05/11/25

Group theory seminar of the ICMAT.

Normalizers of Parabolic subgroups in Artin & Coxeter groups.

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I. Coxeter groups, Artin groups

II. Useful groupoids for us.

III. Tits cone intersection and consequences.

I. Coxeter groups, Artin groups.

1) Definitions, Tits representation.

Let S be a finite set, a Coxeter matrix is a map $m: S\times S \to \mathbb{N}^* \cup \{\infty\}$ such that

$$m(s,t) = m(t,s) \qquad \begin{array}{l} m(s,s) = 1 \\ m(s,t) \geq 2 \text{ if } s \neq t. \end{array}$$

From this data, we define two groups.

$$W = \langle S \mid \forall s\neq t \ \underbrace{sts\ldots}_{m(s,t)} = \underbrace{tst\ldots}_{m(t,s)},\ s^2 = 1\ \forall s \rangle. \quad \text{Coxeter group}$$

$$A = \langle S \mid \text{\_\_\_\_\_\_\_\_\_\_\_\_} \rangle. \quad \text{Artin group}$$

There is a surjective map $A \to W$, we denote its kernel by P and we call P the pure Artin group

$$1 \to P \to A \to W \to 1.$$

Exple: ⊛ All $m(s,t) = \infty \Rightarrow W =$ free prod. of $\mathbb{Z}/2\mathbb{Z}$ $A =$ Free group.

⊛ All $m(s,t) = 2 \Rightarrow W = (\mathbb{Z}/2\mathbb{Z})^n$ $A = \mathbb{Z}^n$.

⊛ $W = \mathfrak{S}_n$ $A =$ braid group on n strands.

⊛ W a Weyl group; affine Weyl group ...

Remark: Our understanding of A is significantly better when W is finite (spherical case).

We restrict our attention to W finite, and specify wich result are known to adapt.

2) Tits representation

We fix (S, m) a Coxeter matrix. $\Delta = \{\alpha_s\}_{s \in S}$ a basis of a $\mathbb{R}$-vector space V.

We define a bilinear form $(\cdot,\cdot)$ on V by setting

$$(\alpha_s, \alpha_t) = -\cos(\pi / m(s,t)).$$

For $s \in S$, we define $r_s \in GL(V)$ by

$$r_s(x) = x - 2\langle x, \alpha_s \rangle \alpha_s.$$

Prop: For all s, r_s is a reflection in V, and the map $\begin{array}{l} S \to GL(V) \\ s \mapsto r_s \end{array}$ extend to a faithful representation $W \to GL(V)$ called the Tits representation.

Theo (Coxeter '34) "A finite subgroup $W \subseteq GL(V)$ generated by reflections is a Coxeter group"

Def: We set $\Phi = \bigcup_{s \in S} W.\alpha_s$ the set of roots of W.

It is partitioned as $\Phi = \underset{\Delta}{\overset{\cup\!\!\mid}{\Phi^+}} \sqcup \Phi^-$ (positive & negative roots).

For $\alpha \in \Phi$, $r_\alpha : x \mapsto x - 2\langle x, \alpha \rangle \alpha$ is a reflection in W.

+ if $\alpha = w\alpha_s$, $r_\alpha = w \circ r_s \circ w^{-1}$. ; $r_{-\alpha} = r_\alpha$.

$$\Phi^+ \overset{1:1}{\longleftrightarrow} \{r \in W,\ r \text{ is a reflection}\}.$$

For r_α a reflection, $H_\alpha = \{v \in V \mid \langle v, \alpha \rangle = 0\} = \mathrm{Ker}(r_\alpha - 1)$ is the reflecting hyperplane of r_α.

Prop: For $v \in V$, $\mathrm{Stab}_W(v) = \{r_\alpha \mid v \in H_\alpha\}$ is generated by the reflections of W it contains

Cor: W acts freely on $X := V \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ and on $X_{\mathbb{C}} = V \otimes \mathbb{C} \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha \otimes \mathbb{C}$

The space X is not connected, but its complexification $X_{\mathbb{C}}$ is, and we have a covering map $X_{\mathbb{C}} \twoheadrightarrow X_{\mathbb{C}}/W$.

Theo (Brieskorn 71)
$\pi_1(X_{\mathbb{C}}) \simeq P$, $\pi_1(X_{\mathbb{C}}/W) \simeq A$

gen to W infinite by Van der Lek 83, but need to replace $X_{\mathbb{C}}$ with the "complexified Tits cone".

$\rightarrow$ $1 \to P \to A \to W \to 1$ is now geometric!

A combinatorial tool to understand $X_{\mathbb{C}}$: the Deligne groupoid (in next section).

Theo (Deligne 72)
$X_{\mathbb{C}}$ (and $X_{\mathbb{C}}/W$) are $K(\pi, 1)$ spaces

gen to W infinite still widely open.

3) Parabolic subgroups

Def: For $I \subseteq S$, we define $W_I = \langle I \rangle \subseteq W$ and $A_I = \langle I \rangle \subseteq A$, and we call those "standard parabolic subgroups".

W_I is again a Coxeter group for m_I (Tits 61)

A_I is again an Artin group for $m_{/I}$ (Van der Lek 83)

The group W_I is also the pointwise stabilizer of

$$\Theta_I = \bigcap_{s\in I} H_{\alpha_s} \qquad \text{Tits cone intersection}$$

The group $N_W(W_I)$ acts on Θ_I, with kernel W_I. We can remove the hyperplanes of W not containing Θ_I and obtain a free action

$\boxed{N(W;I) \text{ the normalizer quotient}}$ acts freely on

$$X_I = \Theta_I \setminus \bigcup_{H\in\mathcal{H}_I} H \qquad \mathcal{H}_I = \{H_\alpha \mid \Theta_I \not\subseteq H_\alpha\}.$$

geometric/topological

We have a short exact sequence $\boxed{1 \to \pi(X_I) \to \pi(X_I/N(W,I)) \to N(W,I) \to 1}$

We can also define normalizer quotients at the level of Artin groups

$$N(A,I) = N_A(A_I)/A_I$$

We have a short exact sequence

algebraic/combinatorial.

$$\boxed{1 \to \underset{\substack{\| \\ P\cap N_A(A_I)/A_I}}{N(P,I)} \to N(A,I) \to N(W,I) \to 1}.$$

II. Groupoids.

1) Groupoid coverings.

A good model for groups is the fundamental group of a space X with base point x.

Similarly, if $F\subseteq E$ is a subset of a top. space E, we can define a fundamental groupoid $\pi_1(E,F)$

Objects: elements of F, $Hom(v,v') =$ htop classes of paths $\gamma : v \rightsquigarrow v'$.

Let $\mathcal{G}$ be a groupoid, Notation for $u, v \in Ob(\mathcal{G})$, $\mathcal{G}(u,v) =$ morphism $u \to v$
$\mathcal{G}(u,-) =$ morphism $u \to ?$

Def: A functor $\mathcal{F} : \tilde{\mathcal{G}} \to \mathcal{G}$ is a groupoid covering if
$\forall u \in Ob\mathcal{G}$, $\hat{u} \in \mathcal{F}^{-1}(u)$, $\gamma \in \mathcal{G}(u,-)$, $\exists! \tilde{\gamma} \in \tilde{\mathcal{G}}(\hat{u},-)$ such that $\mathcal{F}(\tilde{\gamma}) = \gamma$.

In other words, the restriction of $\mathcal{F}$ to $\tilde{\mathcal{G}}(\hat{u},-)$ is a bijection $\tilde{\mathcal{G}}(\hat{u},-) \xrightarrow{1:1} \mathcal{G}(\mathcal{F}(\hat{u}),-)$.

This follows a topological intuition: If $p : E \to B$ is a (topo) covering map, and $b \in B$ is a base point, then p induces a groupoid covering $\pi_1(E, p^{-1}(b)) \longrightarrow \pi_1(B,b)$.

$\to$ "Galois theory of groupoid coverings", Deck transformations, universal covering groupoid.

2) Groupoids and normalizer quotients W, A, I, S as before.

Def: The Brink Howlett groupoid $\mathcal{BH}(\mathcal{F})$ is defined as follows.
- $Ob_j = \{J \subseteq S$ such that W_J and W_I are conjugate$\}$
- $Hom(J,K) = \{x \in W \mid \{x.\alpha_k \mid k \in K\} \overset{\text{in } W}{=} \{\alpha_j \mid j \in J\}\}$.

"action of W on the subsets of roots".

$Hom(I,I) = \{x \in W \mid x.\{\alpha_i \mid i \in I\} = \{\alpha_i \mid i \in I\}\}$. ok for infinite group's

Prop (Brink Howlett 99) $Hom(I,I) \simeq N(W,I)$ the groupoid describes the normalizer quotients + $N_W(W_I) = W_I \rtimes N(W,I)$ (S)

Similarly, if A is spherical

Def: $A^+ = \langle S \rangle^+$, $A_I^+ = \langle I \rangle^+ \subseteq A^+$.

$\mathrm{Ribb}(J,K) = \{ \alpha \in A \mid A_J^\alpha = A_K \}$.

$\mathrm{Ribb}^+(J,K) = \{ \alpha \in A^+ \mid A_J^\alpha = A_K \}$.

$\pi\mathrm{Ribb}^+(J,K) = \{ \alpha \in A^+ \mid A_J^\alpha = A_K$ + α is "coprime with A_J^+" $\}$.

$\pi\mathrm{Ribb}(J,K)$ = formal inverse of the above

Prop (Paris 97, Godelle 03/10).

$\pi\mathrm{Ribb}(I,I) \simeq N(A,I)$ and $N_A(A_I) \simeq A_I \rtimes N(A,I)$

III. Tits cone intersection and consequences.

1) Walls and chambers.

The connected components of $V \setminus \bigcup H_\alpha$ are the chambers, among those the fundamental chamber $C = \{ v \mid (v, \alpha_s) > 0 \ \forall s \in S \}$.

and $X = \bigsqcup_{w \in W} w.C$.

We can define $C_J = \left\{ v \;\middle|\; \begin{array}{ll} (v \mid \alpha_s) = 0 & s \in I \\ (v, \alpha_s) > 0 & s \notin I \end{array} \right\}$ the fund. chamb of the Tits cone intersection O_J.

$$V = \bigsqcup_{J \subseteq S} \bigsqcup_{x \in W/W_J} x.C_J$$

$$O_I = \bigsqcup_{L \subseteq S} \bigsqcup_{\substack{x \in W/W_L \\ x.C_L \subseteq O_I}} x.C_L.$$

This gives a labelling of a stratification of O_I.

2). The Deligne groupoid.

Objects = chambers. Arrows = Galeries (seq. of adjacent chambers)

+ we identify the galleries which are "geodesic". (a path of minimal length between two chambers). → Category D^+

Theo (Deligne 72) The groupoid D is equivalent to $\pi_1(X_c)$.

. where D is the groupoid of formal inverses of D^+.

Condition: chambers are open simplicial cone.

3). Our results.

* The chambers of X_I can be parametrized combinatorially.

$\mathrm{Cham}(I)$ $\{(x, J) \mid W_I x = x W_J$ and x has min. length in $x W_J\}$

$(x, J) \mapsto x \circ C_J$.

Theo: Fix $\widetilde{BH}$ a groupoid with $Ob = \mathrm{Cham}(I)$, a singleton b/w each pair as a morphism.

The map $(x, J) \mapsto J$ extends to a universal groupoid covering $\widetilde{BH} \to BH$ whose Deck transformation are $N(W, I)$.

Cor: "atomic Matsumoto theorem" for simple wall crossings in $\widetilde{BH}$.

* . Even thoug X_I is not the complement of a reflection arr.

($N(W, I)$ is not a refl. group in general).

We can still apply the Deligne groupoid.

$N(W,I) \curvearrowright$ Deligne groupoid $D(X_J)$ and we obtain a groupoid covering $D(X_J) \to D(X_J)/N(W,I)$.

+ $(x,I) \mapsto I$ induces a Funtor $D(X_I) \to \pi \mathrm{Ribb}_I$.

<u>Theo:</u> The functor $D(X_I) \to \pi \mathrm{Ribb}_I$ is a covering with induces an isomorphism $D(X_J)/N(W,I) \simeq \pi \mathrm{Ribb}_I$.

the sequences

$$1 \to \pi_1(X_J) \to \pi_1(X_J/N(W,I)) \to N(W,I) \to 1$$

$$1 \to N(P,I) \to N(A,I) \to N(W,I) \to 1$$

are isomorphic

+ $X_J/N(W,I)$ is a $K(\pi,1)$ for $N(A,I)$.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \widetilde{BH} \arrow[d] \\ R \arrow[r] & BH \end{tikzcd}$$