Regular theory in complex braid groups

PhD students' seminar of the LAMFA

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September 21, 2022

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Reflections, reflection group

Let V be a complex vector space of dimension $n < \infty$.

An element $r \in \mathrm{GL}(V)$ is a ***reflection*** if $\mathrm{Ker}\,(r-1)$ is a hyperplane of V and if r has finite order.

We call $H_r := \mathrm{Ker}\,(r-1)$ the ***reflecting hyperplane*** of r.

A subgroup $W \leqslant \mathrm{GL}(V)$ is a ***complex reflection group*** if it is generated by reflections.

- → $W = \mu_n \leqslant \mathrm{GL}_1(\mathbb{C})$
- → $W = \mathfrak{S}_n \leqslant \mathrm{GL}_n(\mathbb{C})$ as the permutation matrices (with reflections as transpositions)

We denote by $\mathcal{A}$ the set of reflecting hyperplanes of (reflections of) W.

Irreductibility

A CRG $W \leqslant \mathrm{GL}(V)$ is ***irreducible*** if the representation $W \hookrightarrow \mathrm{GL}(V)$ is irreducible (i.e the only W-stable subspaces of V are $\{0\}$ and V).

Every CRG decomposes as a product of irreducible reflection group. For $W = \mathfrak{S}_n$, the line $D = \mathbb{C}(1, \ldots, 1)$ is W-invariant. So W is not irreducible, but $W' = \mathfrak{S}_n$ acting on

$$H := \left\{(x_1, \ldots, x_n) \mid \sum x_i = 0\right\} = D^\perp$$

is irreducible

Theorem (Shephard, Todd 54)

Every irreducible CRG is isomorphic to either

1. A member of an ***infinite series*** $G(de, e, n)$ $(d, e, n \in \mathbb{N})$ of groups of monomial matrices.
2. An ***exceptional group*** $G_4, \ldots, G_{37}$.

Steinberg

Let $X = V \setminus \bigcup \mathcal{A}$ be the complement in X of the reunion of the reflecting hyperplanes. It is path-connected. We say that a vector $v \in X$ is ***regular***. For $\mathfrak{S}_n$, the reflecting hyperplane for the transposition $(i\ j)$ is given by $x_i = x_j$. So in this case

$$X := \{(x_1, \cdots, x_n) \in \mathbb{C}^n \mid \forall i \neq j, x_i \neq x_j\}$$

Theorem (Steinberg)

The action of W on X is free.

In particular $w \in W$ is trivial if and only if there is some $x \in X$ such that $w.x = x$.
As W is discrete (finite), this has nice consequences on the topology of X and X/W.

Regular elements

From now on, $k > 1$ is an integer. We denote $\zeta_k := \exp\left(\frac{2i\pi}{k}\right) \in \mu_k^* \subset \mu_k$

Definition

An element $g \in W$ is called ζ-***regular*** if it admits a ζ-eigenvector which is a regular vector. We say k-regular instead of ζ_k-regular.

If g is ζ-regular with $\zeta \in \mu_d^*$, then g is of order d (Steinberg).
We denote $V(g,\zeta)$ the ζ-eigenspace of g. By construction, the centralizer $C_W(g)$ acts on $V(g,\zeta)$. We have

Theorem (Springer 74)

The group $C_W(g)$ acts on $V(g,\zeta)$ as a complex reflection group.
Furthermore all ζ_k elements (should they exist) are conjugate.

Existence of regular elements

To a CRG of rank n one can associate two sequences of integers

$$d_1, \cdots, d_n \quad \text{and} \quad d_n^*, \cdots, d_1^*$$

That we call the ***degrees*** and the ***codegrees*** of W.
For $k > 0$, define

$$A(k) := \{d_i \mid k \text{ divides } d_i\} \text{ and } B(k) := \{d_i^* \mid k \text{ divides } d_i^*\}$$

Also, $a(k)$ and $b(k)$ their respective cardinalities.

Theorem (Springer 74, Broué 88, Lehrer-Michel 03)

We have that ζ_k-regular elements exist if and only if $a(k) = b(k)$. If so, the degrees (resp. codegrees) of the associated centralizer are the elements of $A(k)$ (resp. $B(k)$).

The degrees (resp. codegrees) of G_{30} are $2, 12, 20, 30$ (resp. $0, 10, 18, 28$). For $k = 4$, we have $A(4) = 12, 20$ and $B(4) = 0, 28$. So 4 is regular and the centralizer of a 4-regular element in G_{30} is isomorphic to G_{22}.

Braid group and pure braid group

Let $x \in X$ be a basepoint in X. The orbit $W.x$ will be the basepoint in the orbit space X/W.

The ***pure braid group*** is the fundamental group $P(W) := \pi_1(X, x)$.
The ***braid group*** is the fundamental group $B(W) := \pi_1(X/W, W.x)$.

Since the action of W on X is free, we get a covering $X \to X/W$.
An element $b \in B(W)$ lifts to a unique (up to homotopy) path $\widetilde{b} : x \rightsquigarrow w.x$ for some $w \in W$.

The map $b \mapsto w$ induces a group morphism $B(W) \to W$, and we have a short exact sequence

$$1 \to P(W) \to B(W) \to W \to 1$$

For $b \in B(W)$, we have $b \in P(W)$ if and only if b lifts to a loop from x to itself in X.

Centers of irreducible braid groups

From now on, W is assumed to be irreducible.

Let $m := |Z(W)|$, we have $Z(W) = \{\zeta\mathrm{Id} \mid \zeta \in \mu_m\}$.

Let $z_W = \zeta\mathrm{Id}$ be a generator of the center. Consider the path

$$\widetilde{z_W} : t \mapsto e^{\frac{2i\pi t}{m}} x$$

from x to $z_W.x$ in X. It induces an element z_B in $B(W)$, sent to z_W by the projection $B(W) \twoheadrightarrow W$.
The element z_B^m is represented by the loop

$$z_P : t \mapsto e^{2i\pi t} x$$

which lies inside $P(W)$. The element z_P is called the ***full-twist***.

Theorem

The elements z_P, z_B, z_W generate the centers of P, B, W.

Invariant regular orbits and regular regular orbits

Let g be a k-regular element, $V_g := V(g, \zeta_k)$, $G := C_W(g)$.
Denote by X_g the space of regular vectors in V_g.

Proposition

The map $G.x \mapsto W.x$ induces a homeomorphism $X_g/G \simeq (X/W)^{\mu_k}$

From this we get a morphism

$$\begin{array}{ccccc} B(G) & & \overset{\iota}{\dashrightarrow} & & B(W) \\ \big\downarrow{=} & & & & \big\uparrow{=} \\ \pi_1(X_g/G, G.y) & \xrightarrow{\simeq} & \pi_1((X/W)^{\mu_k}, W.y) & \longrightarrow & \pi_1(X/W, W.x) \end{array}$$

We want to understand the kernel and image of this morphism.

Regular lifts

We have a path in X_g

$$\gamma : t \mapsto e^{\frac{2i\pi t}{k}} y$$

Which goes from y to $g.y$.

- It induces a (central) element in $B(G) := \pi_1(X_g/G, G.y)$, which is sent to some $\widetilde{g} \in B(W)$ by ι.
- The element $\widetilde{g}$ is a k-th root of z_P in $B(W)$.
- The element $\widetilde{g}$ is sent to some conjugate of g in W.
- The morphism ι has image inside $C_{B(W)}(\widetilde{g})$.

From this we deduce the definition:

We call a k-***regular braid*** a k-root of z_P in $B(W)$.

Main theorem

Theorem (Bessis 15, G. 22)

Let W be an irreducible complex reflection group.

- (A) *There exists k-regular braids in $B(W)$ if and only if k is regular for W.*
- (B) *The k-regular braids form a conjugacy class of $B(W)$. They are mapped to ζ_k-regular elements in W.*
- (C) *If ρ a k-th root of z_P, and w is its image in W. Then $C_{B(W)}(\rho) \simeq B(W')$ where $W' := C_W(w)$.*

This was shown by Bessis for 'well-generated' irreducible groups. The missing cases are done using various techniques and case by case analysis. This only covers the lifts of ζ_k-regular elements, what about others regular elements ?

Periodic elements ?

In W, if g is ζ-regular, then g^d is ζ^d-regular.

Lemma

If g is a regular element of order k, then g is a power of some k-regular element.

Suppose that g is ζ-regular. By assumption we have $\zeta, \zeta_k \in \mu_k^*$. There is integers m, n such that $\zeta^m = \zeta_k$ and $\zeta_k^n = \zeta$. We have $g' := g^m$ is ζ_k regular and $g'^n = g$.
So we consider powers of k-regular braids in $B(W)$. Such a braid respect an equation of the form

$$\rho^p = z_P^q$$

for some integers (p, q). In particular (as z_P^q is central) those elements admit a central power.

Roots of elements in the center

Definition

An element in $B(W)$ which admits a central power is called a ***periodic element***

Corollary (G. 22)

Let W be an irreducible complex reflection group. Periodic elements in $B(W)$ are exactly the powers of the regular braids.
In particular the image of γ in W is a regular element.

Corollary (G. 22)

If $z \in Z(B(W))$, then roots of z are unique in $B(W)$ up to conjugacy.

This is a bit more general than uniqueness of roots of z_P. The problem of uniqueness of roots up to conjugacy remains an open problem in general.

Thank you for your attention