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Seminario de Algebra de la US

Normalizers of Parabolic subgroups in Artin & Coxeter groups.

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I. Coxeter groups, Artin groups.

II. Our tool: groupoids.

I. Coxeter groups, Artin groups.

1) Definitions and general results.

Fix S a finite set. A Coxeter matrix is a map $S \times S \to \mathbb{N}^* \cup \{\infty\}$. with

- $m(s,t) = m(t,s)$ - $m(s,s) = 1$ - $m(s,t) \geq 2$ for $s \neq t$.

From this data we define two groups.

$$W = \langle S \mid \forall s,t \quad \underbrace{sts\ldots}_{m(s,t)} = \underbrace{ts\ldots}_{m(t,s)},\ s^2 = 1 \rangle$$ Coxeter group

$$A = \langle S \mid \underline{\hspace{6em}} \rangle$$ Artin group.

The kernel of $A \twoheadrightarrow W$ is the pure Artin group P.

Algebraic SEC $\quad 1 \to P \to A \to W \to 1.$

Exple: ∘ $m(s,t) \equiv \infty$ $\quad$ ~~W~~ W = free prod of $\mathbb{Z}/2\mathbb{Z}$ $\quad$ A = Free group.

∘ $m(s,t) \equiv 2$ $\quad$ W = direct ______ $\quad$ A = Free abelian group.

∘ (Affine) Weyl groups $\sim$ (euclidean)/Spherical Artin groups.

∘ $W = \mathfrak{S}_n$ $\quad$ A = braid grp on n-strands.

Exple of today. D_4
$$\begin{tikzcd} s \arrow[dr, dash, "3"] \arrow[d, dash, dotted, "2"'] \arrow[drr, dash, dotted, "2"] & & \\ t \arrow[r, dash, "3"'] \arrow[rr, dash, dotted, bend right, "2"'] & u \arrow[r, dash, "3"'] & v \end{tikzcd}$$

Theo (Coxeter 34) If $n := |S|$, then there is a faithful rep $\rho : W \to GL_n(\mathbb{R})$ mapping S to reflections.
Cvsl, any finite reflection group over $\mathbb{R}$ has a Coxeter presentation.

Assume W finite from now on.

Fix V an n dimensional $\mathbb{R}$-vector space, and identify W with its image in $GL(V)$.

Prop: $\forall v \in V$, $\mathrm{Stab}_W(v) = 1 \iff v$ is on a reflecting hyperplane
$X := V \setminus \bigcup \mathrm{refl.hp} = \{ v \in V \mid \mathrm{Stab}_W(v) = 1 \}$.

$\Rightarrow W$ acts freely on $X_{\mathbb{C}} = V \otimes \mathbb{C} \setminus \bigcup (\mathrm{refl\,hp} \otimes \mathbb{C})$ and we have a covering $X_{\mathbb{C}} \twoheadrightarrow X_{\mathbb{C}}/W$.

Theo (Brieskorn 71) $\pi_1(X_{\mathbb{C}}) = P$ and $\pi_1(X_{\mathbb{C}}/W) = A$.

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\mathbb{C}}) \arrow[r] \arrow[d, equal] & \pi_1(X_{\mathbb{C}}/W) \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, equal] & 1 & \text{topological} \\ 1 \arrow[r] & P \arrow[r] & A \arrow[r] & W \arrow[r] & 1 & \text{algebraic.} \end{tikzcd}$$

Theo (Deligne 72, Brieskorn Saito 72) $X_{\mathbb{C}}$ and $X_{\mathbb{C}}/W$ are $K(\pi,1)$ spaces.

Exple: D_4

$$s = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad t \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad u \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \quad v = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \in GL_4(\mathbb{R}).$$

$$X_{\mathbb{C}} = \{ (x_i) \in \mathbb{C}^4 \mid \forall i \neq j,\ x_i \notin \pm x_j \}.$$

3) Parabolic subgroups.

Def: $I \subseteq S \rightsquigarrow W_I = \langle I \rangle \subseteq W$, $A_I = \langle I \rangle \subseteq A$
standard parabolic subgroups.
Parabolic subgroup = conjugates of standard parabolic subgroups.

W_I is a Coxeter group for $m_{|I}$ (Tits 61)
A_I — Artin ———— (Van der Lek 83)

Theo:
$$N_W(W_I) = W_I \rtimes N_W(W_I)/W_I = W_I \rtimes N(W,I)$$
$$N_A(A_I) = A_I \rtimes N_A(A_I)/A_I = A_I \rtimes N(A,I)$$
Brink Howlett 99

Paris 97

← Normalizer quotient.

→ Algebraic SEC $1 \to N(P,I) \to N(A,I) \to N(W,I) \to 1$.

→ give it a geometric meaning?

Def: $\Theta_I = \{v \in V \mid W_I \subseteq \mathrm{Stab}_W(v)\} = \bigcap_{s \in I} H_s$ Tits cone intersection.
$\mathcal{H}_I = \{$refl hyp not containing $\Theta_I\}$. $X_I = \Theta_I \setminus \bigcup \mathcal{H}_I$.

lemme: $N_W(W_I) \curvearrowright \Theta_I \rightsquigarrow N(W,I) \curvearrowright \Theta_I$, this action on X_I and $(X_I)_{\mathbb{C}}$ is free.
with kernel W_I

Topological SEC $1 \to \pi_1((X_I)_{\mathbb{C}}) \to \pi_1\left((X_I)_{\mathbb{C}}/N(W,I)\right) \to N(W,I) \to 1$.

Theo (G. Heng, Licata, Yacobi 25)
The two sequences are isomorphic in a natural way.
$(X_I)_{\mathbb{C}}$ and $(X_I)_{\mathbb{C}}/N(W,I)$ are $K(\pi,1)$ spaces.

Expl: $I = \{a\}$ for D_4. $W_I = \{1, a\} \simeq \mathbb{Z}/2\mathbb{Z}$. $\Theta_I = \{x_2 = x_3\}$ is a hyperplane.

take $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$ as a basis for Θ_I.

$\mathcal{H}_I = \{a \pm c = 0,\ a \pm b = 0,\ b \pm c = 0,\ c = 0\}$ 7 hyperplanes.

$A_3 : 6$, $B_3 : 9$, $H_3 : 15$.

$N_W(W_I) = \pm\left\{ \begin{pmatrix} 1 & & \\ & \pm X & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & \pm X & 1 \\ 1 & & \end{pmatrix} \right\}$ with $X \in \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \right\}$

$N(W, I) = \pm\left\{ \begin{pmatrix} 1 & & \\ & \pm 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \cancel{1} & & 1 \\ & \pm 1 & \\ & & 1 \end{pmatrix} \right\}$. <u>not a reflection group</u>

II. Our tool: groupoids.

1) Groupoid for X_I: the Deligne groupoid.

Even if $(\Theta_I, \mathcal{H}_I)$ is not a reflection arrangement, it is still a simplicial arrangement (chambers are simplicial cones).

$\mathcal{H}_I$ cuts Θ_I in different connected components, the chambers.

Def: Gallery = sequence of adjacent chambers. $\mathcal{D}^+$ is a category

- $Ob(\mathcal{D}^+)$ = chambers
- $Hom(C, C')$ = galleries + identification of "geodesic" galleries.

Theo (Deligne 72) let $\mathcal{D}$ be the groupoid obtained from $\mathcal{D}^+$ by inverting all the arrows. $\mathcal{D}$ is equivalent to $\pi_1((X_I)_\mathbb{C})$ and $(X_I)_\mathbb{C}$ is $K(\pi, 1)$

galleries $\simeq$ paths in $(X_I)_\mathbb{C}$ between chambers

2) Groupoid for $N(A,I)$: the ribbon groupoid.

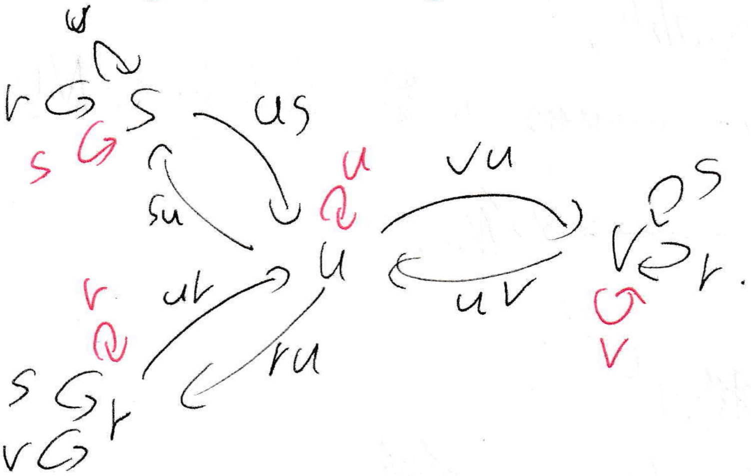
W finite $\Rightarrow$ $A^+ = \langle S\rangle^+ \subseteq A$ is a Garside monoid.

- $\exists$ lcms, gcds, - Word and conjugacy problem solution. ...

Def: positive ribbon $A_I \to A_J$ = $\alpha \in A^+$ s.t. $\alpha^{-1} A_I^+ \alpha = A_J^+$.

$\rightsquigarrow$ ribbon category; ribbon groupoid. Ribb.

Ex:



$\mathrm{Ribb}(A_I, A_I)$ is slightly bigger than $N(A,I)$.

Def: $\alpha : A_I \to A_J$ is reduced if it is "coprime" with A_I.

$\rightsquigarrow$ $r\mathrm{Ribb}^+$ category of reduced ribbons.

Theo (Paris 97, Godelle 03, 10)

$r\mathrm{Ribb}(A_I, A_I) = N(A,I)$ (and $N_A(A_I) = A_I \rtimes N(A,I)$)

3) A relation between the 2: Groupoid coverings

The action of $N(W,I)$ on X_I preserves chambers and adjacency.

$\rightsquigarrow$ it induces an action on $\mathcal{D}^{(+)}$ by automorphisms.

$\Rightarrow$ quotient groupoid $\mathcal{D}/N(W,I)$, by construction equivalent to $\pi_1((X_I)_{\mathbb{C}})/W$.

groupoid (Galois) covering $\mathcal{D} \longrightarrow \mathcal{D}/N(W,I)$.

$\underline{\mathrm{Ex}}$ple: In the picture.

$$N(W,I) = \left\langle \underset{\text{symmetry}}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}, \underset{\text{symmetry}}{\begin{pmatrix} -1 & & \\ & & \\ & & 1 \end{pmatrix}}, \underset{\text{inversion}}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}} \right\rangle.$$

On the other hand, using the combinatorics of walls & chambers in the Tits cone intersection, we construct a functor

$$\mathcal{D} \longrightarrow \pi\mathrm{Ribb}(A_I)$$

which is also a Galois covering with Deck tf $\simeq N(W,I)$.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{D}/N(W,I) \arrow[dl, dashed, "\simeq"] \\ \pi\mathrm{Ribb}(A_I) & \end{tikzcd}$$

Which is exactly what our theorem states.

<u>Bonus</u>: There is an equivalent of ribbons for Coxeter groups: the Brink-Howlett groupoid $\mathcal{BH}$. We construct its universal covering, for all Coxeter groups.

$\mathcal{H} \subseteq \mathbb{R}^3$ a set of hyperplanes (all through 0).

$\mathcal{H} \cap \mathbb{S}^2$ a set of great circles on $\mathbb{S}^2$.

$\mathbb{S}^2 \setminus \begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow \mathbb{R}^2$ stereographic projection

$\mathcal{H} \cap \mathbb{S}^2 \longrightarrow$ a set of lines and circles in $\mathbb{R}^2$.