Regular theory in complex braid groups

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First definitions

- $W \leqslant \mathrm{GL}(V)$ be an irreducible complex reflection group of rank $n \geqslant 2$
- $d_1, \cdots, d_n$ the reflection degrees of W
- $d_n^*, \cdots, d_1^*$ the reflection codegrees of W
- For $k > 0$, $\zeta_k := \exp\left(\frac{2i\pi}{k}\right) \in \mu_k^* \subset \mu_k$

For $k > 0$, define

$$A(k) := \{d_i \mid k \text{ divides } d_i\} \text{ and } B(k) := \{d_i^* \mid k \text{ divides } d_i^*\}$$

Also, $a(k)$ and $b(k)$ their respective cardinalities.
Recall that an element $g \in W$ is called ζ_k-***regular***, or k-***regular*** if the eigenspace $\mathrm{Ker}\,(g - \zeta_k)$ contains regular vectors.

Centralizers of regular elements

Theorem (Springer 74, Broué 88, Lehrer-Michel 03)

- *We have that ζ_k-regular elements exist if and only if $a(k) = b(k)$.*
- *All ζ_k-regular elements in W are conjugate.*
- *If $g \in W$ is ζ_k-regular, then the centralizer $W' := C_W(g)$ acts on $\mathrm{Ker}\,(g - \zeta_k)$ as a complex reflection group.*
- *The degrees (resp. codegrees) of W' are the elements of $A(k)$ (resp. $B(k)$).*

What about braid groups ?
We want an analogue of k-regular elements inside of $B(W)$

Full twist and center of irreducible complex braid groups

Let $B(W)$ be the braid group of W, $P(W)$ its pure braid group.

Theorem (Broué, Malle, Rouquier 98, Bessis 15, Digne, Marin, Michel 11)

$Z(B(W))$ is cyclic, generated by an element z_B.
$Z(W)$ is cyclic generated by the image of z_B in W.
$Z(P(W))$ is cyclic and generated by $z_P := z_B^{|Z(W)|}$.
We have a short exact sequence

$$1 \longrightarrow Z(P(W)) \longrightarrow Z(B(W)) \longrightarrow Z(W) \longrightarrow 1$$

The element z_P is called the ***full-twist***. It is defined topologically as the element of $B(W)$ represented by the loop $t \mapsto e^{2i\pi t}x_0$, where x_0 is the basepoint for $P(W)$.

A $\rho \in B(W)$ such that $\rho^k = z_P$ will be called a k-***regular braid***.

Bessis' work

An irreducible complex reflection group W is ***well-generated*** if it can be generated by n reflections. Otherwise, W is ***badly-generated*** (and it can be generated by $n+1$ reflections).

Theorem (Bessis 15)

Assume that W is well-generated.

- (A) *There exists k-regular braids in $B(W)$ if and only if k is regular for W.*
- (B) *The k-regular braids form a conjugacy class of $B(W)$. They are mapped to ζ_k-regular elements in W.*
- (C) *If ρ is a k-regular braid, and w is its image in W. Then $C_{B(W)}(\rho) \simeq B(W')$ where $W' := C_W(w)$.*

The proof is done by extracting combinatorial properties from subtle topology tricks, which only exist in the well-generated cases
But the combinatorial mindset carries on to almost all cases

A bit of Garside theory: Definition

Recall that a ***homogeneous Garside monoid*** is a monoid M endowed with a particular element Δ, satisfying the following conditions:

- M is ***homogeneous****** and ***cancellative***.
- There are gcds and lcms (for both left and right divisibility).
- The element Δ is be ***balanced***. The set S of its divisors is finite, and it generates M.

Such a monoid embeds in its group of fractions $G(M)$, which we call a ***Garside group***.

Theorem

Let W be an irreducible complex reflection group. The group $B(W)$ is a Garside group, such that z_B is some power of Δ. Except in the cases

$$G(de, e, n),\ \ d, e \geqslant 2 \text{ and } G_{31}$$

The Garside automorphism

In $G(M)$, conjugation by Δ induces an automorphism. This automorphism restricts to an automorphism of M !
Let $s \in S$ be a simple. There is a $\overline{s}$ be such that $s\overline{s} = \Delta$.
We also have some $\phi(s)$ such that $\overline{s}\phi(s) = \Delta$.
We have $\phi(s) = s^{\Delta}$, indeed

$$s\Delta = s\overline{s}\phi(s) = \Delta\phi(s)$$

Furthermore, we see that $\phi(s) \in S$. Since S is finite, ϕ has finite order d.

Periodic elements

In a complex braid group $B(W)$ admitting a convenient Garside structure, a regular braid ρ is such that $\rho^p = z_P = \Delta^q$ for some integers p and q.
In general We want to study elements ρ of $G(M)$ such that $\rho^p = \Delta^q$.
Call such an element a (p, q)-***periodic element***.
Of course, any (p, q)-periodic element is also (pn, qn)-periodic for all $n > 0$. **But there is also a converse !**

Theorem

Any (p, q)-periodic element in $G(M)$ is conjugate to some ρ is in M such that $\rho^{p'} = \Delta^{q'}$ where $p' = \frac{p}{p \wedge q}$ and $q' = \frac{q}{p \wedge q}$.

From now on, we will suppose that p and q are coprime.

For given (coprime) p and q, we have three things to do:

1. State whether or not (p, q)-periodic elements exist.
2. Determine the conjugacy classes of (p, q)-periodic elements.
3. Compute the centralizer of (p, q)-periodic elements.

All of this can be achieved through the construction of the ***category of periodic elements***.

Decompositions of Δ

For any positive integer n, one can consider

$$D_n = D_n(\Delta) := \{(u_0, \cdots, u_{n-1}) \mid u_0 \dots u_{n-1} = \Delta\}$$

There is an action of ϕ on such tuples, given by

$$(u_0, \dots, u_{n-1})^{\phi} := (u_1, \cdots, u_{n-1}, \phi(u_0))$$

One can then define

$$D_n^m := \{(u_0, \dots, u_{n-1}) \in D_n \mid (u_0, \dots, u_{n-1})^{\phi^m} = (u_0, \dots, u_{n-1})\}$$

Case of coprime integers

If p and q are coprime. One can construct a well-defined sequence of integers $k_1, \dots, k_{n-1}$ such that

$$D_p^q \approx \{u \in S \mid u\phi^{k_1}(u)\dots\phi^{k_{n-1}}(u) = \Delta\}$$

In particular, D_p^q now only depends on one parameter u.
This helps a lot in reducing notation: we have

$$D_{2p}^{2q} = \left\{(a,b) \mid ab \in D_p^q\right\} \text{ and } D_{3p}^{3q} = \left\{(x,y,z) \mid xyz \in D_p^q\right\}$$

The category $\mathcal{C}_p^q$ of (p, q)-periodic elements

The sets D_p^q, D_{2p}^{2q} and D_{3p}^{3q} give a presentation of a category $\mathcal{C}_p^q$:

- The objects are the elements of D_p^q
- $(a, b) : ab \to b\phi^{k_1}(a)$ is a generating morphism, that we call a simple morphism
- $(x, y, z) \rightsquigarrow (x, yz) \circ (y, z\phi^{k_1}(x)) = (xy, z)$ is a relation

From this we get:

- Divisors of (a, b) are the same thing as divisors of a: if $a_1 a_2 = a$, we have a triple (a_1, a_2, b) which gives the relation

$$(a, b) = (a_1, a_2 b)(a_2, b\phi^{k_1}(a_1))$$

- The composition of $(a, b)(c, d)$ is a simple morphism if and only if $c \preceq b$. Indeed if $cx = b$, we have a triple (a, c, x) giving

$$(a, b)(c, d) = (ac, x)$$

Greedyness

Let $(a, b)(c, d)$ be a two path. A divisor of (c, d) that can be composed with (a, b) is the same as a common divisor of b and c.
So the biggest such element is given by $x = b \wedge_G c$. Denoting $xb' = b, xc' = c$ we have

$$(a, b)(c, d) = (a, b)(x, c'd)(c', d\phi^{k_1}(x)) = (ax, b')(c', d\phi^{k_1}(x))$$

One can check that this last path is greedy. The simple elements make up a Garside family in $\mathcal{C}_p^q$.
For every object u of $\mathcal{C}_p^q$, define

$$\Lambda_u := (u, 1) : u \to \phi^{k_1}(u)$$

It is the lcm of the simple morphisms starting at u. The Garside family we consider is bounded by Λ

Classification of periodic elements up to conjugacy

The map $(a, b) \to a$ induces a collapse functor $F : \mathcal{C}_p^q \to M$, which extends to a functor F : from $\mathcal{G}_p^q := \mathrm{Env}(\mathcal{C}_p^q)$ to $G(M)$

Theorem (Bessis 07)

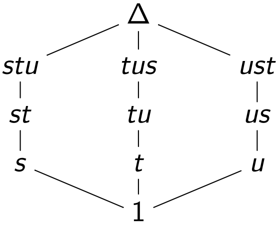
The connected components of $\mathcal{C}_p^q$ are in one to one correspondance with conjugacy classes of (p, q)-periodic elements in $G(M)$.
Furthermore, for $x \in \mathrm{Obj}(\mathcal{C}_p^q)$, the functor F sends $\mathcal{G}_p^q(x, x)$ to the centralizer in $G(M)$ of $F(\Lambda^q)$, which is a (p, q)-periodic element.

From this we deduce everything we want: a computation of conjugacy classes and of centralizers of periodic elements.

Consider the usual presentations of B_{12}:

$$B_{12} := \langle s, t, u \mid stus = tust = ustu \rangle$$

It can be seen as a monoid presentation for some monoid M_{12}. It is a homogeneous Garside monoid for $\Delta = stus$.
The simple form the following lattice (for left divisibility)



Furthermore, we have $z_{B_{12}} = \Delta^3$, and $z_{P_{12}} = \Delta^6$
The Garside automorphism is given by $\phi(s) = t, \phi(t) = u, \phi(u) = s$.

Examples of calculations for B_{12}

The element $z_{P_{12}}$ has length 24, so there can be k-th regular braids in B_{12} only if k divides 24. We give two examples.
For $k = 24$, we study $(24, 6)$-regular elements in M_{12}. This amounts to studying $(4, 1)$-regular elements. We have

$$\begin{aligned} D_4^1 &= \{(u, v, w, x) \in S \mid uvwx = \Delta \text{ and } (u, v, w, x) = (v, w, x, \phi(u)\} \\ &= \{u \mid u^4 = \Delta\} = \varnothing \end{aligned}$$

The category $\mathcal{C}_4^1$ is empty: there is no $(24, 6)$-regular elements in B_{12}, that is there is no 24-regular braids, which is coherent with the fact that 24 is not a regular number for G_{12}.

For $k = 4$, we study $(2,3)$ regular elements in M_{12}. We have

$$\begin{aligned} D_2^3(\Delta_{12}) &= \{(u,v) \mid uv = \Delta_{12} \text{ and } (u,v) = (\phi(v), \phi^2(u))\} \\ &= \{u \mid u\phi^2(u) = \Delta_{12}\} = \{st, us, tu\} \end{aligned}$$

The category $\mathcal{C}_2^3$ is given by

$$\begin{tikzcd}
& st \arrow[dl, bend right, "{(s,t)}"'] \arrow[dr, bend left=10, "{(st,1)}"'] & \\
tu \arrow[ur, bend right=10, "{(tu,1)}"'] \arrow[rr, bend right=10, "{(t,u)}"'] & & us \arrow[ul, bend right, "{(u,s)}"'] \arrow[ll, bend right=10, "{(us,1)}"']
\end{tikzcd}$$

With relations

$$(u,s)(s,t) = (us,1);\ (s,t)(t,u) = (st,1);\ (t,u)(u,s) = (tu,1)$$

The relations imply that the atoms of this category are $(s,t),(u,s)$ and (t,u). We could represent it by the following graph

$$\begin{tikzcd}
 & st \arrow[dl, bend right, "{(s,t)}"'] & \\
tu \arrow[rr, bend right, "{(t,u)}"'] & & us \arrow[ul, bend right, "{(u,s)}"']
\end{tikzcd}$$

Take the object st, we have $\Lambda^3 = (st,1)(us,1)(tu,1)$. Its image in B_{12} is $(stu)^2$. Its centralizer in B_{12} is cyclic and generated by

$$stu = F((s,t)(t,u)(u,s))$$

Maximal roots

Consider the set R of regular numbers for W. It is finite, so we can consider the subset R_{max} of regular numbers which are maximal for divisibility.

A k-regular braid for $k \in R_{max}$ is called ***maximal***.

Lemma

Every k-regular braid is a power of some maximal regular braid.

Let ρ be a k-regular braid. By assumption k divides some $k' \in R_{max}$. Let ρ' be a k'-regular braid. We have that $\rho'^{k'/k}$ is a k-regular braid. So there is some $c \in B(W)$ such that

$$\rho = c^{-1}\rho'^{k'/k}c$$

Thus ρ is a power of $c^{-1}\rho' c$.

Roots of elements in the center

Corollary (G. 22)

Let W be an irreducible complex reflection group. If $\gamma \in B(W)$ admits a central power, then $\gamma = \rho^p$ for ρ a maximal regular braid and $p \in \mathbb{N}$.
In particular the image of γ in W is a regular element.

Corollary

If $z \in Z(B(W))$, then roots of z are unique in $B(W)$ up to conjugacy.

If x, y are such that $x^n = y^n \in Z(B(W))$. We have $x = \rho^p$, $y = \sigma^q$ where ρ, σ are k, k'-regular respectively.
Define $d = k \wedge k'$, $dk_1 = k, dk_1' = k'$. So ρ^{k_1} and $\sigma^{k_1'}$ are d-regular braids.
Writing $\ell(x)$ for the length of x, we have

$$\ell(x) = \ell(y) = \ell(\rho)p = \ell(\sigma)q \Rightarrow \ell(z_P)pk' = \ell(z_P)qk \Rightarrow pk' = qk$$

So k_1 divides p and k_1' divides q: x and y are powers of ρ_1^k and $\sigma^{k_1'}$, which are conjugate.

Torsion elements in $B(W)/Z(B(W))$

Every element of R_{max} is divisible by $|Z(W)|$. One can then define $R_{mod} := \frac{1}{|Z(W)|} R_{max}$

Proposition (Shvartsman 96, G.22)

Let W be any irreducible complex reflection group. If γ is an element of torsion in $B(W)/Z(B(W))$, then the order of γ divides an element of R_{mod}. Conversely, every divisor of an element of R_{mod} is the order of some element of $B(W)/Z(B(W))$.

Thank you for your attention