16/12/25
Séminaire de Topologie
de l'IMJ PRG

Normalisers of Parabolic subgroups in Artin & Coxeter groups.

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I. Coxeter groups, Artin groups.
II. Useful groupoids today.
III. Tits cone intersections and convexness.

I. Coxeter groups, Artin groups.

1) Definitions.

Fix S finite. Coxeter matrix = map $m : S\times S \to \mathbb{N}^* \cup \{\infty\}$ such that

$m(s,t) = m(t,s)$ $\quad m(s,s) = 1 \quad m(s,t) \geq 2$ if $s \neq t$.

From this data, define two groups

$$W = \langle S \mid \forall s,t \ \underbrace{sts\ldots}_{m(s,t)} = \underbrace{ts\ldots}_{m(s,t)},\ s^2 = 1\ (\forall s)\rangle \quad \text{Coxeter group}$$

$$A = \langle S \mid \ \text{—} \ \text{———} \ \text{—} \rangle \quad \text{Artin group.}$$

$+ \ P = \mathrm{Ker}(A \twoheadrightarrow W)$. pure Artin group.

We have an algebraic/combinatorial SES $1 \to P \hookrightarrow A \twoheadrightarrow W \to 1$

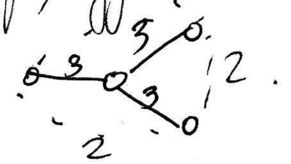
Exple: $\circledast$ $m(s,t) \equiv \infty \Rightarrow W =$ Free prod of $\mathbb{Z}_2$ $\quad A =$ Free group.

$\circledast$ $m(s,t) \equiv 2 \Rightarrow W =$ Direct ——— $\quad A =$ Free abelian.

$\circledast$ $W = \mathfrak{S}_n$ $\quad A =$ braid group on n-strands.

$\circledast$ W Weyl grp, affine Weyl group.

$\circledast$ $W = D_4$



Theo (Coxeter 34) If $|S| = n$, there is $\rho : W \to GL_n(\mathbb{R})$ faithful and which sends s to reflections.

CVSC, every (finite) reflection group in $\mathbb{R}$ admits a Coxeter presentation

Fix V and identify W with its image in $GL(V)$. Define

$$\mathcal{A} = \{ \operatorname{Ker}(r-1) \mid r \in W \text{ is conj to an element of } S \}$$

$\hookrightarrow$ reflections (S = simple reflection).

We assume that W is finite from now on. (simplicity)

Prop: W acts freely on $X = V \setminus \bigcup \mathcal{A}$ and on $X_{\mathbb{C}} = V_{\mathbb{C}} \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$.

We have a short exact sequence (topological). $1 \to \pi_1(X_{\mathbb{C}}) \to \pi_1(X_{\mathbb{C}}/W) \to W \to 1$

Theo (Brieskorn 71) $\pi_1(X_{\mathbb{C}}) = P$ $\pi_1(X_{\mathbb{C}}/W) \simeq A$.

Adapted to W infinite by Van der Lek 83. Replacing $X_{\mathbb{C}}$ with the "complexified Tits Cone".

Theo (Deligne 72) $X_{\mathbb{C}}$ and $X_{\mathbb{C}}/W$ are $K(\pi,1)$ spaces

Open to W infinite still widely open.

2) Parabolic subgroups.

Def: For $I \subseteq S$, we define $W_I = \langle I \rangle \subseteq W$ and $A_I = \langle I \rangle \subseteq A$, we call these "standard parabolic subgroups.

W_I (resp A_I) is again a Coxeter (resp Artin) group for $m_{|I}$
(Tits 61) (Van der Lek 83).

Def: $\Theta_I = \{ v \in V \mid \forall w \in W_I \; w.v = v \} = \bigcap_{s \in I} H_s$ $= \operatorname{Ke}(s-1)$.

The Tits cone intersection.

for W infinite, need to work in V^*.

Lemma: $\operatorname{Stab}(\Theta_I) = N_W(W_I)$ acts linearly on Θ_I with kernel W_I.

Define $\mathcal{H}_I = \{ H \in \mathcal{A} \mid \Theta_I \not\subset H \}$. $X_J = \Theta_J \setminus \bigcup_{H \in \mathcal{H}_I} H$.

Def: The normalizer quotient $N(W, I) = N_W(W_I)/_{W}$ acts faithfully on Θ_I and freely on X_J

We have a topological SEC $1 \to \pi_1(X_J) \to \pi_1(X_J/W) \to N(W,I) \to 1$.

At the Artin grp level, we define $N(A, I) = N_A(A_I)/A_I$.

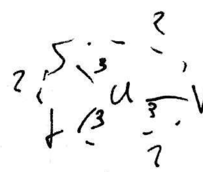
Theo: (Brink Howlett 98) $N_W(W_I) = W_I \rtimes N(W,I)$
(Paris 97) $N_A(A_I) = A_I \rtimes N(A, I)$.

We have a short exact seq. $1 \to \underset{P \cap N(A,I)}{\overset{\shortparallel}{N(P,I)}} \longrightarrow N(A,I) \longrightarrow N(W,I) \to 1$.

Theo: (G.Heng, Licata, Yacobi 25)
The two sequences are isomorphic, X_J and X_J/W are $K(\pi,1)$ spaces for $N(P\,I)$ and $N(A,I)$.

Conj: This could hold for arbitrary W (implies "classical" $K(\pi,1)$ conjecture).

3) If needed: an example

Recall the Coxeter diagram D_4  . We can set

$$s = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad t = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad u = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \quad v = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \in GL_4(\mathbb{R}).$$

$$\mathcal{A} = \{ (x_1 \dots x_4) \in \mathbb{R}^4 \mid x_i \in \pm x_j \text{ for a couple } i \neq j \}.$$

$W_I = \langle u \rangle = \{1, u\}$ is a standard parabolic subgroup

$\Theta_I = \{ x_2 = x_3 \}$ is a hyperplane. (we use basis $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right)$ a, b, c for coordinates.

$\mathcal{H}_I = \{ a \pm c = 0, \ a \pm b = 0, \ b \pm c = 0, \ c = 0 \}$ 7 hyperplanes.

Now, we have

$$N_W(V_I) = \pm\left\{\begin{pmatrix}1 & \\ \pm X & 1\end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \pm X\end{pmatrix}\right\} \quad X \in \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$$

$$N(W,I) = \pm\left\{\begin{pmatrix}1 & \\ \pm 1 & 1\end{pmatrix}\right\} \subseteq GL(\Theta_I).$$

→ $N(W,I)$ is not a reflection group

$\mathcal{H}_I$ is not a reflection arrangement. A_3: 6hp ; B_3: 9hp; H_3: 15hp.

Now, if $I = \{s,t,v\}$. $\Theta_I = \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle$ and $N_W(W_I) = \left\langle W_I, \begin{pmatrix}1 & & \\ & 0 & -1 \\ & -1 & 0\end{pmatrix}\right\rangle$.

II. Useful groupoids.

1) Groupoid coverings.

Let E be a top. space, $F \subseteq E$. The fundamental groupoid $\Pi_1(E,F)$ is def by

- Obj $= F$, $\mathrm{Hom}(v,v')$ = htopy classes of paths $v \to v'$ in E.

Def: A functor $F: \tilde{\mathcal{G}} \to \mathcal{G}$ between two groupoids is a covering if every arrow $\gamma \in \mathcal{G}(a,-)$ lifts uniquely to $\tilde{\gamma} \in \tilde{\mathcal{G}}(\tilde{a},-)$ for a given $\tilde{a} \in F^{-1}(a)$.

In other words. F induces $\tilde{\mathcal{G}}(\tilde{a},-) \xrightarrow{1:1} \mathcal{G}(a,-)$

This follows the topological intuition: $p: E \xrightarrow{\text{cov. map}} B$ induces a groupoid covering $\Pi_1(E, p^{-1}(b)) \to \Pi_1(B,b)$. (= path lifting property for covering maps)

→ Deck transformation; Galois covering; universal covering groupoid (= $\tilde{\mathcal{G}}$ equivalent to trivial group).

[R. Brown, Topology & groupoids].

Idea: relate topological cover $X_I \to X_I/N(W,I)$

with a groupoid cover $\underset{?}{D} \longrightarrow \underset{?}{{}_x\mathrm{Ribb}}(A,I)$.

2) Who is D?

Let $(V, \mathcal{A})$ be a real hyperplane arrangement wich is <u>simplicial</u> (it is the case for us)

Chambers = connected components of $X = V \setminus \cup \mathcal{A}$.

2 chambers are <u>adjacent</u> if they share a wall, wich is then unique.

<u>Def</u>: A gallery is a sequence of adjacent chambers. The Deligne category

D^+ is the category.

$Ob(D^+)$ = chambers

$Hom(C, C')$ = Galleries + identification of geodesic (= shortest) galleries

<u>Theo</u> (Deligne 72)

The groupoid D obtained by formally inverting all arrows in D^+ is equivalent to $\pi_1(X_{\mathbb{C}})$, + $X_{\mathbb{C}}$ is a $K(\pi, 1)$

3) Who is qRibb?

Since W is finite, the monoid $A^+ = \langle S \rangle^+ \subseteq A$ has strong algebraic properties (Garside monoid):

- existence of lcm and gcds for monoid divisibility
- word problem solution in A
- conjugacy problem in A.

A_I^+ is a submonoid of A^+ wich is also a Garside monoid.

<u>Def</u>: A ribbon $A_I \to A_J$ is an element $\alpha \in A$ s.t $\alpha^{-1} A_I^+ \alpha = A_J^+$.

+ positive $\alpha \in A^+$.

$\Rightarrow$ Ribbon groupoid Ribb, s.t Ribb(A_I, A_I) is the <u>quasi centraliser</u> of A_I ($\subseteq N_A(A_I)$).

$$QZ_A(A_I) A_I = N_A(A_I).$$

Problem: $QZ_A(A_I) \cap A_I \neq \{1\}$ in general. If we want a smaller group giving a semidirect, we need a smaller groupoid

Def: reduced positive ribbon $A_I \to A_J$ = positive ribbon α + coprime with A_I^+

- $r\mathrm{Ribb}^+$: category of reduced ribbon.
- $r\mathrm{Ribb}$ groupoid of Ribb generated by $r\mathrm{Ribb}^+$

Theo (Paris 97, Godelle 03-10)

$r\mathrm{Ribb}(A_I, A_I) = N(A, I)$ and $N_A(A_I) \simeq N(A, I) \ltimes A_I$.

III. Tits cone intersection.

Even if $(\Theta_I, \mathcal{H}_I)$ is not a reflection arrangement, we can still use the Deligne groupoid construction $\mathcal{D}$.

× The action $N(W,I) \curvearrowright X_I$ induces an action $N(W,I) \curvearrowright \mathcal{D}^+$ and a groupoid covering $\mathcal{D} \twoheadrightarrow \mathcal{D}/N(W,I) \to$ equivalent to $\pi_1\left(X_I/N(W,I)\right)$

× By Coxeter combinatorics, we have a description of the Chambers of X_I.

$$\mathrm{Cham}(X_I) = \{(x, J) \mid W_I x = x W_J \text{ and } x \text{ has minimal length in } xW_I\}$$

$$(x, J) \mapsto x \cdot \left\{ v \;\middle|\; \begin{array}{l} \langle v, \alpha_s \rangle = 0 \; s \in I \\ \langle v, \alpha_s \rangle > 0 \; s \notin I \end{array} \right\}.$$

+ adjacent chambers are related by "simple wall crossings".

We obtain a functor $\mathcal{D} \to r\mathrm{Ribb}$ sending $(x, J) \to J$

Theo: (G, Heng, Licata, Yacobi) This functor induces an isomorphism

$\mathcal{D}/N(W,I) \to r\mathrm{Ribb}_I$ (connected component of A_I in $r\mathrm{Ribb}$)

We also describe $\widetilde{BH}$, universal cover of a groupoid BH, similar to ribbons for Coxeter groups.

This last thm holds for arbitrary Coxeter groups.