19/02/26

Workshop: Geometric Topology

Seifert link groups & their torsion quotients.

j w/ I. Haladjian.

I. Link, link groups

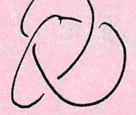
II. Torsion quotients.

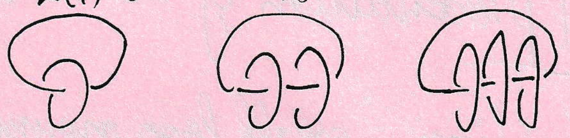
III. Some examples.

I. Link, link groups.

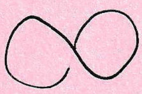
1) Links

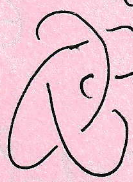
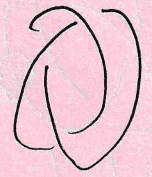
Def: A knot is the image of a map $S^1 \to \mathbb{R}^3$ which is a diffeo onto its image.
A link with n components is the same thing but replacing S^1 with the disjoint union of n copies of S^1.

Ex:
- The trivial knot O
- The trefoil 
- The trivial link with 2 components O O
- The figure 8 knot 
- The keychains $k(1)$ $k(2)$ $k(3)$

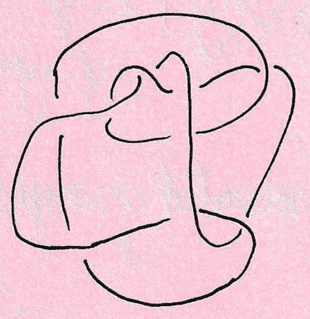
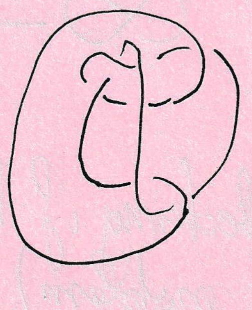
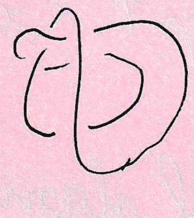
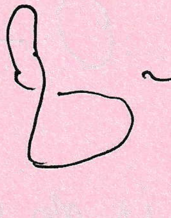


We consider links up to deformation (isotopy)

 ~ O

 ~ 

Given the diagrams of two links, hard to know if the links are isotopic.

 ~  ~  ~  ~ O

General method: Attach to L an invariant $f(L)$ so that
$$L \sim L' \Rightarrow f(L) = f(L')$$
↳ should be easy to check

Rq: $f(L)=$ "does the diagram look like the unknot" is not an invariant.

2) Link group.

Fix L a link.

The complement $\mathbb{R}^3 \setminus L$ is a topological space (a 3 manifold, in fact)

Def: The link group $G(L)$ is the fundamental group $\pi_1(\mathbb{R}^3 \setminus L)$

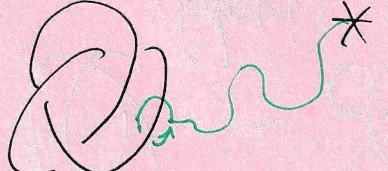
⚠ This is different from the def of Milnor, but we consider isotopy and not homotopy of links.

We have $L \sim L' \Rightarrow \mathbb{R}^3 \setminus L \cong \mathbb{R}^3 \setminus L' \Rightarrow G(L) \cong G(L')$ thus $G(L)$ is an invariant.

Prop: Artin 25, Alexander 23, Birman 72.

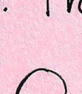
There is an algorithm $\boxed{\text{Algo}}$ which, given a link L, outputs a presentation of $G(L)$. Whose generators are meridians

Meridian: = small loop around a component of L.

 $\in G(L) = \pi_1(\mathbb{R}^3 \setminus L, *)$

Meridians form a distinguished set of generators of $G(L)$.

Problem ①: The presentation depends on the diagram.

 $\longrightarrow$ Algo 1: $\underset{\mathbb{Z}}{\underset{\wr\wr}{\langle x \mid \emptyset \rangle}}$  $\longrightarrow$ Algo 1: $\left\langle x, y \,\middle|\, \begin{matrix} x = y \\ y^{-1}xy = y \end{matrix} \right\rangle$

There is no general algorithm checking if two presented groups are isomorphic (undecidable problem).

Problem 2: We could have $G(L) \simeq G(L')$ and $L \not\sim L'$ (example later).

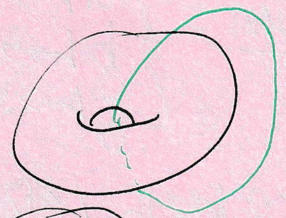
3) Seifert links

A particularly good (to me) class of links.

2 types of Seifert links.

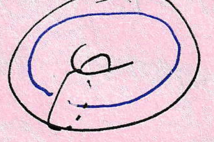
- * Keychains $K(k)$ $k \geq 0$.
- * Torus necklaces $L(n,m)$, $L^*(n,m)$ $L_*(n,m)$, $L_*^*(n,m)$.

$L(n,m)$: Start from n lines on a cylinder. Make them do $\underline{m}$ nth of a turn, and close the cylinder into a torus. You obtain a $n \wedge m$ components link $L(n,m)$.



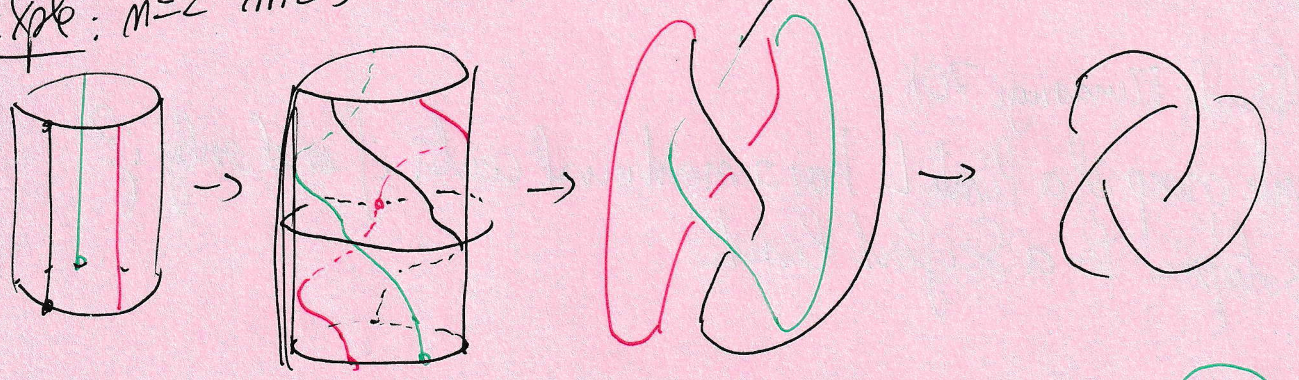
$L^*(n,m)$ add a circle around the outside of the torus.

$L_*(n,m)$ add a circle around the inside of the torus

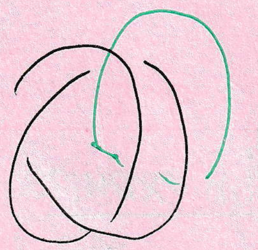


$L_*^*(n,m)$ add both.

Expl: $n=2$ $m=3$



$L(2,3)$



$L^*(2,3)$

List a few isotopies: (Budney 06)

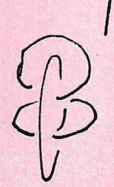
$L_*(n,m) \sim L^*(m,n)$ $\quad L(n,m) \sim L(m,n)$

$L(1,n) \sim L(1,m)$ $\quad L_*(n,m) \sim L(n+\frac{n}{m}, m+1)$ if $m|n$.

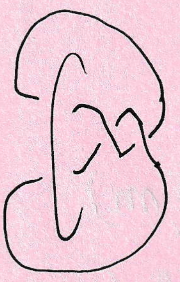
$L(2,2) \sim K(1) \sim L_*(n,1)$

$L(1,1) \sim K0$

Example: By Algo 1, we have. $\langle a,b\rangle$ $\langle z\rangle$.



$G(K(2)) = \langle a,b,z \mid az=za,\ bz=zb\rangle = F_2\times\mathbb{Z}$
↳ free group

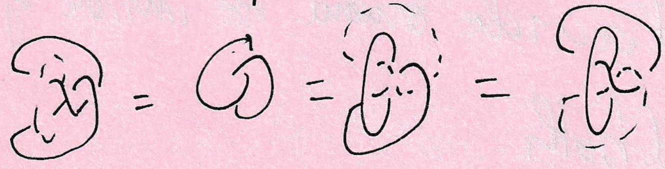


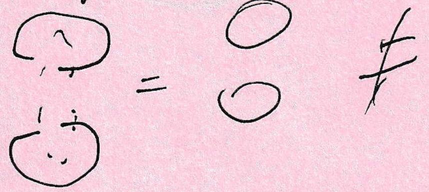
$G(L^*(2,2)) = \langle s,t,u \mid stu = tus = ust\rangle$.
$\quad\ \ \| $
$G^*(2,2)$.

Lemma: $K(2)\not\sim L^*(2,2)$ but $G^*(2,2)\simeq G(K(2))$.

dem: $\begin{cases} a\mapsto s \\ b\mapsto t \\ z\mapsto stu\end{cases}$ $\begin{cases} s\mapsto a \\ t\mapsto b \\ u\mapsto (ab)^{-1}z\end{cases}$ give isomorphisms.

If $K(2)\sim L^*(2,2)$ then $K(2)$ minus a component is isot to $L^*(2,2)$ minus some component. However





Theorem (Burde Murasugi 70)
The link group of a link L has a nontrivial center if and only if L is isotopic to a Seifert link.

II. Torsion quotients.

1) An example.

We consider $G^*(2,2) = \langle s\, t\, u \mid stu = tus = ust\rangle$. And consider the map ρ
$G^*(2,2)\to GL_2(\mathbb{C})$ $\quad s\mapsto\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ $\quad t\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad u\mapsto\begin{pmatrix}0&i\\-i&0\end{pmatrix}$.
We have $\rho(stu)=\rho(tus)=\rho(ust)=\begin{pmatrix}i&0\\0&i\end{pmatrix}$. So ρ defines a morphism $G^*(2,2)\to W\subseteq GL_2(\mathbb{C})$.

The three element $\sigma = \rho(s)$, $\tau = \rho(t)$ $\mu = \rho(u)$ are reflections of order 2.
In fact, we have

$$W = \langle \sigma, \tau, \mu \mid \sigma\tau\mu = \tau\mu\sigma = \mu\sigma\tau \,,\ \underbrace{\sigma^2 = \tau^2 = \mu^2 = 1}_{\text{torsion relations}} \rangle.$$

and $\operatorname{Ker} \rho = \langle\langle s^2, t^2, u^2 \rangle\rangle$ normal closure in $G^*(2,2)$.

2) Complex reflection groups.

Def: $r \in GL_n(\mathbb{C})$ is a reflection if r has finite order and if $\operatorname{Ker}(r-1)$ is a hyperplane (equiv, the minimal polynom of r is $(\zeta - X)(1-X)$ with ζ root of unity)
a complex reflection group is a finite group $W \subseteq GL_n(\mathbb{C})$ wich is generated by reflections (CRG)

- Real reflection groups are Coxeter groups (Coxeter 1934). Can be studied with the strong combinatorics of Coxeter groups.
- Broué Malle Rouquier 98 - find "nice" presentations for (GRCs), trying to imitate Coxeter $\rightarrow$ no real combinatorial tool out of this.
- Achar Aubert 08 Introduce so called J-groups. CRG of rank 2 are exactly finite J-groups. Is there a good combinatorics of J groups?
- Gobet 21 Some J-groups are related to torus knot groups (as tors. quot).
- Haladjian 25 true for many J-groups, replacing torus knot with torus necklaces
- G, Haladjian 25 True for all generalized J-groups, and it goes both ways!

3). Torsion quotients

We fix G a group and S a finite generating set of G. Write $S = \{s_1, \dots, s_d\}$.
We fix $k \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^d$.

Def: The torsion quotient G_K is the quotient $G/\langle\langle s_i^{k_i} \mid i \in [\![1,d]\!] \rangle\rangle$.

If $G = \langle S \mid R \rangle$ is a presentation of G, then

$$G_K = \langle S \mid R + s_i^{k_i} = 1 \ \forall i \in [\![1,d]\!] \rangle.$$

- If $k_i = 1$, then we just delete s_i from the presentation: torsion quotient of some other group G'
- If $g s_i g^{-1} = s_j$, we should have $k_i = k_j$ (otherwise we can reduce to this case).

⚠ G_K is not finite in general (example below)

Def: We call (abstract) reflections the conjugates of (nontrivial) powers of the s_i in G_K.

We say that $G_K \simeq_{Ref} H_L$ if there is a group isomorphism $\varphi : G_L \to H_K$ such that $\varphi(R(G_K)) = R(H_L)$ (isomorphism preserving the reflections)

(We abusively call reflection the powers of meridians in link groups to homogeneize statements).

Theo (Haladjian 25)

Every complex reflection group of rank 2 is refl. isom. to a torsion quotient of a Seifert link

$$G_{15} = (G_*(1,2))_{(2,2)} = \langle s,t,u \mid stu = tus,\ ustut = stutu,\ s^2 = t^2 = u^2 = 1 \rangle$$

$$G_{13} = (G_*(2,3))_{\cancel{22}} = \langle s,t,u \mid susts = tusts,\ tust = ustu,\ ustus = stust,\ s^2 = t^2 = u^2 = 1 \rangle$$

$$G_4 = (G(2,3))_3 = \langle s\,t \mid sts = tst,\ s^3 = t^3 = 1 \rangle.$$

Theo (G, Haladjian 25)

(1) Torsion quotients of Torus necklaces coincide with generalized J-groups.

(2). The family of finite torsion quotients of Torus necklaces coincide with the family of CRG of rank ≤ 2 up to refl. isom.

(3) Classification of torsion quotients of Seifert link up to refl. isom.

Example infinite torsion quotient

$G = \langle s,t,u \mid stu = tus = ust \rangle$.

$G_3 = \langle s,t,u \mid stu = tsu = ust,\ s^3 = t^3 = u^3 \rangle$ is infinite.

$[G_3 : D(G_3)] = 27 \rightarrow H = D(G_3) = \langle a,b,c \mid bc = cb,\ cab = bac,\ c^2ac = b^3c \rangle$.

H^{ab} is infinite, thus $[H : DH] = \infty \Rightarrow [G_3 : D^2(G_3)] = \infty$.

Theo (G, Haladjian 25)

Let L, L' be two Seifert links, G, G' the link groups, W, W' two torsion quotients. We have

$$W \simeq_{Rg} W' \Rightarrow G \simeq_{Rg} G' \Rightarrow L \sim L'$$

Thus $G(L)$, endowed with meridians, is a perfect invariant for Seifert links.

III. Two examples

Back to the figure 8 knot  $=: L$. We have

$$G = G(L) = \langle x, y \mid xyx^{-1}yx = yxyx^{-1}y \rangle.$$

Prop: L is not a Seifert link.

dem: Compute G_2 = Dihedral (pentagon).

Since L has only one component (L is a knot) it must be isotopic to a torus knot $T(n,m)$. ($n \wedge m = 1$). The only torus knot K' such that $G(L')_2$ is finite and has order 60 is $T(2,5)$.

(+ we have $G_2^{ab} = \mathbb{Z}/2\mathbb{Z}$ and $G(L')_k^{ab} = \mathbb{Z}/k\mathbb{Z}$ so $k=2$)

However, $G(T(2,5))_3$ is finite (order 360). While

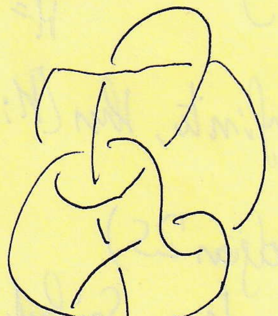
G_3 is infinite ($[G_3 : D^2(G_3)]$ is infinite).

So this is a contradiction.

In particular we have $Z(G) = \{1\}$!

<u>This does not always work so easy</u>

Let L be the Conway knot



$G(L)$ is ugly. If we try to show that $G_2 = G(L)_2$ is infinite, we have

$[G_2 : D(G_2)] = 2$, and $D(G_2)$ is perfect: $[G_2 : D^i(G_2)] = 2 \ \forall i$

$D(G_2)$ "looks infinite" but this is not a proof.

Rq: If L' is the unknot, then $G' = \mathbb{Z}$, $G'_n = \mathbb{Z}/n\mathbb{Z}$, and we have $[G'_2 : D(G'_2)] = 2$, while $D(G'_2)$ $(=1)$ is perfect.

again, the Conway knot "looks like" the unknot.