alpha$. The covering $\mathcal{F}$ is universal if the vertex group of $\widetilde{\mathscr{G}}$ is trivial. Note that this is equivalent to $\mathrm{Hom}_{\widetilde{\mathscr{G}}}(\widetilde{x}, \widetilde{y})$ being a singleton for all objects in $\widetilde{\mathscr{G}}$.

The deck transformation group $\mathrm{Deck}(\mathcal{F})$ consists of automorphisms $\varphi : \widetilde{\mathscr{G}} \to \widetilde{\mathscr{G}}$ such that $\mathcal{F} = \mathcal{F} \circ \varphi$. (Note here that we require actual equality of functors, and not just isomorphisms.) There is a natural action of $\mathrm{Deck}(\mathcal{F})$ on the fibers $\mathcal{F}^{-1}(x)$, and if this action is transitive (for some, hence any, choice of x), we say that $\mathcal{F}$ is a normal cover. In this case, for any $x \in \mathscr{G}$ and $\widetilde{x} \in \mathcal{F}^{-1}(x)$, we have a morphism $\mathscr{G}_x \to \mathrm{Deck}(\mathcal{F})$ which fits into a short exact sequence

$$1 \to \widetilde{\mathscr{G}}_{\widetilde{x}} \to \mathscr{G}_x \to \mathrm{Deck}(\mathcal{F}) \to 1. \tag{2.6}$$

An action of a group N on $\mathscr{G}$ consists of a compatible pair of actions of N on the set of objects and the set of morphisms of $\mathscr{G}$ (i.e. the action is a map $N \to \mathrm{Aut}(\mathscr{G})$ into the group of automorphisms of $\mathscr{G}$). We let $[x], [\alpha]$, etc. denote the equivalence classes of objects and morphisms in $\mathscr{G}$ with respect to the action of N.

Suppose we are given a group N acting on $\mathscr{G}$ such that the action is free on the objects. Then we can consider the quotient category $\mathscr{G}/N$, whose objects are the equivalences classes of objects of $\mathscr{G}$, and where morphisms are given by

$$\mathrm{Hom}_{\mathscr{G}/N}([x], [y]) \coloneqq \Big\{ [\alpha] \ \mid \ g, g' \in N, \alpha \in \mathrm{Hom}_{\mathscr{G}}(g \cdot x, g' \cdot y) \Big\}.$$

Note that since N acts freely on the objects of $\mathscr{G}$, a morphism $[\alpha] \in \mathrm{Hom}_{\mathscr{G}/N}([x], [y])$ admits a unique representative in $\mathscr{G}$ whose source is x. Composition in $\mathscr{G}/N$ is given in the following way: Consider $[\alpha], [\beta]$ two morphisms in $\mathscr{G}/N$ such that the source of $[\beta]$ is the target of $[\alpha]$. Again, there is a unique representative β' of $[\beta]$ whose source in $\mathscr{G}$ is the target of α. We can then define $[\alpha] \circ [\beta]$ to be $[\alpha \circ \beta']$. One easily checks that this definition is independent of the choice of representative, and that $\mathscr{G}/N$ is again a groupoid. Moreover, the natural quotient $\mathcal{Q} : \mathscr{G} \to \mathscr{G}/N$ is a normal covering with $\mathrm{Deck}(\mathcal{Q}) \cong N$, and we have a short exact sequence $1 \to \mathscr{G}_x \to (\mathscr{G}/N)_{[x]} \to N \to 1$.

Finally, we recall that a groupoid completion of a small category $\mathscr{G}^+$ is the category $\mathscr{G}$ obtained by formally inverting all morphisms. The groupoid completion is equipped with a canonical functor

$\kappa : \mathscr{G}^+ \to \mathscr{G}$ satisfying the following universal property: every functor from $\mathscr{G}^+$ to a groupoid factors through κ.

3. Tits cone intersection

In this section, we recall the definition of Iyama–Wemyss' Tits cone intersection and its associated combinatorics following [20]. We also prove that the hyperplane arrangement in its interior is locally-finite, and study the action of the normaliser quotient $N(W, J)$ (see (1.2)) on the Tits cone and its associated complexified hyperplane complement.

3.1. **Basic definition.** Let $I \subseteq \Gamma_0$. We define the following:

$$\Theta_I \coloneqq \{\varphi \in \Theta \mid \varphi(\alpha_i) = 0 \text{ for all } i \in I\}, \tag{3.1}$$

$$C_I \coloneqq \{\varphi \in \Theta_I \mid \varphi(\alpha_j) > 0 \text{ for all } j \notin I\}. \tag{3.2}$$

For each $J \subseteq \Gamma_0$, the Tits cone intersection of J, or the J-cone, is defined as

$$\mathrm{Cone}(J) \coloneqq \mathrm{Cone} \cap \Theta_J,$$

with its interior in Θ_J (not in Θ) denoted by $\mathrm{Cone}(J)^\circ$. The J-roots are defined as $\Phi_J \coloneqq \{\alpha \in \Phi \mid \Theta_J \nsubseteq H_\alpha\}$. The decomposition of Φ into positive and negative roots restricts to a decomposition of Φ_J into positive and negative J-roots, which we denote by Φ_J^+ and Φ_J^- respectively. We also consider the set of hyperplanes in Θ_J:

$$\mathcal{H}_J \coloneqq \{H_\alpha \cap \Theta_J \mid \alpha \in \Phi_J^+\}.$$

When the context is clear, we will simply refer to the hyperplanes in $\mathcal{H}_J$ as H_α (instead of $H_\alpha \cap \Theta_J$).

Note that $\mathrm{Cone}(J)$ is a convex cone (it is the intersection of two convex cones), and it has the following natural stratification:

$$\mathrm{Cone}(J) = \bigsqcup_{L \subseteq \Gamma_0} \bigsqcup_{\substack{x \in W/W_L \\ x \cdot C_L \subseteq \Theta_J}} x \cdot C_L. \tag{3.3}$$

In terms of the hyperplane arrangement $(\mathrm{Cone}(J), \mathcal{H}_J)$, the chambers are $x \cdot C_L \subseteq \Theta_J$ with $|L| = |J|$, which are also the connected components of the hyperplane complement $\mathcal{X}_J \coloneqq \mathrm{Cone}(J) - \bigcup_{\alpha \in \Phi_J^+} H_\alpha$. The faces of each chamber $x \cdot C_L$ are $x \cdot C_{L+i}$ for each $i \in \Gamma_0 - L$. Each face $x \cdot C_{L+i}$ is supported on the hyperplane $H_{x \cdot \alpha_i}$, where $\overline{x \cdot C_{L+i}} = H_{x \cdot \alpha_i} \cap \overline{x \cdot C_L}$ is the wall.

Remark 3.4. In the case that Γ is spherical, Deligne [7, Proposition 1.5] studied a hyperplane arrangement dual to $(\mathrm{Cone}(J), \mathcal{H}_J) = (\Theta_J, \mathcal{H}_J)$. More precisely, consider the quotient map $q : \Theta \to \Theta/\Theta_J$. Deligne studied the set of hyperplanes in Θ/Θ_J given by

$$\mathcal{H}'_J = \{U \subset \Theta/\Theta_J \mid \mathrm{codim}(U) = 1,\ q^{-1}(U) \in \mathcal{H}\}.$$

Note that q induces an isomorphism of hyperplane arrangements

$$(\Theta_{J^c}, \mathcal{H}_{J^c}) \cong (\Theta/\Theta_J, \mathcal{H}'_J),$$

where $J^c \coloneqq \Gamma_0 \setminus J$.

In general, the hyperplane arrangement $(\mathrm{Cone}(J), \mathcal{H}_J)$ need not be locally finite – this is already the case in the classical setting, i.e. for an infinite Coxeter group W with $J = \varnothing$ (there are infinitely many hyperplanes and the origin is in $\mathrm{Cone}(J) = \mathrm{Cone}$). Analogous to the classical case, the following proposition shows that the hyperplane arrangement in its interior: $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$, is a locally-finite hyperplane arrangement.

Proposition 3.5. *Every point $u \in \mathrm{Cone}(J)^\circ$ is contained in an open ball that intersects only finitely many hyperplanes in $\mathcal{H}_J$. In particular, $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is a locally-finite hyperplane arrangement.*

Proof. Firstly, note that the interior of a convex space is convex, so $\mathrm{Cone}(J)^\circ$ is also convex.

Now let $u \in \mathrm{Cone}(J)^\circ$. Since the chambers in $\mathcal{X}_J$ are open, it is clear that if u is in some chamber in $\mathcal{X}_J$ then we can always find a small open ball which avoids all hyperplanes in $\mathcal{H}_J$. Thus we are left to consider the points u that lie on some hyperplane in $\mathcal{H}_J$. By viewing $u \in \mathrm{Cone}$, we can choose a sufficiently small open ball $B'_u \subseteq \mathrm{Cone}$ such that B'_u only intersects with hyperplanes $H' \in \mathcal{H}$ that pass through u. Then $B_u \coloneqq B'_u \cap \Theta_J$ is an open ball in $\mathrm{Cone}(J)$ that contains u and only intersects with hyperplanes $H \in \mathcal{H}_J$ that pass through u. It is therefore sufficient to show that there are only finitely many hyperplanes in $\mathcal{H}_J$ that pass through u.

To this end, note that we can always find two points $v, v' \in \mathrm{Cone}(J) \setminus \bigcup_{H \in \mathcal{H}_J} H$ such that the closed segment $[v, v']$ passes through u, where $[v, v'] \subseteq \mathrm{Cone}(J)$ by convexity of $\mathrm{Cone}(J)$. Since $v, v' \in \mathrm{Cone}(J) \subseteq \mathrm{Cone}$, by [28, Lemma 9], $[v, v']$ intersects only finitely hyperplanes in $\mathcal{H}$ (Vinberg calls these mirrors) that do not contain both v and v'. In particular, u can only be contained in finitely many hyperplanes in $\mathcal{H}_J$ (as none of them passes through v and v'), as required. □

Remark 3.6. While $\mathrm{Cone}(J) \subseteq \mathrm{Cone}$ by definition, note that $\mathrm{Cone}(J)$ (and hence $\mathrm{Cone}(J)^\circ$) can be completely disjoint from Cone°. For example, if J is such that W_J is an infinite parabolic subgroup, then $\mathrm{Cone}(J) \cap \mathrm{Cone}^\circ = \varnothing$. In particular, the arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ need not be a restriction from $(\mathrm{Cone}^\circ, \mathcal{H})$, so the locally-finiteness of $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ does not immediately follow from the locally-finiteness of $(\mathrm{Cone}^\circ, \mathcal{H})$.

Remark 3.7. Since chambers are open subsets, it is clear that the set of chambers in $(\mathrm{Cone}(J), \mathcal{H}_J)$ are the same as the set of chambers in $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$. Moreover, two chambers in $(\mathrm{Cone}(J), \mathcal{H}_J)$ are adjacent (i.e. they share a common wall) if and only if they are adjacent in $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$. As such, when referring to chambers and paths between chambers (see Section 3.4), we will work with $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ instead of $(\mathrm{Cone}(J), \mathcal{H}_J)$. This is easier to do due to the locally-finite property of $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$. For example, local-finiteness of $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ immediately implies that any two chambers in $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ (hence also in $(\mathrm{Cone}(J), \mathcal{H}_J)$) are connected by a finite path.

The following Labelling Theorem in [20] shows us how to algebraically label the chambers.

Theorem 3.8 ([20, Theorem 1.15])**.** *Fix $J \subseteq \Gamma_0$. There is a bijection between the set of chambers of $(\mathrm{Cone}(J), \mathcal{H}_J)$ and the following set*

$$\mathrm{Cham}(J) \coloneqq \{(x, I) \mid W_J x = x W_I \text{ and } x \text{ has minimal length in } xW_I\}, \tag{3.9}$$

given by sending $(x, I) \mapsto x \cdot C_I$.

Remark 3.10. Note that we can have $x \cdot C_I = x' \cdot C_I$ for distinct x and x', and indeed this is the case for any $x' \in xW_I$. The length constraint that appears in (3.9) removes this ambiguity.

From here on, we will abuse notation and denote both the set in (3.9) and the set of chambers in $\mathrm{Cone}(J)$ by $\mathrm{Cham}(J)$, and accordingly we will sometimes write $(x, I) \in \mathrm{Cham}(J)$ and other times $x \cdot C_I \in \mathrm{Cham}(J)$ depending on the context. Note that by Remark 3.7 the set $\mathrm{Cham}(J)$ is also the set of chambers in $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$.

3.2. **Simple wall crossings.** To declutter notation below, for a set X and an element a we sometimes write:

$$X + a \coloneqq X \cup \{a\},$$
$$X - a \coloneqq X \setminus \{a\}.$$

Recall that a wall of a chamber $x \cdot C_I$ is a codimension-one subspace of $\overline{x \cdot C_I}$ given by its intersection with some hyperplane $H \in \mathcal{H}_J$. Note that such a hyperplane can always be described by $H_{x\alpha_a}$ for some $a \in \Gamma_0 - I$, and the wall is given by $\overline{x \cdot C_{I+a}}$. In [20], Iyama–Wemyss provide us with explicit formulae that relate two adjacent chambers sharing a common wall. To do so, we need to introduce some notation.

Let $I \subseteq \Gamma_0$ and let $a \in \Gamma_0 - I$. Following [20], let $\Gamma(I+a)$ denote the induced subgraph of Γ on the vertices $I+a$. We define $[I,a]$ to be the set of vertices of the connected component of $\Gamma(I+a)$ that contains a.

Now let $(x,I) \in \mathrm{Cham}(J)$ and suppose that we have $a \notin I$ such that $W_{[I,a]}$ is a finite parabolic subgroup. We set

$$v_{a,I} := w_{[I,a]-a} w_{[I,a]} \in W. \tag{3.11}$$

Set $K := I + a - \iota_{[I,a]}(a)$. The **simple wall crossing** at a is given by

$$\omega_{a,I}(x,I) := (xv_{a,I}, K). \tag{3.12}$$

Note that $v_{a,I}$ can be alternatively characterised as the unique element for which $v_{a,I}(\Phi^+_{I+a} \setminus \Phi^+_K) \subseteq \Phi^-_K$ and $v_{a,I}(\Phi^+_K) \subseteq \Phi^+_{I+a}$ (cf. paragraph after Proposition 2.1 in [3]).

This terminology is justified by the following:

Proposition 3.13 ([20, Theorem 1.32])**.** *Let $(x,I),(y,K) \in \mathrm{Cham}(J)$ be two adjacent chambers. Then there exists a unique $a \in \Gamma_0 - I$ and a unique $a' \in \Gamma_0 - K$ such that $I + a = K + a'$. Furthermore,*

(1) the common wall is given by

$$\overline{x \cdot C_{I+a}} = \overline{y \cdot C_{K+a'}},$$

(2) $W_{[I,a]} = W_{[K,a']}$ is a finite parabolic subgroup, and

(3) the two chambers are related by the simple wall crossings formula (3.12)*:*

$$\omega_{a,I}(x,I) = (y,K),$$
$$\omega_{a',K}(y,K) = (x,I).$$

Note also that $v_{a,I}v_{a',K} = v_{a',K}v_{a,I} = e$.

Remark 3.14. In [20] the simple wall crossing is denoted ω_a. In situations where no confusion can arise, we will also adopt this notation, but in general we record also I to avoid ambiguity.

Remark 3.15. The elements $v_{a,I}$ defined here are the inverse of $v[a,I]$ in the notation of [3]. In fact, Brink and Howlett define more generally elements $v[I,J]$ for disjoint subsets I and J satisfying certain conditions; see [3, Section 2].

We recall some crucial results from [20].

Lemma 3.16. *(1) If $|\Phi_{I+a} - \Phi_I| < \infty$, then $W_{[I,a]}$ is finite.*

(2) Let (x,I) and $\omega_a(x,I)$ be adjacent chambers. Then $\ell(xv_{a,I}) = \ell(x) + \ell(v_{a,I})$ if and only if (e,J) and (x,I) are on the half space separated by $H_{x\cdot\alpha_a}$.

(3) Any two chambers are related by a finite sequence of wall-crossings.

(4) If W_{I+a} is finite (so $W_{[I,a]}$ is also finite), then the simple wall crossing formula (3.12) *reduces to*

$$\omega_{a,I}(x,I) = (xw_I w_{I+a}, \iota_{I+a}(a)) \tag{3.17}$$

Proof. The first statement is [20, Lemma 1.40]. The second and third statements are Proposition 1.42 and 1.43 in [20] respectively. The final statement is a straightforward exercise. □

Remark 3.18. Statement (3) in Lemma 3.16 is actually an immediate consequence of the locally-finiteness of the hyperplane arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ (whose chambers are also chambers of $(\mathrm{Cone}(J), \mathcal{H}_J)$; see Remark 3.7).

3.3. **Normaliser actions and complexified hyperplane complements.** Note that $\Theta_J \subseteq \Theta$ is invariant under the action of $\mathrm{Norm}_W(W_J) \subseteq W$. Indeed, for $g \in \mathrm{Norm}_W(W_J)$, we have that $g \cdot \alpha_j = \sum_{j' \in J} c_{j'} \alpha_{j'}$ for all $j \in J$, and so $g \cdot \Theta_J = \Theta_J$. It follows that the Tits cone intersection $\mathrm{Cone}(J)$ is also invariant under this action (use e.g. (3.3)). The following lemma describes the action on its set of chambers $\mathrm{Cham}(J)$:

Lemma 3.19. *(1) The normaliser* $\mathrm{Norm}_W(W_J)$ *acts on* $\mathrm{Cham}(J)$ *by sending* $x \cdot C_I \overset{g}{\mapsto} (gx) \cdot C_I$ *for each* $g \in \mathrm{Norm}_W(W_J)$*. In terms of labels* $(x, I) \in \mathrm{Cham}(J)$*, the action is given by* $g \cdot (x, I) = (y, I)$*, where* y *is the minimal length representative in the coset* gxW_I*.*

(2) This action is transitive on the first label: any two chambers of the form (x, I) *and* (y, I) *lie in the same* $\mathrm{Norm}_W(W_J)$*-orbit.*

(3) The action factors through the quotient $N(W, J) := \mathrm{Norm}_W(W_J)/W_J$*, which acts faithfully.*

(4) The action preserves chamber adjacency; precisely if $\omega_a(C) = C'$*, where* $C, C' \in \mathrm{Cham}(J)$*, then* $\omega_a(g \cdot C) = g \cdot C'$ *for any* $g \in N(W, J)$*.*

Proof. Since $g \cdot \Theta_J = \Theta_J$ for all $g \in \mathrm{Norm}_W(W_J)$, if $x \cdot C_I \subseteq \Theta_J$, then $g \cdot (x \cdot C_I) = (gx) \cdot C_I$ is still contained in Θ_J, and hence is a chamber in $\mathrm{Cham}(J)$. By the bijection in Theorem 3.8, it follows that the second label of the chambers are constant under the action, i.e. a chamber (x, I) is mapped by g to another chamber of the form (x', I), for some $x' \in W$. This proves (1).

To see that the action is transitive, suppose $x \cdot C_I, y \cdot C_I \in \mathrm{Cham}(J)$ are two chambers. Then $yx^{-1} \in \mathrm{Norm}_W(W_J)$; indeed, we have $W_J x = xW_I$ and $W_J y = yW_I$, which implies

$$W_J yx^{-1} = yW_I x^{-1} = yx^{-1} W_J.$$

Then yx^{-1} maps $x \cdot C_I$ to $y \cdot C_I$, proving transitivity.

We now show that the action factors through $N(W, J)$ and the resulting action is faithful. It is immediate that any $g \in W_J \subseteq \mathrm{Norm}_W(W_J)$ acts trivially, since for a chamber (x, I), $W_J x = xW_I$ implies that $gx = xg'$ for some $g' \in W_I$, so that

$$(gx) \cdot C_I = (xg') \cdot C_I = x \cdot C_I.$$

For faithfulness, if $g \in \mathrm{Norm}_W(W_J)$ such that $g \cdot (x, I) = (x, I)$, then

$$W_J x = xW_I = gxW_I = gW_J x,$$

which implies that $g \in W_J$ as required.

Finally, (4) is immediate since the action of $\mathrm{Norm}_W(W_J)$ on Θ_J is linear. Moreover, if two chambers $x \cdot C_I$ and $y \cdot C_K$ shares a common wall supported on the hyperplane $H_{x \cdot \alpha_a}$ (so that $\omega_{a,I} : x \cdot C_I \to y \cdot C_K$), then $gx \cdot C_I$ and $gy \cdot C_K$ shares a common wall supported on the hyperplane $H_{gx \cdot \alpha_a}$. Hence the algebraic description also follows. □

Remark 3.20. Unlike the $J = \varnothing$ case, the action of $N(W, J)$ need not be transitive on the whole set of chambers.

Recall from Proposition 3.5 that the hyperplane arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ in the interior of $\mathrm{Cone}(J)$ is a locally-finite hyperplane arrangement. We can therefore associate to it its complexified hyperplane complement $\mathcal{X}_J^{\mathbb{C}}$, which is defined as

$$\mathcal{X}_J^{\mathbb{C}} := \Big(\mathrm{Cone}(J)^\circ \times \mathrm{Cone}(J)^\circ \Big) \setminus \bigcup_{\alpha \in \Phi_J^+} H_\alpha \times H_\alpha. \tag{3.21}$$

Note that when Γ is spherical, we have that $\mathrm{Cone}(J) = \Theta_J$, and

$$\mathcal{X}_J^{\mathbb{C}} \cong \Theta_J \otimes_{\mathbb{R}} \mathbb{C} \setminus \bigcup_{\alpha \in \Phi_J^+} H_\alpha \otimes_{\mathbb{R}} \mathbb{C}.$$

Remark 3.22. Since $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is a locally-finite arrangement in an open convex cone, one can also associate to it a Salvetti complex that is homotopy equivalent to $\mathcal{X}_J^{\mathbb{C}}$; see [25, Section 3.1].

Since $N(W, J)$ acts on $\mathrm{Cone}(J)$, and hence on $\mathrm{Cone}(J)^\circ$, we have a well-defined action of $N(W, J)$ on $\mathcal{X}_J^{\mathbb{C}}$, which is moreover free and properly discontinuous by Lemma 3.19. Thus the associated quotient map $\mathcal{X}_J^{\mathbb{C}} \to \mathcal{X}_J^{\mathbb{C}}/N(W, J)$ is a normal covering, which induces the following short exact sequence, generalising the one in (2.3) (where $Z \in \mathcal{X}_J^{\mathbb{C}}$):

$$1 \to \pi_1(\mathcal{X}_J^{\mathbb{C}}, Z) \to \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W, J), Z) \to N(W, J) \to 1. \tag{3.23}$$

3.4. **Geodesics paths.** We recall some familiar notions from the study of locally-finite hyperplane arrangements. A path p in the hyperplane arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is a sequence of chambers

$$p = ((x_1, I_1), (x_2, I_2), ..., (x_m, I_m))$$

such that for each i, (x_i, I_i) and (x_{i+1}, I_{i+1}) are adjacent. The source of p is $s(p) = (x_1, I_1)$, the target of p is $t(p) = (x_m, I_m)$, and the length of p is $m - 1$.

If p and q are two paths such that $t(p) = s(q)$, we let $p * q$ denote their concatenation, which is a path from $s(p)$ to $t(q)$. If (x, I) and (y, K) are adjacent chambers related by $\omega_{a,I}$, we let $\omega_{a,I} : (x, I) \to (y, K)$ denote the length one path from (x, I) to (y, K) which we also call a **simple wall crossing**. As before, if no confusion can arise we may denote the simple wall crossing by ω_a.

To a path p we can associate by Proposition 3.13 the nodes $a_i \in \Gamma_0$ satisfying $\omega_{a_i,I_i}(x_i, I_i) = (x_{i+1}, I_{i+1})$, i.e.

$$p = \omega_{a_1,I_1} * \cdots * \omega_{a_{m-1},I_{m-1}}.$$

We define

$$v_p \coloneqq v_{a_1,I_1} \cdots v_{a_{m-1},I_{m-1}} \in W$$

Finally, p is a **geodesic path** if it crosses a minimal number of walls between the chambers (x_1, I_1) and (x_m, I_m). Equivalently, p crosses any hyperplane in $\mathcal{H}_J$ at most once.

Proposition 3.24. *A path $p = ((x_1, I_1), (x_2, I_2), ..., (x_m, I_m))$ is a geodesic if and only if*

$$\ell(v_p) = \sum_{i=1}^{m-1} \ell(v_{a_i,I_i}).$$

Proof. First the consider the case where p is a path starting at (e, J). We induct on the length of p. The statement is trivial if p has length 1. Now assume for induction that the statement is true for length $n - 1$, and consider a path p of length n with $(x, I) \to \omega_a(x, I)$ being the last step of p, i.e. $p = q * \omega_a$ for some path q of length $n - 1$. Then p is a geodesic if and only if q is a geodesic that does not cross the hyperplane $H_{x\cdot\alpha_a}$. Therefore p is a geodesic if and only if q is a geodesic and (x, I) and (e, J) are on the same half space separated by $H_{x\cdot\alpha_i}$. The statement now follows from Lemma 3.16(2).

Now consider instead that we have a path starting at an arbitrary chamber (z, L). Then $z^{-1}\cdot\Theta_J = \Theta_L$, and this induces a bijection from $\mathrm{Cham}(J)$ to $\mathrm{Cham}(L)$. This bijection preserves and reflects geodesic paths. Moreover, the simple wall crossing $\omega_a : (x, I) \to (xv_{a,I}, I + a - \iota_{[I+a]}(a))$ is sent to the simple wall crossing $\omega_a : (x', I) \to (x'v_{a,I}, I + a - \iota_{[I+a]}(a))$, with x' the minimal length representative in $z^{-1}xW_I$. It follows that any path p is sent to a path $z^{-1} \cdot p$ with the property that $v_{z^{-1}\cdot p} = v_p \in W$. In particular, whether or not v_p is length additive with respect to each simple wall crossing is invariant under the bijection. Thus we can apply z^{-1} so that our path starts at (e, L) and the result follows from the argument above. $\square$

3.5. **The Deligne groupoid.** The Deligne groupoid was introduced in [7] to study the topology of finite simplicial arrangements (and prove the $K(\pi,1)$ conjecture for finite reflection groups). Our setting is more general than that of Deligne, but the construction can be carried out similarly (although the important properties are only proven in the setting of Deligne).

To the locally-finite arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$, we associate an oriented graph on the set of vertices $\mathrm{Cham}(J)$. To every pair of adjacent chambers we associate a 2-cycle, i.e. given two adjacent chambers (x, I) and (y, K) there is one arrow $(x, I) \to (y, K)$ and another opposite arrow $(y, K) \to (x, I)$. We continue to use the notation $\omega_{a,I}$ as above to label these arrows.

A path in this graph from (x, I) to (y, K) is geodesic if it is a path of minimal length between (x, I) and (y, K). Equivalently, a path is geodesic if and only if it crosses every hyperplane $H \in \mathcal{H}_J$ at most once.

Define $\mathscr{D}^+ := \mathscr{D}^+(J)$ as the category with:

- objects given by $\mathrm{Cham}(J)$; and
- morphisms $\mathrm{Hom}_{\mathscr{D}^+}((x, I), (y, K))$ are paths from (x, I) to (y, K) modulo the relation generated by the identification of any two geodesic paths with the same source and target.

The arrangement groupoid $\mathscr{D} := \mathscr{D}(J)$ associated to $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$ is defined as the groupoid completion of $\mathscr{D}^+$. If Γ is of finite-type, then $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J) = (\Theta_J, \mathcal{H}_J)$ is a finite simplicial arrangement, and we call $\mathscr{D}$ the **Deligne groupoid**. In this case, $\mathscr{D}$ is equivalent to the fundamental groupoid of $\mathcal{X}_J^{\mathbb{C}}$, denoted $\pi_1(\mathcal{X}_J^{\mathbb{C}})$ [7].

Note that the action of $N(W, J)$ on $\mathrm{Cham}(J)$ from Lemma 3.19 induces an action of $N(W, J)$ on $\mathscr{D}$. Indeed, an element $g \in N(W, J)$ acts on the simple wall crossing $\omega_{a,I} : (x, I) \to (y, K)$ by mapping it to the simple wall crossing $\omega_{a,I} : g \cdot (x, I) \to g \cdot (y, K)$ (see Lemma 3.19(4)). This action clearly preserves geodesics, since if a path p crosses some hyperplane more than once clearly the same is also true for the path $g \cdot p$. Therefore we have a well-defined action of $N(W, J)$ on $\mathscr{D}$.

4. The Brink–Howlett groupoid

In [3] Brink and Howlett introduced a groupoid in order to study the normaliser $\mathrm{Norm}_W(W_J)$. We recall their construction here, and also give an explicit realisation of its universal cover in terms of the geometry of the Tits cone intersection. In Corollary 4.11 we use this to give a new proof to parts of one of Brink and Howlett's main results (Theorem A in [3]). We then explain in Corollary 4.15 how to use the connection between Brink and Howlett's work and the Tits cone intersection to answer questions of Iyama and Wemyss about "atomic Matsumoto relations".

To begin, fix $J \subseteq \Gamma_0$. Recall that the reflection representation V of W has basis given by the simple roots α_k for $k \in \Gamma_0$. The **Brink–Howlett groupoid** $\mathscr{BH} := \mathscr{BH}(J)$ is defined as follows:

- The objects are subsets $I \subseteq \Gamma_0$ which are associates of J.
- The morphism space $\mathrm{Hom}_{\mathscr{BH}}(I, K)$ consists of $x \in W$ such that
$$\{\alpha_i \mid i \in I\} = \{x \cdot \alpha_k \mid k \in K\}.$$
Composition is given by multiplication in W on the right.

Remark 4.1. Note that $\mathscr{BH}$ is the opposite category of the groupoid in [3]. Namely, our composition convention follows path composition which reads from left to right:

$$\mathrm{Hom}_{\mathscr{BH}}(I, K) \times \mathrm{Hom}_{\mathscr{BH}}(K, L) \to \mathrm{Hom}_{\mathscr{BH}}(I, L)$$
$$(x, y) \mapsto xy.$$

This is chosen so that it matches the geometric description of the universal cover of $\mathscr{BH}$ below.

The following lemma is immediate from the definition of $v_{a,I}$ (see (3.11) and (3.12)).

Lemma 4.2. *Suppose $I \in \mathscr{BH}$ and $a \in \Gamma_0 - I$ such that $W_{[I,a]}$ is finite. Then*
$$v_{a,I} \in \mathrm{Hom}_{\mathscr{BH}}(I, I + a - \iota_{[I,a]}(a)).$$

Definition 4.3. Define the groupoid $\widetilde{\mathscr{BH}} \coloneqq \widetilde{\mathscr{BH}}(J)$ as follows.

- The set of objects is $\mathrm{Cham}(J)$.
- The morphism space $\mathrm{Hom}_{\widetilde{\mathscr{BH}}}((x,I),(y,K))$ is the singleton consisting of $x^{-1}y \in W$. Composition is induced by multiplication in W (on the right).

Note that if (x,I) and (y,K) are adjacent chambers related by wall crossings $\omega_{a,I}$ and $\omega_{a',K}$, then necessarily $\mathrm{Hom}_{\widetilde{\mathscr{BH}}}((x,I),(y,K)) = \{v_{a,I}\}$ and $\mathrm{Hom}_{\widetilde{\mathscr{BH}}}((y,K),(x,I)) = \{v_{a',K}\}$. More generally, the following result shows that the morphism space between any two chambers is given by some sequence of wall crossings.

Proposition 4.4. *The groupoid $\widetilde{\mathscr{BH}}$ is generated by morphisms $v_{a,I} \in \mathrm{Hom}_{\widetilde{\mathscr{BH}}}((x,I),(y,K))$, where (x,I) and (y,K) are adjacent chambers related by the simple wall crossing $\omega_{a,I}$.*

Proof. Since the space of morphisms between any two object is a singleton, this is an immediate consequence of the fact that any two chambers are related by a finite sequence of simple wall crossings (Lemma 3.16(3)). ☐

Remark 4.5. There is an obvious functor $\mathscr{D}^+ \to \widetilde{\mathscr{BH}}$ (hence also from $\mathscr{D}$) which is the identity on objects and sends each length 1 path $(x,I) \to \omega_a(x,I)$ in $\mathscr{D}^+$ to the morphism $v_{a,I} : (x,I) \to \omega_a(x,I)$ in $\widetilde{\mathscr{BH}}$; this is clearly well-defined since all hom spaces in $\widetilde{\mathscr{BH}}$ are singletons. By construction, this functor is full and surjective on objects, but it is not faithful in general (it is faithful in the degenerate case where there is only one chamber).

Proposition 4.6. *(1) If $(x,I),(y,K) \in \mathrm{Cham}(J)$ then $x^{-1}y \in \mathrm{Hom}_{\mathscr{BH}}(I,K)$.*
(2) If $w \in \mathrm{Hom}_{\mathscr{BH}}(I,K)$ and $(x,I) \in \mathrm{Cham}(J)$, then $(xw,K) \in \mathrm{Cham}(J)$, so that $w \in \mathrm{Hom}_{\widetilde{\mathscr{BH}}}((x,I),(xw,K))$.

Proof. For (1), it suffices to show for all $(y,K) \in \mathrm{Cham}(J)$ that $y \in \mathrm{Hom}_{\mathscr{BH}}(J,K)$, since the result above then follows via composition (and taking inverses).

By the definition of the labelling of chambers, we have

$$yC_K \subseteq \Theta_J,$$

where $|J|=|K|$ and moreover y has minimal length in $W_J y = yW_K$. Thus for all $\varphi \in C_K$ and all $j \in J$, $(y \cdot \varphi)(\alpha_j) = \varphi(y^{-1} \cdot \alpha_j) = 0$. Since $y^{-1} \cdot \alpha_j$ is a root, which is either positive or negative, $y^{-1} \cdot \alpha_j$ must be a linear combination of α_k for $k \in K$. In particular, $y^{-1} \cdot \alpha_j$ is a root of W_K for any $j \in J$.

We claim that for any $j \in J$, $y^{-1} \cdot \alpha_j$ is a positive root. Indeed, since y has minimal length in $W_J y$, y^{-1} has minimal length in $y^{-1}W_J$. If $y^{-1} \cdot \alpha_j$ is a negative root, then $\ell(ys_j) < \ell(y)$, contradicting length-minimality. The same argument using length-minimality in yW_K instead shows that $y \cdot \alpha_k$ are also all positive roots of the parabolic W_J. Now y and y^{-1} are linear isomorphisms between the vector spaces $\mathrm{Span}_{\mathbb{R}}\{\alpha_j\}_{j\in J}$ and $\mathrm{Span}_{\mathbb{R}}\{\alpha_k\}_{k\in K}$, whose matrices in terms of these fixed bases are both non-negative matrices. This implies that the matrix of y (and y^{-1}) is a permutation of a diagonal matrix with non-negative entries. It follows that the matrix of y must be a permutation matrix, as non-identity positive scalar multiple of a simple root is not a root. As such, $y \cdot \{\alpha_k \mid k \in K\} = \{\alpha_j \mid j \in J\}$, which says that $y \in \mathrm{Hom}_{\mathscr{BH}}(J,K)$ by definition. This completes the proof of (1)

For (2), let $w \in \mathrm{Hom}_{\mathscr{BH}}(I,K)$. Then for all $i \in I$, there exists $k \in K$ such that $w^{-1} \cdot \alpha_i = \alpha_k$. It follows that $w \cdot C_K \subseteq \Theta_I$. Since $(x,I) \in \mathrm{Cham}(J)$, we get $x \cdot \Theta_I = \Theta_J$, which implies that $xw \cdot C_K \subseteq \Theta_J$. We claim that xw is minimal in xwW_K. Indeed, if xw is not minimal, then there is some $k \in K$ such that $xw(\alpha_k) = x(\alpha_i) \in \Phi^-$, which contradicts the length-minimality of x in xW_I. ☐

We define a functor

$$\begin{aligned}\mathcal{F} : \widetilde{\mathscr{B}\mathscr{H}} &\to \mathscr{B}\mathscr{H}, \\ (x, I) &\mapsto I,\end{aligned} \tag{4.7}$$

and on morphisms $x^{-1}y \in \mathrm{Hom}_{\widetilde{\mathscr{B}\mathscr{H}}}((x, I), (y, K))$ is mapped to $x^{-1}y \in \mathrm{Hom}_{\mathscr{B}\mathscr{H}}(I, K)$. By Proposition 4.6(1), $\mathcal{F}$ is well-defined.

Theorem 4.8. *The functor $\mathcal{F}$ in* (4.7) *is a universal groupoid covering and*

$$\mathrm{Deck}(\mathcal{F}) \cong N(W, J).$$

Proof. Existence of path lifting follows from Proposition 4.6(2). Note that uniqueness comes for free since morphism spaces in $\widetilde{\mathscr{B}\mathscr{H}}$ are singletons, from which it follows too that the covering is universal.

Next we show that the deck transformation group is $N(W, J)$. By Lemma 3.19 we can endow $\widetilde{\mathscr{B}\mathscr{H}}$ with an action of $N(W, J)$. Indeed, the lemma describes the action on objects, and since morphism spaces are singletons this extends uniquely to an action on the category. Then $\mathcal{F}$ is $N(W, J)$-equivariant, where $N(W, J)$ acts trivially on $\mathscr{B}\mathscr{H}$, i.e. $N(W, J)$ acts as deck transformations of $\mathcal{F}$. This shows that $N(W, J) \leq \mathrm{Deck}(\mathcal{F})$.

Conversely, suppose $\varphi \in \mathrm{Deck}(\mathcal{F})$. Then we obtain maps on the fibres $\varphi : \mathcal{F}^{-1}(I) \to \mathcal{F}^{-1}(I)$ for all associates I of J. Set $(g, J) \coloneqq \varphi(e, J) \in \mathcal{F}^{-1}(J)$, so that g has minimal length in gW_J. We claim that

$$\varphi(x, I) = g \cdot (x, I) = (gx, I)$$

for all associates I of J.

Set $(x', I) \coloneqq \varphi(x, I)$. Consider the morphism $x \in \mathrm{Hom}_{\widetilde{\mathscr{B}\mathscr{H}}}((e, J), (x, I))$. Under the composite functor $\mathcal{F} \circ \varphi$, x is mapped to $g^{-1}x' \in \mathrm{Hom}_{\mathscr{B}\mathscr{H}}(J, I)$. Since φ is a deck transformation this morphism must be equal to $x \in \mathrm{Hom}_{\mathscr{B}\mathscr{H}}(J, I)$. Hence $x' = gx$, and we deduce that $\varphi(x, I) = (gx, I) = g \cdot (x, I)$ as claimed. It follows that φ agrees with the action of $gW_J \in N(W, J)$ on $\widetilde{\mathscr{B}\mathscr{H}}$, and so $N(W, J) = \mathrm{Deck}(\mathcal{F})$. □

As a corollary we deduce a classical result of Lusztig [22, Lemma 5.2] (which was also independently proved by Howlett [18]). Set $N_J \coloneqq \mathscr{B}\mathscr{H}_J$, i.e.

$$N_J = \{x \in W \mid (\forall j \in J)(\exists j' \in J)\ x \cdot \alpha_j = \alpha_{j'}\}.$$

Note that $N_J \subseteq \mathrm{Norm}_W(W_J)$ by definition.

Corollary 4.9. *The vertex group N_J is isomorphic to $N(W, J)$ and moreover the normaliser is a semidirect product* $\mathrm{Norm}_W(W_J) \cong W_J \rtimes N_J$

Proof. Applying (2.6) to the universal cover $\mathcal{F} : \widetilde{\mathscr{B}\mathscr{H}} \to \mathscr{B}\mathscr{H}$, we obtain that $N_J \cong \mathrm{Deck}(\mathcal{F})$. By Theorem 4.8 this implies that $N_J \cong N(W, J) = \mathrm{Norm}_W(W_J)/W_J$, which gives us the short exact sequence

$$1 \to W_J \to \mathrm{Norm}_W(W_J) \to N_J \to 1.$$

We claim that the sequence above splits. For this it suffices to show that the inclusion $N_J \subseteq \mathrm{Norm}_W(W_J)$ followed by the quotient onto $N(W, J)$ agrees with the isomorphism $N_J \cong \mathrm{Deck}(\mathcal{F}) \cong N(W, J)$. To this end, note that the isomorphism $N_J \cong \mathrm{Deck}(\mathcal{F})$ sends $g \in N_J \mapsto \varphi \in \mathrm{Deck}(\mathcal{F})$ with the defining property that $\varphi(e, J) = (g, J)$. This in turn dictates that $\varphi \mapsto gW_J \in N(W, J)$ (see proof of Theorem 4.8), as required. □

Definition 4.10. Following [3, Section 2] we say an expression of $w \in W$ of the form

$$w = v_{a_1, I_1} v_{a_2, I_2} \cdots v_{a_m, I_m}$$

is a **standard expression** of length m if $\ell(w) = \sum_{i=1}^{m} \ell(v_{a_i, I_i})$.

We now deduce (parts of) a result by Brink–Howlett:

Corollary 4.11 (c.f. [3, Theorem A]). *(1) The groupoid $\mathscr{BH}$ is generated by morphisms*

$$v_{a,I} : I \to I + a - \iota_{[I,a]}(a),$$

for $I \in \mathscr{BH}$ and $a \in \Gamma_0 - I$ such that $W_{[I,a]}$ is a finite parabolic subgroup.

(2) Any two standard expressions of a morphism $w \in \mathrm{Hom}_{\mathscr{BH}}(I, K)$ are of the same length, i.e. if

$$v_{a_1,I_1} v_{a_2,I_2} \cdots v_{a_m,I_m} = v_{b_1,L_1} v_{b_2,L_2} \cdots v_{b_n,L_n}$$

are two standard expressions with $I_1 = L_1$, then $m = n$.

Proof. The first statement follows immediately from the groupoid covering $\mathcal{F} : \widetilde{\mathscr{BH}} \to \mathscr{BH}$ and the fact that elements $v_{a,I}$ generate $\widetilde{\mathscr{BH}}$. Note that the finiteness condition arises from Proposition 3.13(2).

For the second statement, Proposition 3.24 shows that a standard expression of $w \in \mathrm{Hom}_{\mathscr{BH}}(I, K)$ lifts to a geodesic path in the hyperplane arrangement $(\mathrm{Cone}(J)^\circ, \mathcal{H}_J)$. More precisely, consider the standard expression $v_{a_1,I_1} \cdots v_{a_m,I_m}$ and choose x_1 such that $(x_1, I_1) \in \mathrm{Cham}(J)$. Then we can consider the path $p = \omega_{a_1,I_1} * \cdots * \omega_{a_{m-1},I_{m-1}}$ starting at (x_1, I_1). By Proposition 3.24, p is a geodesic path. Similarly, we can construct a geodesic path q using the standard expression $v_{b_1,L_1} \cdots v_{b_n,L_n}$, also starting at (x_1, I_1) (recall $I_1 = L_1$). Since the two standard expressions are equal, p and q also have the same endpoint. By uniqueness of lengths of geodesics, p and q must therefore have the same length, and hence $m = n$. □

The complete statement of [3, Theorem A] is stronger: the relations in $\mathscr{BH}$ are generated by certain "rank two relations"; or in the language of [21], $\mathscr{BH}$ satisfies an atomic Matsumoto theorem. Using Theorem 4.8 we can interpret Brink–Howlett's result on the level of the hyperplane arrangement in $\mathrm{Cone}(J)$. We obtain a generalisation of [21, Theorem 5.1] which removes the finitary assumption. This also generalises [20, Theorem 1.60] to the uniformly weakly Dynkin setting; see [20, Remark 1.62].

We begin with the following result from [3]; here, a left-divisor of $w \in W$ is an element $u \in W$ such that $u \leq w$ with respect to the left weak Bruhat order on W:

Lemma 4.12. *Let I be an associate of J. Suppose we have two distinct elements $a, b \in I^c$ such that $|\Phi_{I+a+b} - \Phi_I| < \infty$. Then there exists $v_{a,b,I} \in \mathrm{Hom}_{\mathscr{BH}}(I, K)$ such that:*

(1) $v_{a,I}$ and $v_{b,I}$ are left divisors of $v_{a,b,I}$ in W, and $v_{a,b,I}$ has exactly two standard expressions (of necessarily the same length):

$$v_{a,b,I} = \begin{cases} v_{a_1,I_1} v_{a_2,I_2} \cdots v_{a_m,I_m}, \\ v_{b_1,L_1} v_{b_2,L_2} \cdots v_{b_m,L_m}, \end{cases} \tag{4.13}$$

with $a_1 = a, b_1 = b$ and $I_1 = I = L_1$;

(2) $v_{a,b,I}$ is a left divisor for any $w \in \mathrm{Hom}_{\mathscr{BH}}(I, K')$ which has $v_{a,I}$ and $v_{b,I}$ as left divisors.

In other words, $v_{a,b,I}$ is minimal with respect to the standard expression length such that $v_{a,I}$ and $v_{b,I}$ are both left divisors.

Proof. Under the assumption $|\Phi_{I+a+b} - \Phi_I| < \infty$, the existence of $v_{a,b,I}$ and its two standard expressions in (4.13) follow from the discussion on "*Type R2*" relations in [3, pg 326] (precisely, $v_{a,b,I}^{-1} = v[\{a, b\}, I]$ in the notation of [3]; see Remark 3.15). This combined with [3, Proposition 2.4] gives statement (1). Statement (2) is a translation of [3, Lemma 4.1]. □

Remark 4.14. The inverse of $v_{a,b,I}$ (i.e. $v[\{a, b\}, I]$) can also be characterised as the unique element $w \in W_{I+a+b}$ satisfying $w(\Phi^+_{I+a+b} - \Phi^+_I) \subseteq \Phi^-_{I+a+b}$ and $w(\Phi^+_I) \subseteq \Phi^+_{I+a+b}$. See discussion after Proposition 2.1 in [3].

Corollary 4.15. *Let $a, b \in I^c$ be distinct elements such that $|\Phi_{I+a+b} - \Phi_I| < \infty$, and let $(x, I) \in \mathrm{Cham}(J)$. Then:*

(i) In the hyperplane arrangement in $\mathrm{Cone}(J)$, the simple wall crossings $\omega_{a,I}$ and $\omega_{b,I}$ starting at (x, I) have a meet, i.e. there are paths

$$\begin{tikzcd}[column sep=small] & (xv_{a,I}, I_1) \arrow[r] & (x_2, I_2) \arrow[r] & \cdots \arrow[r] & (x_{n-1}, I_{n-1}) \arrow[dr] & \\ (x, I) \arrow[ur, "\omega_{a,I}"] \arrow[dr, "\omega_{b,I}"'] & & & & & (xv_{a,b,I}, I_n) \\ & (xv_{b,I}, K_1) \arrow[r] & (y_2, K_2) \arrow[r] & \cdots \arrow[r] & (y_{n-1}, K_{n-1}) \arrow[ur] & \end{tikzcd} ; \tag{4.16}$$

(ii) The two paths above are geodesics and their common length n is minimal among all possible meets (i.e. $v_{a,b,I}$ is the lowest common multiple of $v_{a,I}$ and $v_{b,I}$ in the groupoid sense).

Moreover, any two sequences of wall crossings with the same starting and ending chambers are related via the relation above.

Proof. Theorem A of [3] states that $\mathscr{BH}$ is generated as a groupoid by $v_{a,I}$, with relations generated by those of the form (4.13) (as we range over all I and the possible a and b satisfying the condition in Lemma 4.12).

Combining this with the fact that $\widetilde{\mathscr{BH}}$ is a universal cover of $\mathscr{BH}$ (Theorem 4.8) gives part (i). Part (ii) follows from Lemma 4.12 and the identifications of geodesics and standard expressions given in Proposition 3.24. □

5. The reduced ribbon groupoid

In this section we recall the reduced ribbon groupoid, which will play a crucial role in our proof of Theorem 1.8. This groupoid was introduced by Godelle as part of a more general study of Garside categories. We stress that it is here that we need to impose the finite-type assumptions.

We first recall the definition of an auxiliary (larger) groupoid:

Definition 5.1 ([13])**.** The **ribbon groupoid** $\mathscr{R}' := \mathscr{R}'(J)$ is defined as follows.

- The objects are $I \subseteq \Gamma_0$ which are associates of J.
- $\mathrm{Hom}_{\mathscr{R}'}(I, K) := \{\beta \in A \mid A_I \beta = \beta A_K\}$. Composition is given by multiplication in A (on the right).

Let $I \subseteq \Gamma_0$ and $a \in \Gamma_0 - a$ such that $\Phi_{I+a} - \Phi_I$ is finite. By Lemma 3.16(1), $W_{[I,a]}$ is finite. We define

$$\mu_{a,I} := \Delta^{-1}_{[I,a]-\iota_{[I+a]}(a)} \Delta_{[I,a]} \in A^+. \tag{5.2}$$

Note that we are taking the reduced expression in (5.2), so that it is an element in A^+. Moreover, (5.2) is the positive lift of (3.11), i.e. it is the image under the set-theoretic section $W \to A^+$ of the projection map $\pi : A \to W$.

From here, *we will be restricting to Γ being of finite-type.* Note that now (5.2) reduces to the following

$$\mu_{a,I} = \Delta_I^{-1} \Delta_{I+a} \in A^+; \tag{5.3}$$

c.f. (3.17) in Lemma 3.16.

Definition 5.4 ([13])**.** Suppose that Γ is a finite-type diagram. Let $\mathscr{R}$ denote the subgroupoid of $\mathscr{R}'$ whose objects coincide with the objects of $\mathscr{R}'$, and morphisms are generated by

$$\mu_{a,I} : I \to I + a - \iota_{[I,a]}(a) \tag{5.5}$$

for all $a \in \Gamma_0 - I$. Following Godelle, we call $\mathscr{R}$ the **reduced ribbon groupoid** associated to J.

Firstly, we mention that the relevance of $\mathscr{R}$ for us is that its vertex group $\mathscr{R}_J$ is isomorphic to $N(A, J)$:

Theorem 5.6 ([10, Proposition 4.4])**.** *There is an isomorphism of groups*

$$\mathrm{Norm}_A(A_J) \cong A_J \rtimes \mathscr{R}_J.$$

In particular, $\mathscr{R}_J \cong N(A, J)$.

We now recall a presentation of $\mathscr{R}$ due to Godelle. For any I an associate of J, and $a, b \notin I$, we consider the least common multiple (for left divisibility) $\mu_{a,I} \vee \mu_{b,I}$ in A^+. Since A^+ is an Artin–Tits monoid, it admits a ν-structure in the sense of [13] (see [13, Example 3.14]). It follows by [13, Proposition 2.12 and 3.15] that the least common multiple of $\mu_{a,I}$ and $\mu_{b,I}$ in the underlying positive category of $\mathscr{R}'$ is also $\mu_{a,I} \vee \mu_{b,I}$. Then by [13, Corollary 4.18], the least common multiple of $\mu_{a,I}$ and $\mu_{b,I}$ in the underlying positive category of the subgroupoid $\mathscr{R}$ is also $\mu_{a,I} \vee \mu_{b,I}$. Now, by [13, Proposition 4.19], $\mu_{a,I} \vee \mu_{b,I}$ has precisely two expressions as a product of generators (5.5):

$$\mu_{a,I} \vee \mu_{b,I} = \begin{cases} \mu_{a_1,I_1}\mu_{a_2,I_2}\cdots\mu_{a_m,I_m}, \\ \mu_{b_1,L_1}\mu_{b_2,L_2}\cdots\mu_{b_n,L_n}. \end{cases} \tag{5.7}$$

Here $a_1 = a$, $b_1 = b$ and $I_1 = I = L_1$. By [13, Corollary 4.18], a complete set of relations of $\mathscr{R}$ is given by equating the right hand sides of (5.7) as we range over all $I \in \mathscr{R}$ and $a, b \notin I$.

Lemma 5.8. *Suppose Γ is a finite-type diagram. The two expressions of the lcm $\mu_{a,I} \vee \mu_{b,I} \in A^+$ in* (5.7) *agree with the positive lifts of the two standard expressions of $v_{a,b,I}$ in* (4.13) *of Lemma 4.12. In particular, we have $m = n$ and $\pi(\mu_{a,I} \vee \mu_{b,I}) = v_{a,b,I}$. Moreover, $v_{a,b,I}$ is the join of $v_{a,I}$ and $v_{b,I}$ for the weak left Bruhat order on W.*

Proof. Denote by $\widetilde{W} \subset A^+$ the set of positive lifts of elements of W in A^+. Since Γ is a finite-type diagram, by [6, Section IX.1.3] the map $\pi : A \to W$ restricts to an isomorphism between the following posets:

(1) $\widetilde{W}$: endowed with left-divisibility in the monoid A^+, and
(2) W: endowed with left-divisibility from the weak left Bruhat order.

Moreover, the inverse isomorphism is the positive lift $W \to A^+$. In particular, if $w = s_1 s_2 \cdots s_r$ is a length-additive decomposition of w (i.e. a reduced expression), then the product $\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_r}$ in A^+ is a decomposition of the positive lift σ_w in $\widetilde{W} \subset A^+$.

The condition in Lemma 4.12 is always satisfied since Γ is of finite-type by assumption. Now, note that each $\mu_{a,I} \in A^+$ is the positive lift of $v_{a,I} \in W$ by construction. In particular, $\mu_{a,I} \vee \mu_{b,I}$ lies in $\widetilde{W}$. By applying the isomorphism of posets $\pi : \widetilde{W} \to W$ to its two expressions as in (5.7), we obtain two standard expressions of $\pi(\mu_{a,I} \vee \mu_{b,I})$. Thus $\pi(\mu_{a,I} \vee \mu_{b,I})$ is left-divided by both $v_{a,I}$ and $v_{b,I}$, and it is moreover a morphism in $\mathscr{BH}$ (starting from I). By Lemma 4.12, $v_{a,b,I}$ left divides $\pi(\mu_{a,I} \vee \mu_{b,I})$.

Conversely, since the two decompositions of $v_{a,b,I}$ in (4.13) are standard expressions, they both lift to decompositions of the positive lift of $v_{a,b,I}$ in $\widetilde{W} \subset A^+$. In particular, $\mu_{a,I}$ and $\mu_{b,I}$ both left-divide the positive lift of $v_{a,b,I}$ in A^+, and so does their lcm $\mu_{a,I} \vee \mu_{b,I}$. We then obtain that $\mu_{a,I} \vee \mu_{b,I}$ and the positive lift of $v_{a,b,I}$ left-divide one another in A^+, and thus they are equal.

The rest of the statement is an immediate consequence of the isomorphism of posets $\pi : \widetilde{W} \to W$. □

6. The main result

In this section we continue to assume that Γ is a finite-type diagram. Our aim is to prove Theorem 1.8, which will be an immediate consequence of Theorem 6.5 below.

To begin, we construct a functor $\mathcal{G} : \mathscr{D} \to \mathscr{R}$ as follows: on objects we define $(x, I) \mapsto I$, and on the generating morphism $\omega_{a,I} : (x, I) \to (y, K)$ we define $\mathcal{G}(\omega_{a,I}) = \mu_{a,I} \in \mathrm{Hom}_{\mathscr{R}}(I, K)$.

Recall we have an action of $N(W, J)$ on $\mathscr{D}$. Let $\mathcal{Q} : \mathscr{D} \to \mathscr{D}/N(W, J)$ denote the quotient functor. The essential feature of $\mathcal{G}$ which will imply Theorem 1.8 is that it is a groupoid covering isomorphic to $\mathcal{Q}$. We'll build up to this in a series of results.

Lemma 6.1. *The functor $\mathcal{G} : \mathscr{D} \to \mathscr{R}$ is well-defined.*

Proof. Since $\mathscr{D}$ is the groupoid completion of $\mathscr{D}^+$ and $\mathcal{G}$ maps into a groupoid, it is sufficient to prove that $\mathcal{G}$ is well-defined $\mathscr{D}^+$. In particular, we only have to show that any two geodesic paths in $\mathscr{D}^+$ with the same source and target are sent to the same morphism in $\mathscr{R}$. To do so, we will prove that any geodesic path in $\mathscr{D}^+$ starting at (x_1, I_n) and ending at (x_n, I_n) is sent to the positive lift of $x_1^{-1}x_n \in W$. This is sufficient since this description depends only on the source and target of the path.

So suppose p is a geodesic path in $\mathscr{D}^+$ travelling through the following sequence of chambers

$$p : (x_1, I_1) \to (x_2, I_2) \to \cdots \to (x_n, I_n)$$

with $(x_{i+1}, I_{i+1}) = \omega_{a_i}(x_i, I_i)$ for all $1 \leq i \leq n-1$. Under $\mathcal{G}$, p is sent to the element

$$\mathcal{G}(p) = \mu_{a_1,I_1}\mu_{a_2,I_2} \ldots \mu_{a_{n-1},I_{n-1}} \in \mathrm{Hom}_{\mathscr{R}}(I_1, I_n).$$

Recall that each μ_{a_i,I_i} is, by definition, the positive lift of v_{a_i,I_i}. Since p is geodesic, by Proposition 3.24 we have that $v_{a_1,I_1}v_{a_2,I_2} \ldots v_{a_{n-1},I_{n-1}}$ is a standard expression for $x_1^{-1}x_n$. Consequently, $\mathcal{G}(p)$ is indeed the positive lift of $x_1^{-1}x_n$, as required. $\square$

Lemma 6.2. *The functor $\mathcal{G} : \mathscr{D} \to \mathscr{R}$ is a groupoid covering.*

Proof. The existence of path lifting is immediate. Indeed, any generating path

$$\mu_{a,I} \in \mathrm{Hom}_{\mathscr{R}}(I, I + a - \iota_{[I,a]}(a))$$

can be lifted to a one-step path $\omega_{a,I} : (x, I) \to \omega_a(x, I)$ in $\mathscr{D}^+$ for any choice of $(x, I) \in \mathrm{Cham}(J)$.

To prove uniqueness, it suffices to show that the generating relations in (5.7) lift to the same path in $\mathscr{D}^+$ once we fix a starting point.

Suppose we have $\mu_{a,I}$ and $\mu_{b,I}$ as in (5.7), and let $(x, I) \in \mathrm{Cham}(J)$. Then we can lift the two products in (5.7) to paths in $\mathscr{D}^+$ starting at (x, I). By Proposition 3.24, since these products both descend to standard expressions of an element $w \in W$ (Lemma 5.8), both of these paths are geodesic paths. Moreover, these paths have the same endpoint, given by (xw, K) for some $K \in \mathscr{R}$. By the defining relations in $\mathscr{D}^+$, the paths are equal. This proves uniqueness of path-lifting. $\square$

Proposition 6.3. *The functor $\mathcal{G}$ factors through $\mathcal{Q}$, resulting in a functor*

$$\mathcal{G}' : \mathscr{D}/N(W, J) \to \mathscr{R}.$$

Moreover, $\mathcal{G}'$ is an equivalence.

Proof. Set $N \coloneqq N(W, J)$ for simplicity. Endow $\mathscr{R}$ with the trivial N-action. Then showing that $\mathcal{G}$ factors is equivalent to show that $\mathcal{G}$ is N-equivariant. On objects, this is clear since N acts only on the first factor of chamber labels $(x, I) \in \mathrm{Cham}(J)$, and $\mathcal{G} : (x, I) \mapsto I$. On morphisms, equivariance is also immediate since both $\omega_{a,I} : (x, I) \to (y, K)$ and its image under $g \in N$, i.e. $\omega_{a,I} : g \cdot (x, I) \to g \cdot (y, K)$, are mapped by $\mathcal{G}$ to $\mu_{a,I}$. Since these are generating morphisms this implies that $\mathcal{G}$ is N-equivariant.

Next we show that the induced functor $\mathcal{G}' : \mathscr{D}/N \to \mathscr{R}$ is an equivalence. Essential surjectivity is immediate. Fullness follows from the equality $\mathcal{G}' \circ \mathcal{Q} = \mathcal{G}$ and path-lifting. Indeed, given a morphism $\mu : I \to K$ in $\mathscr{R}$, there exists a lift $\widetilde{\mu} : (x, I) \to (y, K)$, since $\mathcal{G}$ is a cover. Set $\mu' \coloneqq \mathcal{Q}(\widetilde{\mu})$. Then $\mathcal{G}(\mu') = \mu$.

Finally, we show that $\mathcal{G}'$ is faithful. Let $[(x,I)],[(y,K)] \in \mathscr{D}/N$, and consider

$$[p],[p'] \in \mathrm{Hom}_{\mathscr{D}/N}([(x,I)],[(y,K)]),$$

such that $\mathcal{G}'([p]) = \mathcal{G}'([p'])$. Since N acts transitively on first labels of elements in $\mathrm{Cham}(J)$ (cf. Lemma 3.19(2)), we can choose representatives p, p' such that

$$p : (x,I) \to g \cdot (y,K),$$
$$p' : (x,I) \to g' \cdot (y,K),$$

where $g, g' \in N$. Then we have that $\mathcal{G}(p) = \mathcal{G}(p')$. But now p and p' are both lifts of $\mathcal{G}(p) = \mathcal{G}(p')$ with the same source. By uniqueness of path lifting, we have $p = p'$, and hence $[p] = [p']$. □

Corollary 6.4. *The covers $\mathcal{G}$ and $\mathcal{Q}$ are equivalent. In particular, $\mathcal{G}$ is a normal covering and its deck transformation group of is isomorphic to $N(W,J)$.*

Proof. By Proposition 6.3 we have a commutative diagram

$$\begin{tikzcd} & \mathscr{D} \arrow[dl, "\mathcal{Q}"'] \arrow[dr, "\mathcal{G}"] & \\ \mathscr{D}/N(W,J) \arrow[rr, "\mathcal{G}'"] & & \mathscr{R} \end{tikzcd}$$

with $\mathcal{G}'$ an equivalence. Hence the two covers are equivalent. Since $\mathcal{Q}$ is normal and its group of deck transformations is isomorphic to $N(W,J)$, the same is true for $\mathcal{G}$. □

We can now prove the main results of this section, which is an isomorphism between the short exact sequence appearing in Lemma 2.4 and 1.4. We note that these results provide the proof of Theorem 1.8.

Theorem 6.5. *Let $x \in \mathcal{X}_J^{\mathbb{C}}$ and denote also by x its image in $\mathcal{X}_J^{\mathbb{C}}/N(W,J)$. The following short exact sequences are isomorphic:*

$$1 \to \pi_1(\mathcal{X}_J^{\mathbb{C}}, x) \to \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W,J), x) \to N(W,J) \to 1 \tag{6.6}$$
$$1 \longrightarrow N(P,J) \longrightarrow N(A,J) \longrightarrow N(W,J) \longrightarrow 1 \tag{6.7}$$

Proof. As explained in Section 3.5, $\mathscr{D}$ is equivalent to the fundamental groupoid $\pi_1(\mathcal{X}_J^{\mathbb{C}})$, and similarly, $\mathscr{D}/N(W,J)$ is equivalent to the fundamental groupoid $\pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W,J))$. Moreover, the groupoid covering $\mathcal{Q} : \mathscr{D} \to \mathscr{D}/N(W,J)$ is isomorphic to $\pi_1(\mathcal{X}_J^{\mathbb{C}}) \to \pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W,J))$. Therefore (6.6) is isomorphic to the short exact sequence induced by $\mathcal{Q}$:

$$1 \to \mathscr{D}_J \to (\mathscr{D}/N(W,J))_J \to N(W,J) \to 1,$$

where for notation simplicity, $\mathscr{D}_J$ signifies the vertex group based at (e,J).

Now, since by Corollary 6.4 $\mathcal{Q}$ is isomorphic to $\mathcal{G}$, the above sequence is isomorphic to:

$$1 \to \mathscr{D}_J \to \mathscr{R}_J \to N(W,J) \to 1. \tag{6.8}$$

By Theorem 5.6, (6.8) can be rewritten as

$$1 \to \mathscr{D}_J \to N(A,J) \to N(W,J) \to 1. \tag{6.9}$$

Note that here the map $N(A,J) \to N(W,J)$ agrees with $\overline{\pi}$ from Lemma 2.4. Indeed, the map $\mathscr{R}_J \to N(W,J)$ from (6.8) maps a loop, which is a product of $\mu_{a,I}$'s, to the corresponding product of $v_{a,I}$'s. By Lemma 2.4, (6.9) is isomorphic to (6.7), completing the argument. □

Corollary 6.10. *The space $\mathcal{X}_J^{\mathbb{C}}/N(W,J)$ is a $K(\pi,1)$ for $N(A,J)$.*

Proof. Note that $\mathcal{X}_J^{\mathbb{C}}$ is a complexified hyperplane complement from a real finite hyperplane arrangement in a vector space (recall that we are in finite-type). Moreover, the chambers of the real arrangement are open simplicial cones. Therefore Deligne's Theorem [7] applies, and $\mathcal{X}_J^{\mathbb{C}}$ is a $K(\pi, 1)$. Since $\mathcal{X}_J^{\mathbb{C}} \to \mathcal{X}_J^{\mathbb{C}}/N(W, J)$ is a normal cover and $\pi_1(\mathcal{X}_J^{\mathbb{C}}/N(W, J), x) \cong N(A, J)$ by Theorem 6.5, the result follows. $\square$

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