0 Rewriting system

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Let A be an alphabet, $F(A)$ the free monoid on A
elmts = words, product = concatenation, unit = empty word.

Def: Rewriting system = $R \subseteq (F(A)\setminus 1) \times F(A)$. An element $(u,v) \in R$ is a "rule" denoted $u \to v$.

$\xrightarrow{R}$: $x_1 u x_2 \xrightarrow{R} x_1 v x_2$ where $u \to v \in R$.

$\xrightarrow{R}{}^*$: transitive reflexive closure of $\xrightarrow{R}$

$\xleftrightarrow{R}{}^*$ sym (congruence).

$M := F(A)/\xleftrightarrow{R}{}^*$ is a presentation of M.

Eg: $R = \{bab \to aba\}$ on $F(\{a,b\})$, $M = \langle a,b \mid aba = bab\rangle$.

Orientation of R (may) give rise to "normal forms" for elements of M.
$x \in M \rightsquigarrow \hat{x} \in F(A)$ such that $\hat{x} \xrightarrow{R} y$ is impossible.

ex: $\widehat{bab} = aba$.

Thm: (Squier 87, Anick 86, Kobayashi 89)
~~Let~~ $F(A)$ been endowed with a complete rewriting system R.
↳ resolution of $\mathbb{Z}$ as $\mathbb{Z}M$ module, free, recursive differential.

Idea: under combinatorial assumption on a monoid (lcms...)
↳ rewriting system → complex.
↳ Dehornoy-Lafont complex for monoids.

I. Gaussian categories

Let C be a (small) category. $Ob(C)$ its objects. morphisms from u to v is denoted by $C(u,v)$.

⚠ convention for composition: ~~composition of maps~~ product.

$$\begin{tikzcd} & v \arrow[dr, "g"] & \\ u \arrow[ur, "f"] \arrow[rr, "fg"'] & & w \end{tikzcd}$$

Def: A left Gaussian category is a right cancellative right Noetherian category which admits left-lcms.

Def: C is right cancellative if every morphism is an monomorphism.

$$fg = hg \Rightarrow f = h \quad \forall f,g,h \in C.$$

Let $u \in Ob(C)$, sets $C(u,-)$ $C(-,u)$.

On $C(u,-)$ $\quad f \preccurlyeq g \overset{def}{\Leftrightarrow} \exists h \mid fh = g$

$$\begin{tikzcd} & \cdot \arrow[dr, dashed, "\exists h"] & \\ u \arrow[ur, "f"] \arrow[rr, "g"'] & & \cdot \end{tikzcd}$$

$C(-,u)$ $\quad g \succcurlyeq f \overset{def}{\Leftrightarrow} \exists h \mid hf = g$

$$\begin{tikzcd} \cdot \arrow[r, dashed, "\exists h"] \arrow[rr, "g"', bend right] & \cdot \arrow[r, "f"] & u \end{tikzcd}$$

We focus on $\preccurlyeq$. Reflexive ✓ transitive ✓. antisymmetric ✗.

if $\varphi \in C(-,u)$ iso $\varphi \succcurlyeq 1_u \succcurlyeq \varphi \succcurlyeq 1_u \cdots$

Prop: If $C^\times = \{\text{identities}\}$ and C is right cancellative. Then $\preccurlyeq$ is always a preorder

proof: Let $f \succcurlyeq g$ and $g \succcurlyeq f$, we have $f = hg$ $\quad g = h'f$.

Thus $f = hh'f$ $\quad g = h'hg$, $\quad 1 = hh'$ $\quad 1 = h'h$ and $h = 1 = h'$

$\Rightarrow f = g$. □

Def: C is right Noetherian if $\geq$ admits no infinite strictly descending chain
↳ necessary for $\geq$ inductions.

Def: An atom is a $\geq$ minimal element (no identity).

Lem: A common left multiple of $f, g \in C(-,a)$ is a commutative square.

left-lcm is the pullback of f, g.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \cdot \arrow[r, dashed, "\exists f'"] \arrow[d, dashed, "\exists g'"'] \arrow[dr, "m"] & \cdot \arrow[d, "f"] \\
& \cdot \arrow[r, "g"'] & \cdot
\end{tikzcd}$$

Notation: $(f/g)g = f \vee g = (g/f)f$ + well defined by cancellativity.

Example: $B_+ = \langle a,b \mid aba = bab \rangle$. $a/b = ba$ $\quad b/a = ab$
$a \vee b = bab$.

Rq: $bab \to aba$ gives a complete rewriting system!

If C is garsian, $<$ linear on A, $f \in C$ can be written uniquely as a composition
$f = a_1 \dots a_m$ of atoms
with $a_i := md(a_1 \dots a_i)$ $\forall i \in [\![1, m]\!]$. This is $NF(f)$.

→ adapt the DL complex to compute $\underbrace{H_*(ZC, A)}_{???}$.

II. Homology of a category — 1) Def. free modules

Def: A $\mathbb{Z}C$ module is a (contravariant) functor $C \to Ab$.
⚠ composition!

$\mathbb{Z}$ as a trivial $\mathbb{Z}C$ mod: every obj to $\mathbb{Z}$, maps to 1.

What are free modules?

The forgetful functor gives $\{A_u\}_{u \in Ob(C)} \in Set^{Ob(C)}$.

Conversely, let $S = \{S_u\}_{u \in Ob(C)}$. Elements of $F(S)_v$ should be of the form

$$\sum_{\substack{n_f \in \mathbb{Z} \\ f \in C(v, u) \\ s \in S_u}} n_f \, f \cdot s.$$

This is free.

Ex: $u_0 \in Ob(C)$. $S_u = \begin{cases} \emptyset & u \neq u_0 \\ \{*\} & u = u_0 \end{cases}$ $\quad F(S_u) \simeq \mathbb{Z}C(-, u_0)$

and the adjunction is simply Yoneda.

$$\mathrm{Hom}_{\mathbb{Z}C}(\mathbb{Z}C(-, u_0), A) \simeq A_{u_0} \simeq \mathrm{Hom}_{Set^{Ob(C)}}(S_u, A).$$

$\mathbb{Z}C$-mod is abelian, we can derive $- \otimes_{\mathbb{Z}C} A : (\mathbb{Z}C\text{-mod})^{dnt} \to Ab$ in $\mathbb{Z}$.
to compute $H_n(C, A) = \mathrm{Tor}_n^{\mathbb{Z}C}(\mathbb{Z}, A)$.

2). Scalar inversion.

For C a cat, there is a universal groupoid $\mathcal{G} := \mathcal{G}(C)$ with

$$\begin{tikzcd} C \arrow[r, "\forall"] \arrow[d] & \mathcal{H} \text{ groupoid} \\ \mathcal{G} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

the functor $C \to \mathcal{G}$ induces $\mathbb{Z}\mathcal{G}\text{-mod} \to \mathbb{Z}C\text{-mod}$. "forgetful".
$\mathbb{Z}\mathcal{G}$ is a $\mathcal{G}$-C bimodule, thus we have. $\mathbb{Z}\mathcal{G} \otimes_C - : \mathbb{Z}C\text{-mod} \to \mathbb{Z}\mathcal{G}\text{-mod}$.
the "scalar inversion" [illegible].

<u>Thm</u> (Squier 94, G22) Scalar inversion $\dashv$ forgetful.
Moreover, if C is left Ore (in part. if C left Garside). The scalar inversion <u>is exact</u>. then $H_*(C, A) \simeq H_*(\mathcal{G}, \mathbb{Z}\mathcal{G} \otimes_C A)$. $A \in \mathbb{Z}C\text{-mod}$

<u>Prop</u>: If G is equivalent to $\mathcal{G}$, $\mathbb{Z}G$-mod and $\mathbb{Z}\mathcal{G}$-mod are equivalent, so $H_*(G, A) = H_*(\mathcal{G}, A)$

↳ example p.7 about posets.

III. The complex.

Let C be a Garsian category with atoms A, $<$ a lin order on A.

Def: A n-cell is $[\alpha_1, \dots, \alpha_n]$ a tuple of atoms with the same target with $\alpha_1 < \dots < \alpha_n$ and

$$\forall i \in [\![1,n]\!],\ \alpha_i = md(lcm(\alpha_1, \dots, \alpha_i))$$

$md(f) = <$ least right div of f

The source of the cell is that of $lcm(\alpha_1 \dots \alpha_n)$

C_n is the free $\mathbb{Z}C$-module associated. Elmts of $(C_n)_u$ are of the form

$$\sum_{f \in C(u,v)} n_f\, f\,[\alpha_1 \dots \alpha_n].$$

Rq: There is one 0 cell $[\emptyset]_u$ by object u. 1 cells = atoms.

∂_n is constructed recursively, along with a contracting htopy s_{n-1} — $\mathbb{Z}$-linear.

$\partial_0([\emptyset]_u) = 1$. $\quad s_{-1}(1) = [\emptyset]$. $\quad \pi_0 = \partial_0 \circ s_{-1} : f[\emptyset] \mapsto [\emptyset]$.

$\partial_1([\alpha]) = \alpha[\emptyset] - [\emptyset]$. $\quad s_0 f[\emptyset] = \sum_{i=1}^{n} a_1 \dots a_{i-1}[a_i]$ où $a_1 \dots a_n = NF(f)$.

$$\partial_{n+1}([\alpha, A]) = \alpha/A[A] - \pi_n(\alpha/A[A]) \qquad s_n f[A] = \begin{cases} 0 & \text{if } md(f\, lcm A) = A_1 \\ g[\alpha, A] + s_n(g\, \pi_n(\alpha/A[A])) & \end{cases} \quad \alpha = md(f\, lcm A),\ g\,\alpha/A = f.$$

$\pi_{n+1} = s_n \circ \partial_{n+1}$

↳ well def by some induction

Prop: $\partial_{n+1} s_n + s_{n+1} \partial_n = 1$ $\forall n \in \mathbb{N}$. The complex (C_n, ∂_n) is exact!

$$\begin{tikzcd}
\cdots \ C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\partial_0"] & \mathbb{Z} \\
C_2 \arrow[r, "\partial_2"'] \arrow[u, "\pi_2"] & C_1 \arrow[r, "\partial_1"'] \arrow[u, "\pi_1"] \arrow[ul, "s_1"'] & C_0 \arrow[r, "\partial_0"'] \arrow[u, "\pi_0"] \arrow[ul, "s_0"'] & \mathbb{Z} \arrow[ul, "s_{-1}"']
\end{tikzcd}$$

Example: $B_4^+ = \langle a\, b \mid aba = bab \rangle$. $a < b$.

0 cell: $[\phi]$, 1 cell: $[a]$, $[b]$ 2 cells: $[a, b]$.

$\partial_1[a] = a[\phi] - [\phi]$. $\partial_1[b] = (b-1)[\phi]$.

$\partial_2([a,b]) = a/b[b] - \pi_1(a/b[b])$.

$= ba[b] - \pi_1(ba[b])$

$\pi_1(ba[b]) = S_0 \partial_1(ba[b]) = S_0(bab[\phi] - ba[\phi])$

$= S_0(aba[\phi]) - S_0(ba[\phi])$

$= ab[a] + a[b] + [a] - b[a] - [b]$.

$\rightarrow \partial_2[a,b] = (ab - b + 1)[a] + (ba - a + 1)[b]$.

If we spe $a = b = 1$, we get matrices. $(0\ 0)$ for ∂_1, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for ∂_2.
$(\otimes \mathbb{Z})$

Thus $H_n(B_4^+, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0]$.

Example. Posets. Let $(P, \leq)$ be a poset, C_P its category ($C(x,y) = \begin{cases} * & \text{if } x \leq y \\ \phi & \text{otherwise} \end{cases}$)

if P is a finite lattice, C_P is left Gaussian. What is $H_*(C_P, \mathbb{Z})$?

$+ C_P$ is an one category: in groupoid, $\mathcal{G}(C_P)(x,x) = \{x \leq x\}$ is trivial.

thus. $H_n(C_P, \mathbb{Z}) \simeq H_n(\mathcal{G}(C_P), \mathbb{Z}) \simeq \underbrace{H_n(\{*\}, \mathbb{Z})}_{\text{useless}}$

IV. Complex braid groups

Def: A complex reflection group (GRC) is a finite subgroup of $GL_n(\mathbb{C})$ generated by reflections (fixing pointwise a hypeplane).

If $W \leq GL_n(\mathbb{C})$ is a reflection group. W acts freely on

$$X = \{ v \in \mathbb{C}^n \mid \forall \text{ reflection } s \in W,\ s.v \neq v \}.$$

The Braid group of W is defined as $\mathcal{B}(W) := \pi_1(X/W)$ ↳ path connected.

Harder to understand than reflection group, without torsion...

Thm: Every CBG is equivalent to the enveloping groupoid of a Garside category — Dehornoy, Paris, Digne, Michel, Bessis, Bouë, Malle, Rouquier...

→ case by case, hard.

! A lot of these are actually Garside groups directly.

The only problematic exception is the group $\mathcal{B}_{31}$. By using DL on categories, we were able to comput.

$H_n(\mathcal{B}_{31}, \mathbb{N})$	0	1	2	3	4	$= n$
$\mathbb{Z}$.	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	known hard
$\mathbb{Z}_\varepsilon$ "sign rep"	$\mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_{20}$	0	
$R\ \mathbb{Q}[t,t^{-1}]$ nilpo.	$\mathbb{Q}$	0	R/ϕ_6	$R/\frac{t^{10}-1}{t+1}\phi_{15}$	0.	

+ part res on $\mathbb{F}_2$ $\mathbb{F}_3$.

$$H_2(\mathbb{F}_2) = \phi_1\phi_6 = H_2(\mathbb{F}_3).$$

$$H_3(\mathbb{F}_2) = (t^{10}-1)\phi_{15} = \frac{t-1}{t+1}(t^{10}-1)\phi_{15}.$$