04/11/25
Reading grp on
McCammond-Sulway.

Garside groups & Interval groups
after Part 1.2

I. Garside & quasi Garside groups
II. Parabolic supgroups] – obviously too long to include.
III. Interval structures.

I. Garside & quasi-Garside groups.

1) Definitions, normal forms.

First, some vocabulary on monoids. Fix a monoid M.

- left cancellative → $ab=ac \Rightarrow b=c$.
- left divisibility $a \leq b \Leftrightarrow \exists c \mid ac=b$.
  right divisibility $b \succcurlyeq a \Leftrightarrow \exists c \mid b=ca$.
  $a \gtrsim b$
  $a \leq^{M} b$.
- left Noetherian: no infinite $<$ sequence (impart, $M^{\times}=\{1\}$).
- strongly Noetherian: $\forall x$, $\sup\{n \mid \exists s_1 \dots s_n = x,\ s_i \neq 1\} < \infty$.
- atom: no proper left/right divisor.

Left divis is reflexive and transitive, If M is left-cancell and $M^{\times}=\{1\}$, then $\leq$ is a partial order on M.

Ex: $\langle a,b,c \mid ac = bc\rangle^+$ is not cancellative

$\langle a,b \mid a = bab\rangle^+$ is neither left- nor Right Noetherian. $b^{n+1}a \leq b^n a$.

$\langle a,b \mid ab = b\rangle^+$ is left Noetherian but not strongly Noetherian.

$(b^m a^n \leq b^p a^q \iff m<p$ or $m=p$ and $n<q$.)

Lemma: M is strongly Noetherian $\iff \exists \lambda : M \to \mathbb{N}$ s.t. $\lambda(xy) \geq \lambda(x)+\lambda(y)$
$+\ x \neq 1 \Rightarrow \lambda(x) \geq 1$.

If we can choose λ s.t. $\lambda(xy) = \lambda(x)+\lambda(y)$. Then we call λ a length function and M is homogeneous.

Def (Classical). A Garside monoid (M,Δ) with $\Delta \in M$ s.t.

- $M^\times = 1$, M is cancellative, strongly Noetherian.
- Lcms and gcd of pair (both left & right) exist in M.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta) = S$ is finite and generates M.

If S is infinite, (M,Δ) is called quasi Garside = infinite type Garside

Δ is the Garside element.

Def (Modern). A Garside monoid (M,Δ) with $\Delta \in M$ s.t.

- $M^\times = 1$, M is left-cancellative.
- $\forall x \in M$, $x \wedge \Delta$ (left gcd) exists.
- $\mathrm{Div}\,\Delta = \mathrm{Div}_R(\Delta) = S$ is finite and gen M. Garside
- $\mathrm{Div}\,\Delta = \mathrm{Div}_R(\Delta) = S$ gen M and $\sup\{r \mid \exists s_1 \dots s_r = \Delta,\ s_i \neq 1\} < \infty$ quasi Gar

Theo : The two definitions coincide

Let (M, Δ) be quasi-Garside

Prop : $\forall x \in M, \exists ! \phi(x) \in M$ s.t. $x\Delta = \Delta\phi(x)$.

$x \mapsto \phi(x)$ is an automorphism of M. It has finite order if M is Garside

dem : $s \in S$, $\exists \bar{s} \mid s\bar{s} = \Delta$. We have $\bar{s} \in Div_R(\Delta) = S$, thus $\exists \bar{\bar{s}} \mid \bar{s}\bar{\bar{s}} = \Delta$ and we have

$$s\Delta = s\bar{s}\bar{\bar{s}} = \Delta\bar{\bar{s}}. \qquad \phi(s) := \bar{\bar{s}}.$$

In gen, $x = s_1 \ldots s_n$ is a product of elements of S and we can do an induction.

$+ \phi$ acts faithfully on S, which is finite if (M, Δ) is Garside.

Cor : A Garside monoid has nontrivial center.

Theo : Any $x \in M$ can be written uniquely as $x = \Delta^d s_1 \ldots s_n$ with

- $s_1 \neq \Delta$, $s_n \neq 1$
- $\forall i \in [\![1, \ldots, n-1]\!]$, $s_i = \Delta \wedge (s_i s_{i+1})$ (s_i is the "biggest possible").

This is the greedy normal form of x

+ actually, we have $s_i = \Delta \wedge (s_i \ldots s_n)$

↳ the gnf is computable if we can compute $(\Delta \wedge -)$.

Actually we only need to compute it on S.

Prop: Let A be the atom set of M. For $a, b \in A$, let $\theta(a,b)$ be a word in A such that $a\,\theta(a,b) = b\,\theta(b,a) = \mathrm{lcm}_{right}(a,b) \in M$.
Then
$$\langle A \mid a\theta(a,b) = b\theta(b,a)\rangle^{+}$$
is a presentation of M.

Another bigger presentation is $M = \langle S \mid \underset{\text{for } st = u \in S}{st = u} \rangle$.

2) Garside groups

Let M be an arbitrary monoid. The restriction to the category of groups of the functor $\mathrm{Hom}_{Mon}(M, -)$ is representable.

In other words, $\exists\, G(M)$ a grp s.t for all group G.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\iota"'] & G \\ G(M) \arrow[ur, dotted, "\exists !"'] & \end{tikzcd}$$

⚠ ι is not an embedding in general.

Theo: If (M, Δ) is quasi Garside, then $\iota : M \to G(M)$ is an embedding and we have $G(M) = \{\iota(x)^{-1}\iota(y) \mid x, y \in M\}$.
is a Group of fractions
We call $G = G(M)$ a quasi-Garside group

in practice, remove the $^{+}$ in $\langle\ \rangle^{+}$.

Prop (Symmetric normal form) Every $x \in G$ can be written uniquely as $a^{-1}b$ where $a \wedge b = 1$.
The symmetric normal form of x is $a_p^{-1} \dots a_1^{-1} b_1 \dots b_q$ where $a_1 \dots a_p$ and $b_1 \dots b_q$ are the greedy normal forms of a and b.

Prop (left normal form) Every $x \in G$ can be uniquely written
$x = \Delta^k s_1 \dots s_r$ with $s_1 \neq \Delta$, $s_1 \dots s_r$ greedy, $k \in \mathbb{Z}$.

In particular, G is generated by M and Δ^{-1}.

Theo: - A quasi Garside group is a torsion free group with an (explicit) finite dimensional classifying space.
- A Garside group has solvable conjugacy problem.

3) At last, examples.

- $(\mathbb{Z}^n, \mathbb{N}^n, (1 \dots 1))$ is a Garside group.
- Let Br_n be the n-strands braid group. Br_n^+ the monoid of positive crossing braids,
(Br_n, Br_n^+, Δ) is a Garside group, where Δ is the half twist
This was the case studied by Garside

$n = 3$: $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ $S =$

$$\begin{tikzcd} & \Delta & \\ \sigma_1\sigma_2 \arrow[ur, dash] & & \sigma_2\sigma_1 \arrow[ul, dash] \\ \sigma_1 \arrow[u, dash] & & \sigma_2 \arrow[u, dash] \\ & 1 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

• For $n \geq 1$. $G(n,n) = \langle a_0 \dots a_n \mid a_i \cdots a_{n-1} a_0 \cdots a_i = a_{i+1} \cdots a_{n-1} a_0 \cdots a_i \ \forall i \rangle$

is a Garside grp. ($G(n,n) \simeq \pi_1(\mathbb{C}^2 \setminus n$ lines through the origin$)$)

The element $a_0 \dots a_{n-1} = \Delta$ generates $Z(G(n,n))$ and

$$G(n,n) \simeq \underbrace{\langle a_0 \dots a_{n-2} \rangle}_{F_{n-1}} \times \underbrace{\langle \Delta \rangle}_{\mathbb{Z}}.$$

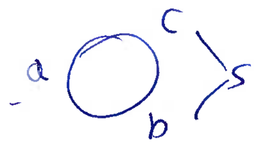
• $\langle a, b \mid a^3 = b^2 \rangle$ is a Garside group $\Delta = a^3 = b^2$.

$a \mapsto \sigma_1 \sigma_2$, $b \mapsto \sigma_1 \sigma_2 \sigma_1$ induces an isomorphism b/w this grp and Br_3.

↳ a Garside monoid for a group is not unique.

• $\langle a, b, c \mid ab = bc = ca \rangle^+$ is also a garside monoid for Br_3.

$a \mapsto \sigma_1$ $b \mapsto \sigma_2$ $c \mapsto \sigma_2^{-1} \sigma_1 \sigma_2$.

• 

$= \left\langle a\, b\, c\, s \;\middle|\; \begin{matrix} abc = bca = cab \\ csc = scs \quad bsb = sbs \\ sa = as \end{matrix} \right\rangle$ is not cancellative:

$sbcsbc \neq bcsbcs$ while $asbcsbc = abcsbcs$.

• If W is a finite Coxeter group, its Artin group is a Garside group with monoid $A^+ = \langle S \rangle^+$ and Garside element the Tits lift of w_0.

• If W is infinite this fails!

no longest element in W = no Δ :(

it's hard (but still true) that $A^+ \hookrightarrow A$ in this case.

. Let's do one quasi Garside.

$n \geq 2$ Let $F_n = \langle a_1 \dots a_n \rangle$ be a free group. $\langle a_1 \dots a_n \rangle^+$ has no chance of being Garside...

$\mathcal{B}_n$ acts on F_n^n by

$$(x_1 \dots x_n) \circ \sigma_i = (x_1 \dots x_{i-1}, \underset{i}{x_i x_{i+1} x_i^{-1}}, \underset{i+1}{x_i}, x_{i+2} \dots x_n)$$

this preserves the fibers of the map

$$F_n^n \longrightarrow F_n^n$$
$$(x_1 \dots x_n) \longmapsto x_1 \cdots x_n .$$

Let $O_n = O_{\mathcal{B}_n}((a_1 \dots a_n))$, and let $S_n = \left\{ g \in F_n \;\middle|\; \begin{array}{l} \exists (s_1 \dots s_n) \in O_n \\ g = s_i \cdots s_j \end{array} \right\}$

then $\langle S_n \rangle^+$ is quasi Garside with Garside elmt $\Delta = a_1 \dots a_n$.

↳ With $n=2$, $F_2 = \mathrm{Art}(\tilde{A}_1)$ $\quad a \overset{\infty}{—} b$ $\quad \langle a, b \mid \emptyset \rangle$ is the Artin presentation.

the other presentation is $\left\langle \begin{array}{l} a_n, b_n \\ n \in \mathbb{Z} \end{array} \;\middle|\; a_i b_i = b_i a_{i+1} = a_{i+1} b_i \right\rangle$.

it is the dual presentat°.

|gen|	2	∞
length relat°	∞	2

not garside ☹ $\qquad$ (quasi) Garside ☺

III. Interval structures

In practice, it is <u>hard</u> to construct Garside monoids / to prove that a monoid is Garside. Interval monoids/groups is a recipe for producing new Garside groups.

1). Definition (classical).

G = Coxeter, S = reflections for instance.

We fix a group G with fixed generating set S which positively gen G. (marked group). We can define

$$\ell_S(g) := \min\{k \mid \exists\, s_1 \dots s_k = g\}$$

the S-length of g, which is the length of a shortest S-decomposition of g.

Using ℓ_S, we define two partial order on G.

$$g \leq g' \Leftrightarrow \ell_S(g) + \ell_S(g^{-1}g') = \ell_S(g').$$

$$g' \succeq g \Leftrightarrow \ell_S(g'g^{-1}) + \ell_S(g) = \ell_S(g').$$

Let $h = g^{-1}g'$, we always have $gh = g'$, and thus $\ell_S(g') \leq \ell_S(g) + \ell_S(h)$ (think of triangle inequality). We say $g \leq g'$ in the equality case.

Def: Interval monoid (G, S, g), denoted M_g is def by

$$\left\langle [1,g]^G_{\leq} \mid s \circ t = (st) \text{ if } s \leq t \right\rangle^+.$$

G_g is the enveloping group of M_g.

Theo: If g is balanced ($[1,g]^G_{\leq} = [1,g]^G_{\geq}$), and if $[1,g]^G_{\leq}$ is a lattice, then (G_g, Γ_g, g) is a quasi Garside group with $S = [1,g]^G_{\leq}$, left divisibility extends $\leq$

Exple: $G = \mathbb{Z}/6\mathbb{Z}$, $S = \{1\}$. $g = 1$, $[1,g]^G = \{0,1\}$ and $G_g \simeq \mathbb{Z}$.

$G = \mathbb{Z}/6\mathbb{Z}$, $S = \{\underset{x}{2}, \underset{y}{3}\}$ $g = 1 = x^2 y$ $[1,g]^G = G$ and $G_g \simeq \mathbb{Z}^2$.

Rq: If S is stable under conjugacy, then ℓ_S is constant on conjugacy classes and all elements are balanced

Exple $G = \mathfrak{S}_3$, $S = \{\overset{(12)}{s}, \overset{(23)}{t}\}$ $g = \overset{w_0}{(1,3)}$. $[1,g]^G = \mathfrak{S}_3$ $G_g = \langle s,t \mid sts = tst \rangle = Br 3$.

$G = \mathfrak{S}_3$ $S = \{\overset{12}{a}, \overset{23}{b}, \overset{13}{c}\}$ $g = ab$ $[1,g]^G = \{1, a, b, c, g\}$ $G_g = \langle ab = bc = ca \rangle$

2) Case of Coxeter groups

Theo: let W be a finite Coxeter group. The three foll. groups

1. $Art(W)$
2. Interval group W_{w_0} with gen. set S (simple refls)
3. Interval group W_c with gen set R (all reflections)

coincide

$2 = 1$ is immediate by looking at the presentation

$2 = 3$ is complicated.

For W infinite Coxeter, w_0 doesn't exist, so no hopes of constructing a Garside group.

<u>However</u>. W still admits Coxeter elements!

Fix a total order on S and let $w = s_1 \dots s_n$. We can define.

$W_w :: \mathrm{Art}^*(\Gamma, w)$ (with gen set all the reflections R).

The injection $S \to R$ induces a morphism $\mathrm{Art}(\Gamma) \to \mathrm{Art}^*(\Gamma, w)$. We can only hope that this is an isomorphism.

<u>Def</u>: If the natural morphism $\mathrm{Art}(\Gamma) \to \mathrm{Art}^*(\Gamma, w)$ is an iso, $[1, w]^W$ <u>encodes</u> a dual presentation for $\mathrm{Art}(\Gamma)$.

<u>Theo</u> C. If Γ is extended Dynkin, $[1, w]^W$ always encodes a dual presentation for $\mathrm{Art}(\Gamma)$.

<u>Theo</u> (Digne) If $\Gamma = \tilde{C}_n$ or $\tilde{G}_2$ or $\tilde{A}_n$, there is at least one choice of w for which $[1, w]^W$ encodes a dual <u>quasi Garside</u> pres for $\mathrm{Art}(\Gamma)$.

<u>Theo</u> (McCammond). For $\tilde{C}_n$, $\tilde{G}_2$, every choice of w works, for $\tilde{A}_n$ we need one specific choice (that of Digne). For other extended Dynkin <u>no choice</u> gives a ^quasi^ Garside group.

$\to$ replace W with something bigger...

3) McCammond Convention, presentation.
Sulway

S is assumed to be symmetric. We can assoc weights (discrete, bounded away from 0) to elements of S.

We replace ℓ_S by $\inf\{\sum_{i=1}^{k} w(s_i) \mid s_1 \dots s_k = g\}$ and the rest works out the same.

⚠ McCammond Sulway work in the Cayley Graph (G,S).

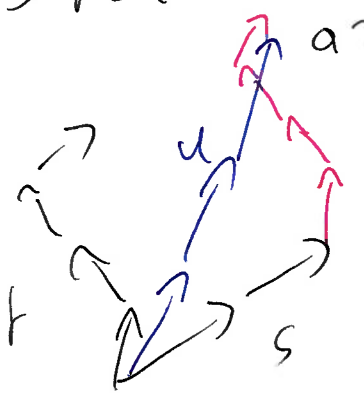
In this convention

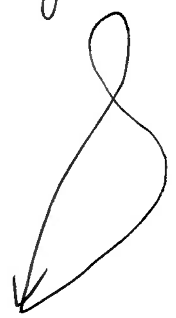
$\ell_S(g) = d(1,g)$. $\quad [g,h]^G = \{x \mid d(g,x) + d(x,h) = d(g,h)\}$.

shortest decomposition $=$ geodesic. "edge labeled poset".

The relations defining G_g can be seen as closed loops in the portion of $\mathrm{Cay}(G,S)$ corresponding to $[1,g]$.

$s \cdot t = u \rightsquigarrow$ concatenating a reduced expr of s,t gives a reduced expr of u.





$=$ identity.

Prop: $S \subseteq G$ stable under conjugacy. $G' \subseteq G$ $\quad G' \cap S = S'$.

Let $w \in G'$ and a weighting s.t

(1) $[1,w]^{G'}$ and $[1,w]^{G}$ are lattices.

(2) $[1,v]^{G'} \hookrightarrow [1,w]^{G}$ is a lattice morphism preserving join & meet.

then the nat. map $G'_w \longrightarrow G_w$ is an injection