Projective complex braid groups

Algebra and topology seminar, ANU

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The set of **regular vectors** is $X := \{v \in V \mid \mathrm{Stab}_W(v) = 1\}$.

Examples

- For $d \geqslant 1$, $\mu_d \leqslant \mathrm{GL}_1(\mathbb{C})$ is a CRG with $X = \mathbb{C}^*$.

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is an irreducible CRG of order 24 with

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- For $m \geqslant 1$ an integer, $G(m, 1, n) \leqslant \mathrm{GL}_n(\mathbb{C})$ is the group monomial matrices with nonzero entries in μ_m. It is a CRG with
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$$X = \{x \in (\mathbb{C}^*)^n \mid x_i \notin \mu_m x_j\}$$
- For $p|m$, $G(m,p,n) \leqslant G(m,1,n)$ is the subgroup of matrices whose product of nonzero entries lies in $\mu_{m/p}$. It is a CRG.

Regular elements (Springer '74)

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If $g \in W$ is d-regular, then $W_g := C_W(g)$ acts on $V(g,\zeta_d)$ as a CRG.

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Examples include

- $B_n \leqslant A_{2n-1}$ as centralizer of a 2-regular element.
- $F_4 \leqslant E_6$ as centralizer of a 2-regular element.
- $G(d,1,n) \leqslant G(1,1,dn)$ as centralizer of a d-regular element.
- $G_{31} \leqslant G_{37} (\simeq E_8)$ as centralizer of a 4-regular element.

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If W is irreducible, then $\widehat{W} = W/Z(W)$.

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Example

We have $Z(G_4) = \{\pm I_2\}$ and $\widehat{G_4}$ has order 12. In fact $\widehat{G_4} \simeq \mathfrak{A}_4$.

Lifts of projective reflection groups and classification

Lemma

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Theorem (Shephard, Todd '54)

If W is an irreducible CRG, then either $W \simeq G(m, p, n)$, or W belongs to the family $G_4, \ldots, G_{37}$ of exceptional groups.

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Half of the exceptional groups have rank 2.

Case of rank 2 groups

Let $W \leqslant \mathrm{GL}_2(\mathbb{C})$ be a CRG.

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Let $s \in \mathrm{PGL}(V)$ be a projective reflection.

- If $\dim V > 2$, there is a unique reflection in $\mathrm{GL}(V)$ which lifts s.
- If $\dim V = 2$, there are two.

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How to distinguish various lifts of $\widehat{W}$?

Maximal lift

Proposition

Let $W \leqslant \mathrm{GL}(V)$ be a CRG. There is a unique maximal $W_f \leqslant \mathrm{GL}(V)$ such that $\widehat{W_f} = \widehat{W}$. We call W_f the *full reflection group* attached to W.

In other words, the family of lifts of a PCRG G admits a maximum $\widetilde{G}$.

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Example

- $\widetilde{\mathfrak{A}_4} = G_7, \widetilde{\mathfrak{S}_4} = G_{11}, \widetilde{\mathfrak{A}_5} = G_{19}$.
- For $i > 22$, then $(G_i)_f = G_i$, except $(G_{25})_f = G_{26}$.
- For $n > 2$, $(G(m,p,n))_f = G(m, n \wedge p, n)$.
- For $n = 2$, $(G(m,p,2))_f = G(\frac{2m}{p \wedge 2}, 2, 2)$.

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Theorem (Steinberg '64)

For any $E \subset V$, $W_E = \{w \in W \mid \forall v \in E, w.v = v\}$ is generated by the reflections it contains.

In particular, $X = V \setminus \bigcup_{H \in \mathcal{A}} H$.

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Definition (Broué, Malle, Rouquier '98)

The **braid group** is $B = B(W) = \pi_1(X/W)$. The **pure braid group** is $P = P(W) = \pi_1(X)$.

We have a short exact sequence $P \hookrightarrow B \twoheadrightarrow W$.

Center of complex braid groups

In X, with basepoint x, consider the paths

$$\beta : t \mapsto e^{\frac{2i\pi t}{|Z(W)|}} x \text{ and } \pi : t \mapsto e^{2i\pi t} x$$

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The path β induces a loop in X/W and an element $\beta \in Z(B)$.

Theorem (BMR '98, Bessis '15, Digne-Marin-Michel '11)

If W irreducible, then $Z(P) = \langle \pi \rangle$ and $Z(B) = \langle \beta \rangle$ and we have

$$Z(P) \hookrightarrow Z(B) \twoheadrightarrow Z(W).$$

First try at projective complex braid group

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We have a commutative diagram of topological space

$$\begin{tikzcd} X \arrow[r, "/W"] \arrow[d, "/\mathbb{C}^*"'] & X/W \arrow[d, "/\mathbb{C}^*"] \\ \widehat{X} \arrow[r, "/W"'] & \widehat{X}/W \end{tikzcd}$$

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Theorem (BMR '98)

There is a morphism $\pi_1(\widehat{X}/\widehat{W}) \to \widehat{W}$ which completes the diagram in a commutative diagram where all short sequence are exact.

A problem

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Example

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- $X = \{(x, y) \in \mathbb{C}^2 \mid y \notin \{x/2, -x, 2x\}\}$,
- $\widehat{X}$ is $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^2$ minus 3 points,
- $\widehat{X}/W$ is $\mathbb{S}^2$ minus one point: it is simply connected.

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Proposition (G. '25)

BMR's result holds only for $G_7, G_{11}, G_{15}, G_{19}$ and $G(m,p,n)$ where $p|n$.

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In other words, we need to remove not only the reflecting hyperplanes, but also the (proper) regular eigenspaces.

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Definition

$X_S = X_S(W) := \{x \in V \mid \mathrm{Stab}_{\widehat{W}}([x]) = 1\}$.
$\widehat{B} := \widehat{B}(\widehat{W}) = \pi_1(\widehat{X_S}/\widehat{W})$ the **projective braid group**.
$\widehat{P} := \widehat{P}(\widehat{W}) = \pi_1(\widehat{X_S})$ the **projective pure braid group**.

Again, we have a short exact sequence $\widehat{P} \hookrightarrow \widehat{B} \twoheadrightarrow \widehat{W}$.

Enlarged braid group

Theorem (Shvartsman '96)

If $W = W_f$ is a Coxeter group, then $\widehat{B} = B/Z(B)$.

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The inclusion $X_S \to X$ induces morphisms $B_S \to B$ and $P_S \to P$.

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- The morphism $B_S \to B$ is an isomorphism if $W = W_f$.
- In general $B_S = \pi^{-1}(W)$, where $\pi : B(W_f) \to W_f$ is the natural projection.

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all arrows are fibrations.

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\langle \pi_S \rangle \arrow[r, hook] \arrow[d, hook] & \langle \beta_S \rangle \arrow[r, two heads] \arrow[d, hook] & Z(W) \arrow[d, hook] \\
P_S \arrow[r, hook] \arrow[d, two heads] & B_S \arrow[r, two heads] \arrow[d, two heads] & W \arrow[d, two heads] \\
\widehat{P} \arrow[r, hook] & \widehat{B} \arrow[r, two heads] & \widehat{W}
\end{tikzcd}
$$

In particular, if $W = W_f$, then $\widehat{B} = B/Z(B)$.

Diagram

The last diagram induces the following one where all sequences are exact.

$$
\begin{tikzcd}
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In particular, if $W = W_f$, then $\widehat{B} = B/Z(B)$.

Theorem (G. '25)

Let $G \subset \mathrm{PGL}(V)$ be a projective complex reflection group.
The projective braid group of G is $\widehat{B} = B(\widetilde{G})/Z(B(\widetilde{G}))$.

Shvartsman results on elements with a central power

Let $R = \{d \geqslant 1 \mid d \text{ is regular for } W \text{ and maximal for divisibility}\}$
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Note that this does not apply to *every* finite Coxeter group.

Regular braids

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The complete answer came more than a decade later:

Theorem (Bessis '15, G. '23)

Let W be a complex reflection group

- *d is regular for W if and only if d-regular braids exist in B.*
- *If so, then all d-regular braids are conjugate in B. The image in W of a d-regular braid is a d-regular element.*
- *If $\rho \in B$ is d-regular, then $C_B(\rho) \simeq B(W_g)$, where g is the image of ρ in W.*

Generalization of Shvartsman's result

Proposition (G. '23)

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$b \in B$ has a central power if and only if $b^d \in Z(B)$ for some $d \in R'$.

This works for arbitrary complex reflection groups, without the assumption that $W = W_f$. However, the proof is case by case.

Consequences on computations ?

Finally, let us try again to compute $\pi_1(\widehat{X}/W)$. Let $K \leqslant B$ denote the subgroup generated by elements having a central power.

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The natural morphism $p : B \to \pi_1(\widehat{X}/W)$ is surjective and $K \subset \mathrm{Ker}(p)$.

In other words, $\pi_1(\widehat{X}/W)$ is a quotient of B/K.

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The quotient B/K is almost always a cyclic groups (often trivial).

Is there a way to compute the fundamental group $\pi_1(\widehat{X}/W)$ in general ?

Thank you !

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