Bessis braid category of the complex braid group B_{31}

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Garside monoids

Let M be a monoid (for instance $\mathcal{M} = \langle a, b \mid aba = bab \rangle^+$).

We say that $x \in M$ ***left-divides*** (resp. ***right-divides***) $y \in M$ if there is some $z \in M$ such that $xz = y$ (resp. $y = zx$).
We denote this by $x \preceq y$ (resp. $y \succeq x$).

In $\mathcal{M}$, we have $a \preceq ab$ and $ab \not\preceq a$. Also $a \preceq bab$ as $a.ba = aba = bab$.

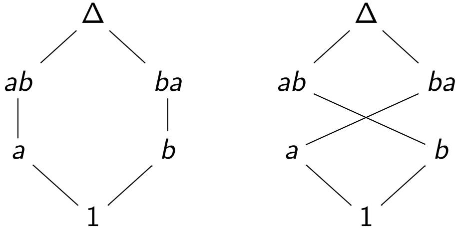
A ***Garside monoid*** is (roughly speaking) a monoid M satisfying combinatorial assumptions (lcms, gcds, cancellativity...) endowed with a special element Δ such that the sets

$$\{s \in M \mid s \preceq \Delta\} \quad \text{and} \quad \{s' \in M \mid \Delta \succeq s'\}$$

are equal, finite, and generate M. This set is denoted by S.

The monoid $\mathcal{M}$

The monoid $\mathcal{M}$ is Garside for the Garside element $\Delta = aba$. We have $S = \{1, a, b, ab, ba, \Delta\}$. The Hasse diagrams of $(S, \preceq)$ and $(S, \succeq)$ are given by



In general, $(S, \preceq)$ and $(S, \succeq)$ are always lattices.

Garside groups

A ***Garside group*** is a group G, endowed with a submonoid $M \leqslant G$, such that M is a Garside monoid which generates G.

Solution to the ***word problem***: normal form of elements of G.
Solution(s) to the ***conjugacy problem***: finite computable subset of every conjugacy class.

For $m \in M$, there is a unique $\phi(m) \in M$ such that $m\Delta = \Delta\phi(m)$.

The map $m \mapsto \phi(m)$ is an automorphism of M, called the ***Garside automorphism***. It induces a permutation of the set S, in particular it has finite order

In $\mathcal{M}$, we have $a\Delta = a(bab) = (aba)b = \Delta b$. In fact ϕ swaps a and b.

Complex braid groups

The braid group on n strands is defined as the fundamental group of the configuration space

$$\begin{aligned} E_n &:= \{(x_i) \in \mathbb{C}^n \mid i \neq j \Rightarrow x_i \neq x_j\} /\mathfrak{S}_n \\ &= \{x \in \mathbb{C}^n \mid \forall \sigma \in \mathfrak{S}_n,\ \sigma.x \neq x\} /\mathfrak{S}_n \end{aligned}$$

Which depends on $\mathfrak{S}_n$. We can replace $\mathfrak{S}_n$ by

1. Finite Coxeter groups: we get spherical ***Artin groups.***
2. Complex reflection groups: we get ***complex braid groups.***

Complex reflection groups are well-understood. In particular the irreducible ones are classified. We get a classification of irreducible complex braid groups with

1. An infinite family depending on integer parameters...
2. A sequence of 34 exceptional cases $B_4, \ldots, B_{37}$.

Exceptional braid groups are Garside groups...

Consider the list of exceptional irreducible complex braid groups.

$$\begin{array}{ccccccccc} B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & B_{10} & B_{11} & B_{12} \\ B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} & B_{20} & B_{21} \\ B_{22} & B_{23} & B_{24} & B_{25} & B_{26} & B_{27} & B_{28} & B_{29} & B_{30} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & & \end{array}$$

Exceptional braid groups are Garside groups...

First, some of these are braid groups of real reflection groups (spherical Coxeter groups). Such groups admits a canonical Garside structure (Dehornoy, Paris).

$$\begin{array}{ccccccccc} B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & B_{10} & B_{11} & B_{12} \\ B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} & B_{20} & B_{21} \\ B_{22} & \cancel{B_{23}} & B_{24} & B_{25} & B_{26} & B_{27} & \cancel{B_{28}} & B_{29} & \cancel{B_{30}} \\ B_{31} & B_{32} & B_{33} & B_{34} & \cancel{B_{35}} & \cancel{B_{36}} & \cancel{B_{37}} & & \end{array}$$

Exceptional braid groups are Garside groups...

Then, a lot of irreducible reflection group have the same braid group as some real reflection group (Shephard groups). In particular they inherit the Garside structure of their real counterpart.

$$\begin{array}{ccccccccc}
\cancel{B_4} & \cancel{B_5} & \cancel{B_6} & B_7 & \cancel{B_8} & \cancel{B_9} & \cancel{B_{10}} & B_{11} & B_{12} \\
B_{13} & \cancel{B_{14}} & B_{15} & \cancel{B_{16}} & B_{17} & \cancel{B_{18}} & B_{19} & \cancel{B_{20}} & \cancel{B_{21}} \\
B_{22} & & B_{24} & \cancel{B_{25}} & \cancel{B_{26}} & B_{27} & & B_{29} & \\
B_{31} & \cancel{B_{32}} & B_{33} & B_{34} & & & & &
\end{array}$$

Exceptional braid groups are Garside groups...

Remaining groups of rank two admit *ad hoc* Garside structures (Picantin).

$$\begin{array}{ccccccccc}
 & & & \cancel{B_7} & & & & \cancel{B_{11}} & \cancel{B_{12}} \\
\cancel{B_{13}} & & \cancel{B_{15}} & & \cancel{B_{17}} & & \cancel{B_{19}} & & \\
\cancel{B_{22}} & & B_{24} & & & B_{27} & & B_{29} & \\
B_{31} & & B_{33} & B_{34} & & & & &
\end{array}$$

Exceptional braid groups are Garside groups...

The groups $B_{24}, B_{27}, B_{29}, B_{33}$ and B_{34} all admit a Garside structure: the ***dual braid monoid*** (Bessis).

$\cancel{B_{24}}$ $\cancel{B_{27}}$ $\cancel{B_{29}}$

B_{31} $\cancel{B_{33}}$ $\cancel{B_{34}}$

Exceptional braid groups are Garside groups... except B_{31}

B_{31}

A new hope

Theorem (Bessis 15)

The braid group B_{37} admits a Garside monoid M with 120 generators $s_1, \dots s_{120}$ and Garside element $\Delta = s_1 \cdots s_8$ (dual braid monoid).

Theorem (Bessis 15)

There is some element $\rho \in B_{37}$ such that $C_{B_{37}}(\rho) \simeq B_{31}$ and $\rho^2 = \Delta^{15}$.

This is wonderful ! It gives rise to a ***Garside category*** $\mathcal{C}_{31}$, and a ***Garside groupoid*** $\mathcal{B}_{31}$ suitable for studying B_{31} !

"One can write down a presentation by generators and relations for $\mathcal{B}_{31}$, and deduce a presentation an relations for the braid group B_{31}."

D. Bessis

Presentation of $\mathcal{C}_{31}$

The defining presentation of the category $\mathcal{C}_{31}$ is given by

$$\begin{aligned}\mathrm{Ob}(\mathcal{C}_{31}) &:= \left\{u \in M(B_{37}) \mid u\phi^8(u) = \Delta\right\}.\\ \mathcal{S} &:= \{(a,b) \in M(B_{37}) \mid ab \in \mathrm{Ob}(\mathcal{C}_{31})\}.\\ \mathrm{Rel} &:= \{(x,y,z) \in M(B_{37}) \mid xyz \in \mathrm{Ob}(\mathcal{C}_{31})\}.\end{aligned}$$

With $(a,b) : ab \to b\phi^8(a)$. An element $(x,y,z) \in \mathrm{Rel}$ expresses the relation

$$(x, yz)(y, z\phi^8(x)) = (xy, z).$$

We get 88 objects, 2603 nontrivial generating morphisms, and 11065 relations.

Atomic loops and centers

Theorem (My computer, 2022)

For every $u \in \mathrm{Ob}(\mathcal{C}_{31})$, the group $\mathcal{B}_{31}(u,u)$ is generated by a set of ***atomic loops***, which are positive elements of minimal length in $\mathcal{C}_{31}(u,u)$. Atomic loops corresponds to braid reflections in B_{31}.

Theorem (G. 22)

Let λ be an atomic loop in $\mathcal{B}_{31}(u,u)$. If $f \in \mathcal{B}_{31}(u,u)$ is such that $f^{-1}\lambda^n f$ is positive, then $f^{-1}\lambda f = \lambda'$ for some atomic loop λ' in $\mathcal{B}_{31}(u,u)$.

Corollary (G. 22)

Let $P \subset B_{31}$ be a finite index subgroup. We have $Z(P) \subset Z(B_{31})$, in particular it is cyclic.

Outline of the proof of the Corollary

Let $f \in Z(P) \subset \mathcal{B}_{31}(u, u)$. We have to show that f commutes with atomic loops.

Let λ be an atomic loop in $\mathcal{B}_{31}(u, u)$. There is some $n \geqslant 1$ such that $\lambda^n \in P$.

We have $f^{-1}\lambda^n f = \lambda^n$ by assumption. By our theorem there is some atomic loop λ' with $f^{-1}\lambda f = \lambda'$ and $\lambda'^n = \lambda^n$.

This implies $\lambda = \lambda'$ (word problem solution). Thus f commutes with all atomic loops.

Presentation(s) of B_{31}

$\mathcal{C}_{31}$ is defined by a categorical presentation. Is it possible to deduce a presentation of $B_{31} = \mathcal{B}_{31}(u, u)$ for some u ?

Answer : YES !

Let $\mathcal{G}$ is a finitely presented groupoid, and let $u \in \mathrm{Ob}(\mathcal{G})$. We can deduce a presentation of $\mathcal{G}(u, u)$ from the presentation of $\mathcal{G}$ using an analogue of the Reidemeister-Schreier method.

Using this method (and a computer !), we deduce, for different objects of $\mathcal{C}_{31}$, different presentations of B_{31}, with atomic loops (braid reflections) as generators !

One example of such presentation

For instance, B_{31} can be generated by 6 elements t, g, n, s, p, m, with relations as follows

$$\begin{cases} tg = nt = gn, \\ ts = st, \;\; pn = np, \;\; gm = mg, \\ psm = smp = mps, \\ tpt = ptp, \;\; tmt = mtm, \;\; pgp = gpg, \\ sns = nsn, \;\; sgs = gsg, \;\; nmn = mnm. \end{cases}$$

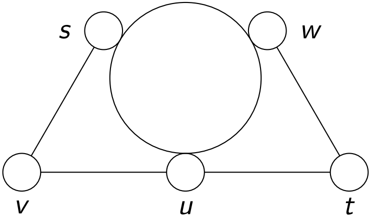
This is a bit redundant as $tg = nt$ implies $tgt^{-1} = n$. We can delete a generator.

The rabbit monoid

Lastly, there is one object for which we recover the "classical" presentation of B_{31}: five generators s, t, u, v, w and relations

$$\begin{cases} ts = st, \ vt = tv, \ wv = vw, \\ suw = uws = wsu, \\ svs = vsv, \ vuv = uvu, \ utu = tut, \ twt = wtw. \end{cases}$$

Summarized in the following diagram:



Thank you for your attention