UNIVERSITÉ de Picardie Jules Verne

Homology of categories & the Dehornoy Lafont order complex

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Ouragan séminar.

I. Homology of categories

1) Notation, definitions.

C is a small category. $C(u,v)$ = morphisms from u to v. 1_u = identity morphism.

⚠ composition denoted as a product.

$$\begin{tikzcd} & v \arrow[dr, "g"] & \\ u \arrow[ur, "f"] \arrow[rr, "fg"'] & & w \end{tikzcd}$$

If $Ob(C)=\{\bullet\}$, then we only have $C(\bullet,\bullet)$, a monoid

Def: A group is a monoid where all elements are invertible.
A groupoid is a category where all morphisms are invertible.

$\forall f: u \to v$, $\exists f^{-1}: v \to u$ with
$ff^{-1} = 1_u$ $f^{-1}f = 1_v$.

Ex: Oriented graphs $\begin{tikzcd} u \arrow[r, bend left, "a"] & v \arrow[l, bend left, "b"] \end{tikzcd}$ $\to$ category of paths. $C(u,u) = \langle ab \rangle^+ \simeq \mathbb{N}$.

Prop: Any category C admits an enveloping groupoid $\mathcal{G}(C)$ obtained by formally inverting morphisms.

Ex: $\begin{tikzcd} u \arrow[r, bend left, "a"] & v \arrow[l, bend left, "b"] \end{tikzcd}$ $\rightsquigarrow$ $\begin{tikzcd} u \arrow[r, bend left=60, "a"] \arrow[r, bend right, "b^{-1}"'] & v \arrow[l, bend right, "a^{-1}"'] \arrow[l, bend left=60, "b"] \end{tikzcd}$ $\mathcal{G}(C)(u,u) = \langle ab, (ab)^{-1} \rangle \simeq \mathbb{Z}$.

Ex: $M = \langle a,b,c \mid ab=ac \rangle^+$, $G(M) = \langle abc \mid ab=ac \rangle$
$= \langle abc \mid b=c \rangle$
$= \langle a,b \mid \emptyset \rangle = F_2$.

Def: A groupoid $\mathcal{G}$ is *connected* if $\mathcal{G}(u,v) \neq \emptyset \ \forall u,v \in Ob(\mathcal{G})$

A group G is *equivalent* to a connected groupoid $\mathcal{G}$ if $G \simeq \mathcal{G}(u,u)$ for some ($\Leftrightarrow$ for all) $u \in Ob(\mathcal{G})$

2) Modules over a Category / groupoid

Let G be a group. A $\mathbb{Z}G$-module (or simply G-module) is an abelian group A, together with a $\mathbb{Z}$-linear action of G.

Let us see $G = \mathcal{G}(\bullet,\bullet)$ as a groupoid with one object. A functor $F : \mathcal{G} \to Ab$ is the data of:

- An abelian group $A = F(\bullet)$.
- $\forall g \in G = \mathcal{G}(\bullet,\bullet)$, a map $\begin{matrix} A \longrightarrow A \\ a \longmapsto g.a \end{matrix}$ of abelian groups.

$$G\text{-modules} \longleftrightarrow \text{functors } \mathcal{G} \longrightarrow Ab.$$

Def: A C-module is a contravariant functor $C \to Ab$.

Equivalently, a C-module A is given by

- $\forall u \in Ob(C)$, an abelian group A_u.
- $\forall f \in C(u,v)$, a morphism of abelian groups $\begin{matrix} A_v \longrightarrow A_u \\ a \longmapsto f.a \end{matrix}$.
- $+ \ 1_u . a = a \qquad (fg).a = f(g.a)$.

⚠ convention for composition.

Ex: trivial module: $\forall u \in Ob(C)$, $A_u = \mathbb{Z}$,
$\forall f \in C(u,v)$, $A_v \to A_u : \mathbb{Z} \xrightarrow{1} \mathbb{Z}$.

Ex: "regular module": $A_u = \mathbb{Z}C(u,-)$.
$\forall f \in C(u,v)$, $f . \sum_{g \in C(u,-)} n_g g = \sum_{g \in C(u,-)} n_g (fg)$

<u>Prop</u>: $\mathcal{C}$-modules behave the same way as modules over algebras:
- direct sum - Kernels - Image - quotions
- exact sequences - resolutions - homology ... - tensor product...

<u>Rq</u>: To define free modules, we need one set of generators for each object of $\mathcal{C}$.

3) Category, groupoid and Group

Let $\mathcal{C}$ be a category. We have natural functors

$$\begin{tikzcd} \mathcal{G}(\mathcal{C})\text{-modules} \arrow[rr, bend right, "\text{restriction, from the ``inclusion'' } \mathcal{C}\to\mathcal{G}(\mathcal{C})"'] \arrow[rr, phantom, "\perp"] & & \mathcal{C}\text{-modules} \arrow[ll, bend right, "\text{``scalar inversion''}\ \mathbb{Z}\mathcal{G}(\mathcal{C})\otimes_{\mathcal{C}} -"'] \end{tikzcd}$$

does the scalar inversion functor preserve homology?

<u>Ex</u>: Consider again $\Pi = \langle a, b, c \mid ab = ac \rangle$. $\Pi \curvearrowright A = \mathbb{Z}/6\mathbb{Z}$ by
$a.x = 3x \quad b.x = 3x \quad c.x = 5x$.
We have $G(\Pi) \otimes_{\Pi} A = \{0\}$ and $H_0(G(\Pi), G(\Pi) \otimes_{\Pi} A) = 0$, whereas
$$H_0(\Pi, A) = A/\langle m.a - a \ \forall m, a \rangle = A/2A = \mathbb{Z}/2\mathbb{Z}.$$

<u>Def</u>: A category $\mathcal{C}$ is cancellative if $fgh = fg'h$ always implies $g = g'$.
A left-Ore category is a cancellative category where common left multiples exist

$$\begin{tikzcd} \cdot \arrow[r, dashed, "y"] \arrow[d, dashed, "x"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd} \qquad xf = yg.$$

Thm (Squier 94, G23)

If C is a left-Ore category, then the scalar inversion functor is exact.
In particular it preserves homology
$$H_*(C,A) \simeq H_*(\mathcal{G}(C), \mathbb{Z}\mathcal{G}(C)\otimes_C A) \qquad \forall C \text{ module } A$$

Cor: $H_*(C,\mathbb{Z}) \simeq H_*(\mathcal{G}(C),\mathbb{Z})$.

Prop: An equivalence between a group G and a groupoid $\mathcal{G}$ induces an equivalence between G-mod and $\mathcal{G}$-mod. In particular.
$$H_*(G,A_G) \simeq H_*(\mathcal{G},A) \qquad \forall A\ \mathcal{G} \text{ module}$$

$\to$ if G is equivalent to $\mathcal{G}(C)$, we have in particular $H_*(G,\mathbb{Z}) \simeq H_*(C,\mathbb{Z})$.

II The Order complex

1) Gaussian monoids

[DP,99] [DL 03]

Let M be a left Gaussian monoid

- $\to$ No invertible element
- $\to$ cancellative
- $\to$ left Noetherian (no infinite descending chain for left-divisibility)
- $\to$ $\exists$ left lcms.

Rq: left Gaussian $\Rightarrow$ left Ore.

$$\begin{tikzcd} \cdot \arrow[r, "b/a"] \arrow[d, "a/b"'] \arrow[dr, "m"] & \cdot \arrow[d, "a"] \\ \cdot \arrow[r, "b"'] & \cdot \end{tikzcd} \qquad (b/a)a = a\vee b = (a/b)b$$

Choose S a generating set for M, an an (arbitrary) linear order $<$ on S.

Def: Let $m \in M$, $md(m) :=$ $<$-smallest right divisor of m in S.

This gives rise to a canonical form
$$NF(m) = a_1 \cdots a_m \qquad \text{where } a_i := md(a_1 \cdots a_i) \in S.$$

Of course, this depends on the choice of $<$.

2) The complex

Def: A n-cell is a tuple $[\alpha_1 \dots \alpha_n]$ with $\alpha_1 < \alpha_2 < \dots < \alpha_n$ $\in S$ such that
$\forall i \in [\![1,n]\!]$, $\alpha_i = \mathrm{md}(\alpha_i \vee \dots \vee \alpha_n)$

X_n is the set of n-cells.

$X_0 = \{[\phi]\}$ $\quad X_1 = \{[\alpha] \mid \alpha \in S\}$ $\quad X_2 = \{[\alpha,\beta] \mid \alpha = \mathrm{md}(\alpha \vee \beta)\}$ . . .

We consider $C_n := \mathbb{Z}M[X_n]$

To construct the differential ∂ ($\mathbb{Z}M$-linear), we need auxiliary maps r_n and s_n (only $\mathbb{Z}$-linear).

$$\begin{tikzcd}
\cdots \; C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\partial_0"] & \mathbb{Z} \\
C_2 \arrow[r, "\partial_2"] \arrow[u, dashed, "s_2"'] & C_1 \arrow[r, "\partial_1"'] \arrow[u, "r_1"] \arrow[ul, dashed, "s_2"] & C_0 \arrow[r, "\partial_0"'] \arrow[u, "r_0"] \arrow[ul, dashed, "s_0"] & \mathbb{Z} \arrow[ul, "s_{-1}"']
\end{tikzcd}$$

$\partial_0 [\phi] = 1 \qquad s_{-1}(1) = [\phi] \qquad r_0(m[\phi]) = s_{-1}(\partial_0(m[\phi])) = s_{-1}(m.1)$
$= s_{-1}(1) = [\phi]$.

$$\partial_{n+1}[\alpha, A] = \alpha/A[A] - r_n(\alpha/A[A]) \qquad s_n(m[A]) = \begin{cases} 0 & \text{if } \mathrm{md}(m\,\mathrm{lcm}(A)) = A_1 \\ g[\alpha, A] + s_n(g\, r_n(\alpha/A[A])) & \alpha = \mathrm{md}(m\,\mathrm{lcm}(A)),\ g\,\alpha/A = m. \end{cases}$$

$s_0([\phi]) = 0$

$r_{n+1} = s_n \circ \partial_{n+1}$.

s_n is well defined thanks to some induction.

Lemma: $\partial([\alpha]) = \alpha[\phi] - [\phi]$.

Prop: Let $[\alpha,\beta] \in \mathcal{X}_2$. We can write

$$\alpha\vee\beta = \underbrace{a_1 \cdots a_{m-1}}_{NF(\beta/\alpha)}\ \underbrace{a_m}_{\alpha} = \underbrace{b_1 \cdots b_{m-1}}_{NF(\alpha/\beta)}\ \underbrace{b_m}_{\beta}$$

We have $\partial_1([\alpha,\beta]) = \sum_{j=1}^{m} b_1 \cdots b_{j-1}[b_j] - \sum_{i=1}^{m} a_1 \cdots a_{i-1}[a_i]$

main ingredient: if $NF(m) = x_1 \cdots x_k$, then $s_0(m[\phi]) = \sum_{i=1}^{k} x_1 \cdots x_{i-1}[x_i]$

Beyond that, how to explicitly compute ∂ ???

Thm: (Dehornoy-Lafont 03)
$\forall n \geq 0$, we have $\partial_{n+1}s_n + s_{n+1}\partial_n = 1_{C_n}$. Then the complex (C_*, ∂_*) is exact.

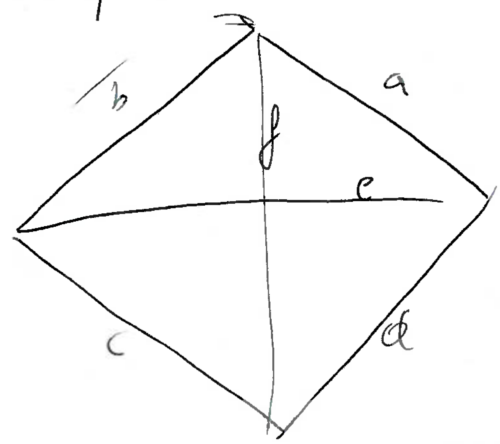
Cor: If M is Garside, then $G(M)$ is FL-type. As $\mathcal{X}_n = \emptyset$ for n big enough.

Thm: (G23). Let $\mathcal{C}$ be a Garside category, the complex (C_*, ∂_*) defined as above is a finite free resolution of the trivial $\mathbb{Z}\mathcal{C}$-module.

Only need to define source and target: for $[\alpha_1 \cdots \alpha_n]$ with same target u, the "source" is that of $\alpha_1 \vee \alpha_2 \cdots \vee \alpha_n$.

3) Order.

An example: consider the dual braid monoid of type A_3.



$ab = be = ef$
$bc = cf = fb$
$ec = cd = de$
$da = af = fd$
$ac = ca$
$bd = db$.

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Consider two orders.

$\underbrace{a < b < c < d}_{\text{not deep.}} \; < \underbrace{e < f}_{\text{deep}}$ and $\underbrace{e < f}_{\text{deep}} < \underbrace{a < b < c < d}_{\text{not deep}}$

In order 1, we have $NF(abc) = eca$.
2, we have $NF(abc) = dbe$.

In order 1, we have 11 2-cells.
2, we have 10 2-cells

[e f] [e a] [e b] [e c] [e d] [f a] [f b] [f c] [f d] [a c] [b d]

[a b] [a c] [a d] [a e] [a f] [b c] [b d] [b f].
[c d] [c e]

Ex: Complex braid gp B_{34}.

			711	3448	7520	7614	2686
21836	1	56			5691	5255	1812
16300.	1	56	646	2839			

How to optimize the ordering $<$ on the set of generators.

Try and minimize the number of 2-cells.

Let $L = \{ a \vee b \mid a \neq b,\ a, b \in S \}$. the set of lcms of 2 elements of S.

For $\ell \in L$, $S_\ell = \{$ elements of S which right divide $\ell \}$.

Let $a \in S_\ell$, we can consider $A_\ell = \{ b \in S_\ell \mid a \vee b = \ell \}$. If a is the $<$minimum of A_ℓ, then we get $n(a, \ell)$ 2 cells whose left lcm is ℓ.
(i.e, $m(\ell)$).

Prop: The number of 2-cells is included in.

$$\left[\sum_{\ell \in L} \min_{a \in A_\ell} n(a, \ell) \text{ and } \sum_{\ell \in L} \max_{a \in A_\ell} n(a, \ell) \right]$$

Ex: For the dual braid monoid of type A_3, bounds are 10 and 11.
For $B(G_{34})$ 630 1071.

Can the (lower) bound always be reached?

In practice, we define the condition $C(a,\ell)$ = "a is the $<$ minimum of S_ℓ".
We then construct iteratively an order by adding conditions with the best $m(a,\ell)$ possible, and such that the conditions are all compatible (i.e are part of an ordering).