24/10/23 Homology computations for complex braid groups with the Dehornoy Lafont complex

I. Complex braid groups.

We consider $W \subseteq GL_n(\mathbb{C})$ a finite group generated by (complex) reflections: $\quad \operatorname{codim} \underbrace{\operatorname{Ker}(s-1)}_{H_s} = 1$ & $\operatorname{Order}(s) < \infty$.

"Complex reflection group"

If $R \subseteq W$ is the set of reflections, we set $\mathcal{H} = \bigcup_{s \in R} H_s$. and $X = \mathbb{C}^n \setminus \mathcal{H}$.

Théo: W acts freely on X, we set $P(W) = \pi_1(X)$, $B(W) = \pi_1(X/W)$. We have a short exact sequence $1 \to P(W) \to B(W) \to W \to 1$

One can restrict our attention to irreducible groups & their braid groups
$G(de,e,n) \to B(de,e,n) \qquad G_4, \dots, G_{37} \to B_4 \dots B_{37}$.

- Understand the lack of injectivity of $W \mapsto B(W)$.
  $\hookrightarrow$ not trivial since $B(de,e,n) \simeq B(e,e,n)$ for $d \geq 2$.
  + several cases in exc groups.

To do this, various methods. $\to$ Alg top, combi, reduction mod prime ...

Prop: $B(W)$ has homological dim the rank of W (Callegaro Marin).

today: a combinatorial tool.

Exp: $A_3 = G_4 \rightsquigarrow \langle s,t,u \mid sts = tst,\ tut = utu,\ su = us \rangle$ 

$B_7 = B_{11} = B_{19} \dots = \langle abc \mid abc = bca = cab \rangle$ 

$B_{31} = \left\langle stuvw \;\middle|\; \begin{matrix} suw = uws = wsu \\ sts = tst \quad vuv = uvu \quad utu = tut \\ twt = wtw \\ st = ts \quad wv = vw \end{matrix} \right\rangle$ 

II. Gaussian categories

We really see categories as monoids with several objects:

→ No size problem: all small.

→ composition as a product.
$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "fg"'] & & \cdot \end{tikzcd}$$

→ Hom-sets denoted by $C(x,y)$.

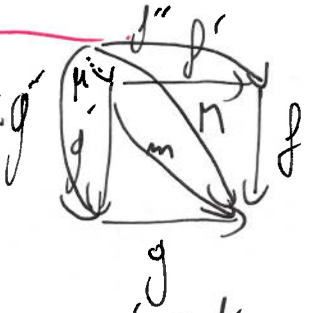
Def: $a : x \to y$ in C is an atom if it has no nontrivial factorisations.

Denoting $f \geq g$ for $\exists\, h \in C$ with $f = hg$.

Prop: If $C^\times = \{1\}$ and every morphism is a mono, then $\geq$ is a preorder on $C(-,u)$ $\forall u \in Ob(C)$. Atoms are minimal elements of $\geq$.

Def: The category C is right Noetherian if there are no infinite strictly descending $>$ chain (i.e $\geq$ well-foundedness).

Def: A category C is left Gaussian if $C^\times = 1$, right Noetherian, every morphism is both mono and epi (cancellative), Pullbacks exist (lcm)

Pullback: 

lcm: $m = ff' = gg'$, $\forall M = ff'' = gg''$ $\exists$ μ tq $m\mu = M$. (+ μ unique by cancell).

Rk: This is a good notion of "divisibility", related to rewriting systems and normal forms.
If $C = M$ is a monoid → Gaussian Monoid [Dehornoy Paris]

Exp: If $(P, \subseteq)$ is a poset, C_P its associated category.
→ cancellative = ok by uniqueness of morphisms.
→ atoms = covering relations of P.
→ Noetherian (right): $\nexists$ infinite sequence $x_1 < x_2 < x_3 \dots < x_\infty$.
→ Pullback exists = two elements with a common upper bound have a "join".

Toy exple, useless for hom comp.

Def: For C a ~~small~~ category ~~we~~ define $\mathcal{G}(C) = C[C^{-1}]$ the enveloping groupoid of C (if $M = C$ is a monoid, $\mathcal{G}(M) = G(M)$ is a group).

Prop: If C is left Garside, C embeds in $\mathcal{G}(C)$, and we have left fractions decomposition (Ore condition)

Theo: Every irreducible complex braid group is isom to the enveloping group of some (several) Garside monoid... except.

$$\begin{array}{c} B_n^*(e) = B(2e, e, n) \\ \text{finite index} \; \downarrow \\ B(2, 1, n) \end{array} \qquad B_{31} \;!!!$$

This is why we need a categorical approach.

III. Categorical homology

Def: Let C be a category, a C-module ($\mathbb{Z}C$-module) is a contravariant functor $C \to Ab$.

Trivial functor ⇝ "constant diagram" $= \mathbb{Z}$. Notion of Free functor (adj to what?)

Notion of tensor product (bimodule).

Def: Let $A \in \mathbb{Z}C$-mod, we define $H_n(C, A) = Tor^n_{\mathbb{Z}C}(\mathbb{Z}, A) = L^n(- \otimes_{\mathbb{Z}C} A)(\mathbb{Z})$.

Can be computed using free (projective) resolutions of $\mathbb{Z}$ as a $\mathbb{Z}C$-module.

Is there a link between $H_n(C, A)$ & $H_n(\mathcal{G}(C), A)$??

We have an obvious functor $\mathbb{Z}\mathcal{G}\text{-mod} \longrightarrow \mathbb{Z}C\text{-mod}$ of scalar restriction.

Using $\mathbb{Z}\mathcal{G} \otimes_{\mathbb{Z}C} -$, we get a scalar inversion functor.

Prop:

$$\begin{tikzcd} \mathbb{Z}\mathcal{G}\text{-mod} \arrow[r, bend right, "\text{restr}"'] \arrow[r, phantom, "\perp"] & \mathbb{Z}C\text{-mod} \arrow[l, bend right, "\mathbb{Z}\mathcal{G} \otimes -"'] \end{tikzcd}$$

Théo: If C is Left Gaussian, then scalar inversion is exact. [Squier] [Cartan Eilenberg]

$\forall A \in \mathbb{Z}C\text{-mod}, \quad H_*(C, A) = H_*(\mathcal{G}, \mathbb{Z}\mathcal{G} \otimes_C A)$.

Furthermore, if $\mathcal{G} \simeq G$ (equivalence) then this induces an equivalence $\mathbb{Z}\mathcal{G}\text{-mod} \simeq \mathbb{Z}G\text{-mod}$, and $H_*(G, A) = H_*(\mathcal{G}, A)$

We can compute homology of a Gaussian category to compute homology of braid groups.

IV. Homology computations: the order complex.

1) A first try: the Charney Meier Whittlesey.

The categories we consider are Garside on top of being Gaussian, the closure of the atoms under $\vee$, division, is finite and well behaved: the set of simples.

From this, a complexe built with the nerve ("Garside nerve") with top. meanings.

Problem: For B_{31}, 660 atoms, 2603' simples, CMW gives a too big complex

	0	1	2	3	4
Rank of the complex.	88	2603	11065	15300	6750

2) The Dehornoy Lafont complex

Fix C Gaussian A its atoms $<$ a lin order on A. Any $f: x \to y$ in C admits a $<$-least div on the right $md(f) = \alpha$. This gives a normal form $f = \underbrace{a_n \dots a_1}_{NF(f)}$ where $a_i = md(a_n \dots a_i)$. $\forall i \in [1, n]$.

Def: A n-cell is a tuple $[\alpha_1 \dots \alpha_n]$ of atoms with the same length x, $\alpha_1 < \dots < \alpha_n$ and
$\forall i \in [1, n] \quad \alpha_i = md(\alpha_i \vee \dots \vee \alpha_n)$
the source is that of the lcm $\alpha_1 \vee \dots \vee \alpha_n$

Ex: $[\emptyset]_u$ for each object u, 1-cells = atoms.
2-cells $= \{\alpha < \beta \mid \alpha = md(\alpha \vee \beta)\}$.

A *chain* will be a linear combination of the form

$$\sum n_f \, f \, [\alpha_1^f \ldots \alpha_n^f]$$

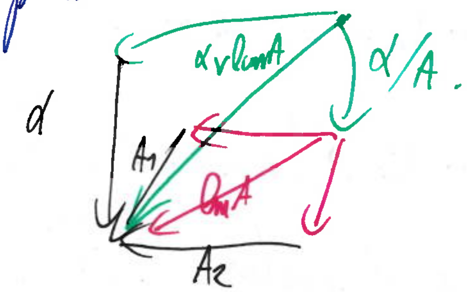
all f have the same target as the source of $\alpha_1^f \vee \ldots \alpha_n^f$ and the same source.

The differential ∂_n is constructed recursively, along with a contracting homotopy s_{n-1} ($\mathbb{Z}$-linear).

$$\partial_0([\emptyset]_u) = 1 \in \mathbb{Z}, \quad s_{-1}(1) = [\emptyset] \qquad r_0 = \partial_0 \circ s_{-1} : f[\emptyset]_{t(f)} \longrightarrow [\emptyset]_{s(f)}.$$

The "reduction map" r is needed to carry on the construction. Assume ∂_n, s_{n-1}, r_n constructed.

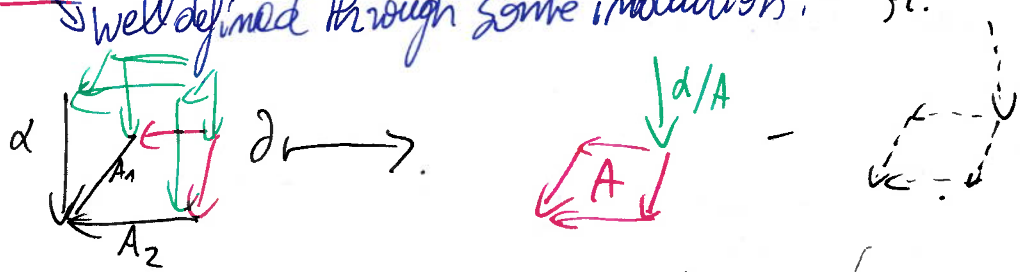
A $(n+1)$-cell is $[\alpha, A]$ where A n-cell, $\alpha = \mathrm{nd}\ \mathrm{lcm}(\alpha, \mathrm{lcm} A)$, let α/A be the unique morphism $\alpha/A \ \mathrm{lcm}(A) = \mathrm{lcm}(\alpha, \mathrm{lcm} A)$.

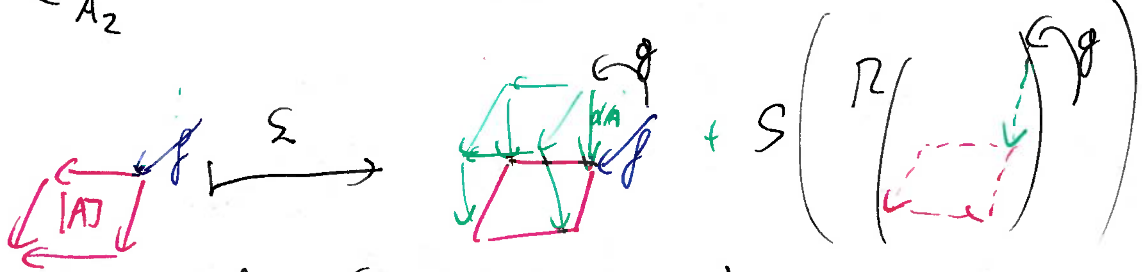


Def: $\partial_{n+1}[\alpha, A] = \alpha/A[A] - r_n(\alpha/A[A])$. $\qquad r_{n+1} = s_n \circ \partial_{n+1}$.

$$s_n(f[A]) = \begin{cases} 0 & \text{if } \mathrm{nd}(f \,\mathrm{lcm} A) = A_1 \\ g[\alpha, A] + s_n(g(r_n \, \alpha/A[A])) & \end{cases} \qquad \begin{array}{l} \alpha = \mathrm{nd}(f\,\mathrm{lcm} A) \\ g\,\alpha/A = f. \end{array}$$

well defined through some induction.





Example: computation of s_0 (see following page).

Theo: ∂_n is a finite free resolution of $\mathbb{Z}$ in $\mathbb{Z}C$, s is a contracting htopy (+ $\partial_n(r_n(f[A])) = \partial_n f[A])$).

example: Let $f : u \to v$ in C, we write $NF(f) = a_n \cdots a_1$. We have $g = a_n \cdots a_2$.

$$
\begin{aligned}
S_0(f[\phi]_v) &= g[a_1] + S_0(g(r_0(a_1/\phi[\phi]))) \\
&= g[a_1] + S_0(g\, r_0(a_1[\phi])) \\
&= g[a_1] + S_0(g[\phi]) \\
&= \sum_{i=1}^{n} a_n \cdots a_{i+1}[a_i]
\end{aligned}
$$

Cor: Let $[\alpha, \beta]$ be a two cell, we write $m_2 = \overbrace{b_m \cdots b_2}^{NF} \beta$, $m_1 = \overbrace{a_n \cdots a_2}^{NF.} \alpha$.

We have
$$
\partial_2[\alpha,\beta] = \sum_{i=2}^{m} b_m \cdots b_{i+1}[b_i] + b_m \cdots b_2[\beta] - \left(\sum_{j=2}^{n} a_n \cdots a_{j+1}[a_j] + a_n \cdots a_2[\alpha] \right).
$$
"comparison between two ways of writing $\alpha \vee \beta$".

For B_{31}, we obtain C (through some optimisation of $<$).

0	1	2	3	4
88	660	1665	1735	642.

This is 10 times less than CMW. (+ for any choice of ordering, $1655 \leq 1665 \leq 1865$).

We compute:

$H_n(B_{31}, M)$	0	1	2	3	4
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Z}_\varepsilon$	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_{20}$	0
$R = \mathbb{Q}[r, r^{-1}]$	$\mathbb{Q}$	0	R/ϕ_6	$R/\left(\frac{r^{10}-1}{r+1}\phi_{15}\right).$	0

– C.M.

+ $H_2(B_{31}, \mathbb{F}_2[r^{\pm 1}]) = \phi_1 \phi_6$ ..
$\mathbb{F}_3[r^{\pm1}]$

$H_3(B_{31}, \mathbb{F}_2[r^{\pm 1}]) = \frac{r^{10}-1}{}\phi_{15} = \frac{r-(r^{10}+1)}{r+1}\phi_{15}.$