Garside monoid for the Artin group associated to finite Coxeter group

Séminaire doctorant

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LAMFA

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This allows the definition of monoid presentation.

Example

$\mathbb{N}^2 = \langle a, b \mid ab = ba \rangle$ (with $a = (0, 1)$ and $b(1, 0)$)
$M = \langle a, b \mid bab = aba \rangle$

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Remark

In a categorical context, this reminds of the definition of monomorphism/epimorphism.

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In a ring monoid $(A, \times)$, we recover the "usual" notion of divisibility.
In $(\mathbb{N}, +)$, we have $n \preceq m$ if and only if $n \leqslant m$ (in the usual order).

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These relations are reflexive and transitive, but they need not be antisymetric : for $x \in M^{\times}$, $x \prec 1 \prec x$.
From now on, we always assume that our monoid are cancellative and $M^{\times} = \{1\}$, then $\preceq$ and $\succeq$ become preorders.

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Since we assume $M^\times = 1$, lcms and gcds, if they exists, are unique.

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Remark

If a presented monoid admits only relations between words of same length is homogeneous. This is the case of $M = \langle a, b \mid aba = bab \rangle$.

Definition of Garside monoids

We say that $x \in M$ is ***balanced*** if the sets of its left-divisors and right-divisors are the same.

Definition

Let M be a homogeneous cancellative monoid. Let $\Delta \in M$, we say that (M, Δ) is a ***Garside monoid*** if Δ is balanced, and $\mathcal{S} := Div(\Delta)$ is finite and generates M.
We also require $(\mathcal{S}, \succeq)$ and $(\mathcal{S}, \preceq)$ to be lattices.
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It is equivalent to ask that two simples admit lcms and gcds, and to ask that every pair of elements does.

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- $(\mathbb{N}^k, +)$ is Garside, with $\Delta = (1, \cdots, 1)$ and $\mathcal{S}$ is made of tuples containings only 0 and 1.
- If (M, Δ) is Garside, then (M, Δ^k) for $k \in \mathbb{N}^*$ is again Garside : there is a choice to make in the Garside structure !

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Remark

Since $H(mm') = H(mH(m'))$, the property 'being greedy' is local : $s_1 \cdots s_r$ is its own normal form if and only if each $s_i s_{i+1}$ is its own normal form.

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Proposition

If (M, Δ) is Garside, then every element of $\mathcal{G}(M)$ admits a unique expression of the form

$$g = \Delta^k s_1 \cdots s_r$$

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This gives a solution to the word problem in Garside groups !

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We have $s\Delta = s\bar{s}s^\phi = \Delta s^\phi$.

Corollary

The mapping $s \mapsto s^\phi$ extends to an automorphism of M, which has finite order, so Δ admits a central power.

Coxeter systems

Definition

Let S be a finite set, we define a ***Coxeter group*** by the presentation

$$W := \langle S \mid \forall s,t \in S, (st)^{m_{s,t}} = 1\rangle$$

With $m_{s,t} \in \mathbb{N} \cup \{\infty\}$, $m_{s,t} = m_{t,s} \geqslant 2$ if $t \neq s$ and $m_{s,s} = 1$.

So the set $m_{s,s} = m_{t,t} = 1$ and $m_{s,t} = 2$ gives the group

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We denote $p(s, t, n)$ the product $stst\ldots$ with n terms.

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If $m_{s,t} = 3$, we get $sts = tst$.

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For $s = t$, we get $s^2 = 1$, so $s = s^{-1}$ (***quadratic relations***).
The relation $(st)^{m_{s,t}} = 1$ is then equivalent to

$$p(s, t, m_{s,t}) = p(t, s, m_{s,t})$$

which will be called the ***braid relation*** between s and t.
If $m_{s,t} = 2$, we get $st = ts$.
If $m_{s,t} = 3$, we get $sts = tst$.
If $m_{s,t} = \infty$, then there is no braid relation between s and t.

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The Coxeter group associated to $M = (m_{s,t})$ is also presented by

$$W := \left\langle S \mid \forall s, t \in S,\ s^2 = 1, p(s, t, m_{s,t}) = p(t, s, m_{s,t}) \right\rangle$$

Classical examples : symetric and dihedral groups

Proposition

Let $n \in \mathbb{N}^*$, we denote $s_i = (i\ i+1)$ for $i \in [\![1, n-1]\!]$. The group $\mathfrak{S}_n$ is a Coxeter group for $S = \{s_i\}_{i \in [\![1,n-1]\!]}$ with the presentation

$$\left\langle s_1, \cdots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j = s_j s_i & \text{for } |i-j| \geqslant 2 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \in [\![1, n-1]\!] \\ s_i^2 = 1 & \text{for } i \in [\![1, n-1]\!] \end{array} \right\rangle$$

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For instance, $\mathfrak{S}_4$ is presented by

$$\langle s, t, u \mid sts = tst,\ tut = utu,\ su = us,\ s^2 = t^2 = u^2 = 1 \rangle$$

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The group D_n of order $2n$ is a Coxeter group for $S = \{s_i\}_{i \in [\![1, n-1]\!]}$ with the presentation

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it is in particular isomorphic to $\mathfrak{S}_3$.
Exercice : show that a subgroup of a Coxeter group generated by two elements of S is always a dihedral group.

Length function

Definition

An element $w \in W$ can always be expressed as a word $\underline{w} = s_1 \cdots s_k$ with $s_i \in S$. We call $\ell_S(w)$ the minimal k for which this is possible, we say that $\ell_S(w)$ is the ***length*** of w, and any expression for w of minimal length is called ***reduced***

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Proposition

1. $\ell_S(w) = 1$ if and only if $w \in S$.
2. $\ell_S(ww') \leqslant \ell_S(w) + \ell_S(w')$ for $w, w' \in W$.
3. $\ell_S(ws) = \ell_S(w) \pm 1$ for $s \in S$.

Main theorems on reduced expressions

Exchange condition

Let $s_1 \cdots s_k = w$ be a reduced expression, and $s \in S$ be such that $\ell_S(sw) < \ell_S(w)$, then there is some $i \in [\![1, k]\!]$ such that $sw = s_1 \cdots \widehat{s_i} \cdots s_k$.

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Theorem

Let $w \in W$ and $s \in S$, we have $\ell_S(sw) < \ell_S(w)$ (resp. $\ell_S(ws) < \ell_S(w)$) if and only if there is some reduced expression of w starting (resp. ending) by s.

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Matsumoto's lemma

Two reduced expression of the same element in W are related by only braid relations (no quadratic relations needed).

Longest element

From now on, W is assumed to be **finite**.
There is an element of maximal length in W, denoted by w_0. This element is unique and defined by $\ell_S(sw_0) < \ell_S(w_0)$ for all $s \in S$.

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Now for $w \in W$, since $\ell_S(w)$ is not maximal, there is some $s \in S$ such that $\ell_S(ws) > \ell_S(w)$, so $\underline{w}.s$ is a subword of some reduced expression of w_0.

Artin groups

For all Coxeter system (W, S), one can consider the ***Artin group*** associated to (W, S), defined by the presentation

$$A(W) := \langle \mathbf{S} \mid p(\mathbf{s}, \mathbf{t}, m_{s,t}) = p(\mathbf{t}, \mathbf{s}, m_{s,t}) \rangle$$

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$A(\mathfrak{S}_3) = \langle \mathbf{s}, \mathbf{t} \mid \mathbf{sts} = \mathbf{tst} \rangle$ $A(D_4) = \langle \mathbf{s}, \mathbf{t} \mid \mathbf{stst} = \mathbf{tsts} \rangle$.

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By definition, we have a morphism $\pi : M(W) \to W$ sending $\mathbf{s}$ to s.

We will see the elements of $M(W)$ as words 'up to braid relations'. This helps to explain the relations $\preceq$ and $\succeq$ in $M(W)$:

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Then the Artin monoid $M(W)$ is a Garside monoid, where Δ is a reduced word for w_0, and the simples are in bijection with W.

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We ommit the proof that $M(W)$ is cancellative.

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Corollary

For $x, y, z \in W$, we have

$$\mathbf{xy} = \mathbf{z} \Leftrightarrow xy = z \text{ and } \ell_S(x) + \ell_S(y) = \ell_S(z)$$

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The element $\Delta = \mathbf{w_0}$ is balanced, and its divisors are the $\mathbf{w}$ for $w \in W$.

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The converse has already been proven.
Thanks to the last corollary, we have a complete description of $(\mathcal{S}, \preceq)$ and $(\mathcal{S}, \succeq)$.

Existence of gcds

Let w_1, w_2 in W, $\mathbf{w}$ the longest word dividing both $\mathbf{w_1}$ and $\mathbf{w_2}$, and $\mathbf{z_1}, \mathbf{z_2}$ such that $\mathbf{wz_i} = \mathbf{w_i}$.

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So there is an expression of $\mathbf{z_i}$ starting by $\mathbf{s}$, but $\mathbf{ws}$ is a common left-subword of $\mathbf{w_1}, \mathbf{w_2}$, longer than $\mathbf{w}$.

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The same reasoning on the right concludes the proof of :

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One can even show that $Z(A(\mathfrak{S}_n))$ is in fact generated by Δ^2.

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- If n is odd, we have

$$s\Delta = sp(t,s,n) = p(s,t,n+1) = p(s,t,n)t = \Delta t$$

so $s^{\phi} = t$ and $t^{\phi} = s$: Δ^2 is central

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Or is there ?

Yes there is, in a weaker sense : there is an infinite Garside family for the monoid $M(W)$.
But this has a cost : we don't know if $M(W)$ embeds in $A(W)$.
So the knowledge of $M(W)$ is not that useful to understand $A(W)$.

Thank you for your attention