Garside groupoids and complex braid groups

Ph.D. thesis

Owen Garnier

Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
Université de Picardie Jules Verne
ED STS 585

November 28th, 2024

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Irreducible CRGs were classified by Shephard and Todd '54:

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Irreducible CRGs were classified by Shephard and Todd '54:

- an infinite series $G(de, e, n)$, where d, e, n are positive integers.

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Irreducible CRGs were classified by Shephard and Todd '54:

- an infinite series $G(de, e, n)$, where d, e, n are positive integers.
- 34 exceptional groups $G_4, \ldots, G_{37}$.

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Irreducible CRGs were classified by Shephard and Todd '54:

- an infinite series $G(de, e, n)$, where d, e, n are positive integers.
- 34 exceptional groups $G_4, \ldots, G_{37}$.

It is possible to use case-by-case proofs:

Corollary

W irreducible can be generated by a set of $n+1$ reflections.

Notations for complex reflection groups

V: complex vector space of dimension $n < \infty$.
W: complex reflection group (CRG).
X: complement in V of the reflecting hyperplanes of W (regular vectors).
X/W: regular orbit space.

Irreducible CRGs were classified by Shephard and Todd '54:

- an infinite series $G(de, e, n)$, where d, e, n are positive integers.
- 34 exceptional groups $G_4, \ldots, G_{37}$.

It is possible to use case-by-case proofs:

Corollary

W irreducible can be generated by a set of $n+1$ reflections.

If W can be generated by n reflections, then W is **well-generated**.

Broué, Malle, Rouquier ’98

Motivated by representation theory of finite reductive groups.

Broué, Malle, Rouquier '98

Motivated by representation theory of finite reductive groups.

- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.

Broué, Malle, Rouquier '98

Motivated by representation theory of finite reductive groups.

- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.
- Short exact sequence $1 \to P \to B \to W \to 1$.

Broué, Malle, Rouquier '98

Motivated by representation theory of finite reductive groups.

- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.
- Short exact sequence $1 \to P \to B \to W \to 1$.
- B is generated by (distinguished) **braided reflections**.

Broué, Malle, Rouquier '98

Motivated by representation theory of finite reductive groups.

- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.
- Short exact sequence $1 \to P \to B \to W \to 1$.
- B is generated by (distinguished) **braided reflections**.
- When W is a Coxeter group, B is the **Artin group** of W.

Broué, Malle, Rouquier '98

Motivated by representation theory of finite reductive groups.

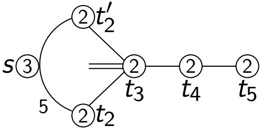
- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.
- Short exact sequence $1 \to P \to B \to W \to 1$.
- B is generated by (distinguished) **braided reflections**.
- When W is a Coxeter group, B is the **Artin group** of W.
- **BMR diagrams**, summarizing presentations of W.

Broué, Malle, Rouquier '98

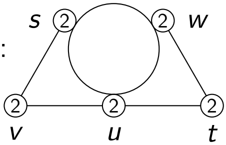
Motivated by representation theory of finite reductive groups.

- (Pure) Braid group: $B := \pi_1(X/W)$ (resp. $P := \pi_1(X)$).
- Notation: $B_i = B(G_i)$ and $B(de, e, n) = B(G(de, e, n))$.
- Short exact sequence $1 \to P \to B \to W \to 1$.
- B is generated by (distinguished) **braided reflections**.
- When W is a Coxeter group, B is the **Artin group** of W.
- **BMR diagrams**, summarizing presentations of W.

$G(12, 4, 5)$:



G_{31} :



Questions raised by BMR

- Determination of the center of B and of P.

Questions raised by BMR

- Determination of the center of B and of P.
  Conjecture: for W irreducible, $Z(B)$ and $Z(P)$ cyclic, and there is a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.

Questions raised by BMR

- Determination of the center of B and of P.
  Conjecture: for W irreducible, $Z(B)$ and $Z(P)$ cyclic, and there is a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.
- Does the BMR-diagram provide a presentation of the braid group ?

Questions raised by BMR

- Determination of the center of B and of P.
  Conjecture: for W irreducible, $Z(B)$ and $Z(P)$ cyclic, and there is a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.
- Does the BMR-diagram provide a presentation of the braid group ?
- Does the monoid defined by the same presentation embed in the braid group ?

Questions raised by BMR

- Determination of the center of B and of P.
  Conjecture: for W irreducible, $Z(B)$ and $Z(P)$ cyclic, and there is a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.
- Does the BMR-diagram provide a presentation of the braid group ?
- Does the monoid defined by the same presentation embed in the braid group ?
- $+$ Is X/W a $K(\pi, 1)$-space ?

Questions raised by BMR

- Determination of the center of B and of P.
  Conjecture: for W irreducible, $Z(B)$ and $Z(P)$ cyclic, and there is a short exact sequence $1 \to Z(P) \to Z(B) \to Z(W) \to 1$.
- Does the BMR-diagram provide a presentation of the braid group ?
- Does the monoid defined by the same presentation embed in the braid group ?
- $+$ Is X/W a $K(\pi, 1)$-space ?

At the time of BMR, most of these questions are solved for all CRGs but

$$G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}.$$

Garside groups

Definition (Dehornoy, Paris '99)

Garside structure on a group G: monoid $M \subset G$ and $\Delta \in M$ such that

- M generates G as a group.
- Left- and right-divisibility are lattice orders on M.
- For all $x \in M$, there is a bound r on the length of a product $x = s_1 \cdots s_r$ with $s_i \neq 1$ in M.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta)$ is finite and generates M.

Garside groups

Definition (Dehornoy, Paris '99)

Garside structure on a group G: monoid $M \subset G$ and $\Delta \in M$ such that

- M generates G as a group.
- Left- and right-divisibility are lattice orders on M.
- For all $x \in M$, there is a bound r on the length of a product $x = s_1 \cdots s_r$ with $s_i \neq 1$ in M.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta)$ is finite and generates M.

If (G, M, Δ) is a Garside group, then

- Solution of conjugacy problem in G $\to$ determination of $Z(G)$.
- Explicit construction of a $K(G, 1)$ (Garside nerve).
- Presentation of G (germ of simple elements, right-complement).

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid**
  (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid** (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).
- "Shephard groups": same monoid as some Coxeter group (Orlik-Solomon '88).

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid** (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).
- "Shephard groups": same monoid as some Coxeter group (Orlik-Solomon '88).
- Rank 2 exceptional groups: *ad hoc* structures (Dehornoy-Paris '99, Picantin '00).

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid** (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).
- "Shephard groups": same monoid as some Coxeter group (Orlik-Solomon '88).
- Rank 2 exceptional groups: *ad hoc* structures (Dehornoy-Paris '99, Picantin '00).

This left out

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid** (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).
- "Shephard groups": same monoid as some Coxeter group (Orlik-Solomon '88).
- Rank 2 exceptional groups: *ad hoc* structures (Dehornoy-Paris '99, Picantin '00).

This left out

- In the infinite series, $B(de, e, n)$ with $n \geqslant 3$ and $d \geqslant 2$ or $e \geqslant 3$.

Garside structures for irreducible complex braid groups

At the beginning of the 2000s, many complex braid groups were already known to admit Garside structures:

- Coxeter group: **Artin-Tits monoid** (Deligne '72, Brieskorn-Saito '72, Dehornoy-Paris '99).
- "Shephard groups": same monoid as some Coxeter group (Orlik-Solomon '88).
- Rank 2 exceptional groups: *ad hoc* structures (Dehornoy-Paris '99, Picantin '00).

This left out

- In the infinite series, $B(de, e, n)$ with $n \geqslant 3$ and $d \geqslant 2$ or $e \geqslant 3$.
- $B_{24}, B_{27}, B_{29}, B_{31}, B_{33}, B_{34}$.

Bessis' dual braid monoid

W well-generated.

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

This excludes

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

This excludes

- $G(de, e, n)$ with $d, e, n \geqslant 2$.

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

This excludes

- $G(de, e, n)$ with $d, e, n \geqslant 2$.
- 7 exceptional groups of rank 2.

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

This excludes

- $G(de, e, n)$ with $d, e, n \geqslant 2$.
- 7 exceptional groups of rank 2.
- G_{31}, which has rank 4.

Bessis' dual braid monoid

W well-generated.

Theorem (Bessis '15 + Ripoll '10 and Douvropoulos '17)

B is a Garside group for the **dual braid monoid** (G, M, Δ).

- *$Z(B)$ is cyclic as conjectured by BMR.*
- *The space X/W is a $K(\pi, 1)$-space.*
- *M provides a presentation of B.*

This excludes

- $G(de, e, n)$ with $d, e, n \geqslant 2$.
- 7 exceptional groups of rank 2.
- G_{31}, which has rank 4.

The questions of BMR are still unanswered for G_{31}.

Regular elements (Springer '74)

d: positive integer.

$V(g,\zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Regular elements (Springer '74)

d: positive integer.

$V(g,\zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Definition (Springer '74)

$g \in W$ is d-**regular** if $V(g,\zeta_d) \cap X \neq \varnothing$, where $\zeta_d := \exp(\frac{2i\pi}{d})$.

Regular elements (Springer '74)

d: positive integer.
$V(g,\zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Definition (Springer '74)

$g \in W$ is d-**regular** if $V(g,\zeta_d) \cap X \neq \varnothing$, where $\zeta_d := \exp(\frac{2i\pi}{d})$.

Theorem (Springer '74, Bessis '15)

If $g \in W$ is d-regular, then $W_g := C_W(g)$ acts on $V(g,\zeta_d)$ as a CRG.

Regular elements (Springer '74)

d: positive integer.
$V(g, \zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Definition (Springer '74)

$g \in W$ is d-**regular** if $V(g, \zeta_d) \cap X \neq \varnothing$, where $\zeta_d := \exp(\frac{2i\pi}{d})$.

Theorem (Springer '74, Bessis '15)

If $g \in W$ is d-regular, then $W_g := C_W(g)$ acts on $V(g, \zeta_d)$ as a CRG. Furthermore, $B(W_g) \simeq \pi_1((X/W)^{\mu_d}) =: B^{\mu_d}$.

Regular elements (Springer '74)

d: positive integer.
$V(g,\zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Definition (Springer '74)

$g \in W$ is d-**regular** if $V(g,\zeta_d) \cap X \neq \varnothing$, where $\zeta_d := \exp(\frac{2i\pi}{d})$.

Theorem (Springer '74, Bessis '15)

If $g \in W$ is d-regular, then $W_g := C_W(g)$ acts on $V(g,\zeta_d)$ as a CRG. Furthermore, $B(W_g) \simeq \pi_1((X/W)^{\mu_d}) =: B^{\mu_d}$.

Examples include

- $G(d,1,n) \subset G(1,1,dn)$ as centralizer of a d-regular element.
- $G_{28}(\simeq F_4) \subset G_{35}(\simeq E_6)$ as centralizer of a 2-regular element.

Regular elements (Springer '74)

d: positive integer.
$V(g,\zeta)$: ζ-eigenspace of $g \in \mathrm{GL}(V)$.

Definition (Springer '74)

$g \in W$ is d-**regular** if $V(g,\zeta_d) \cap X \neq \varnothing$, where $\zeta_d := \exp(\frac{2i\pi}{d})$.

Theorem (Springer '74, Bessis '15)

If $g \in W$ is d-regular, then $W_g := C_W(g)$ acts on $V(g,\zeta_d)$ as a CRG. Furthermore, $B(W_g) \simeq \pi_1((X/W)^{\mu_d}) =: B^{\mu_d}$.

Examples include

- $G(d,1,n) \subset G(1,1,dn)$ as centralizer of a d-regular element.
- $G_{28}(\simeq F_4) \subset G_{35}(\simeq E_6)$ as centralizer of a 2-regular element.
- $G_{31} \subset G_{37}(\simeq E_8)$ as centralizer of a 4-regular element.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.
$\mathcal{U}^{\mu_d}$: a particular dense subset of $(X/W)^{\mu_d}$.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.
$\mathcal{U}^{\mu_d}$: a particular dense subset of $(X/W)^{\mu_d}$.

Definition (Bessis '15)

The relative braid groupoid is $\mathcal{B}_p^q := \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d})$.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.
$\mathcal{U}^{\mu_d}$: a particular dense subset of $(X/W)^{\mu_d}$.

Definition (Bessis '15)

The relative braid groupoid is $\mathcal{B}_p^q := \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d})$.

Problem: $\mathcal{U}^{\mu_d}$ has several path connected components. $\mathcal{B}_p^q$ is not a group, but a **groupoid**.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.
$\mathcal{U}^{\mu_d}$: a particular dense subset of $(X/W)^{\mu_d}$.

Definition (Bessis '15)

The relative braid groupoid is $\mathcal{B}_p^q := \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d})$.

Problem: $\mathcal{U}^{\mu_d}$ has several path connected components. $\mathcal{B}_p^q$ is not a group, but a **groupoid**.

- Objects: path connected components of $\mathcal{U}^{\mu_d}$.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W.
$p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.
$\mathcal{U}^{\mu_d}$: a particular dense subset of $(X/W)^{\mu_d}$.

Definition (Bessis '15)

The relative braid groupoid is $\mathcal{B}_p^q := \pi_1((X/W)^{\mu_d}, \mathcal{U}^{\mu_d})$.

Problem: $\mathcal{U}^{\mu_d}$ has several path connected components. $\mathcal{B}_p^q$ is not a group, but a **groupoid**.

- Objects: path connected components of $\mathcal{U}^{\mu_d}$.
- Morphisms: paths with endpoints in $\mathcal{U}^{\mu_d}$ relative to homotopy leaving enpoints in $\mathcal{U}^{\mu_d}$.

Garside groupoids

$\mathcal{G}$: groupoid. $\mathcal{C} \subset \mathcal{G}$ a category.

Garside groupoids

$\mathcal{G}$: groupoid. $\mathcal{C} \subset \mathcal{G}$ a category.
$\preccurlyeq$ denotes left-divisibility: for $u \in \mathrm{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(u, -)$, we have

$$f \preccurlyeq g \Leftrightarrow \exists h \in \mathcal{C} \mid fh = g.$$

Garside groupoids

$\mathcal{G}$: groupoid. $\mathcal{C} \subset \mathcal{G}$ a category.
$\preccurlyeq$ denotes left-divisibility: for $u \in \mathrm{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(u, -)$, we have

$$f \preccurlyeq g \Leftrightarrow \exists h \in \mathcal{C} \mid fh = g.$$

$\succcurlyeq$ denotes right-divisibility: for $u \in \mathrm{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(-, u)$, we have

$$g \succcurlyeq f \Leftrightarrow \exists h' \in \mathcal{C} \mid g = h'f.$$

Garside groupoids

$\mathcal{G}$: groupoid. $\mathcal{C} \subset \mathcal{G}$ a category.
$\preccurlyeq$ denotes left-divisibility: for $u \in \mathrm{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(u, -)$, we have

$$f \preccurlyeq g \Leftrightarrow \exists h \in \mathcal{C} \mid fh = g.$$

$\succcurlyeq$ denotes right-divisibility: for $u \in \mathrm{Ob}(\mathcal{C})$ and $f, g \in \mathcal{C}(-, u)$, we have

$$g \succcurlyeq f \Leftrightarrow \exists h' \in \mathcal{C} \mid g = h'f.$$

For $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$, we set

$$\mathrm{Div}(\Delta) = \{s \in \mathcal{C} \mid \exists u \in \mathrm{Ob}(\mathcal{C}), s \preccurlyeq \Delta(u)\},$$
$$\mathrm{Div}_R(\Delta) = \{s \in \mathcal{C} \mid \exists u \in \mathrm{Ob}(\mathcal{C}), \Delta(u) \succcurlyeq s\}.$$

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.
- For all $u \in \mathrm{Ob}(\mathcal{C})$, $(\mathcal{C}(u, -), \preccurlyeq)$ and $(\mathcal{C}(-, u), \succcurlyeq)$ are lattices.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.
- For all $u \in \mathrm{Ob}(\mathcal{C})$, $(\mathcal{C}(u, -), \preccurlyeq)$ and $(\mathcal{C}(-, u), \succcurlyeq)$ are lattices.
- For all $f \in \mathcal{C}$, there is a bound r on the length of a composition $f = s_1 \cdots s_r$ with $s_i \neq 1$ in $\mathcal{C}$.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.
- For all $u \in \mathrm{Ob}(\mathcal{C})$, $(\mathcal{C}(u, -), \preccurlyeq)$ and $(\mathcal{C}(-, u), \succcurlyeq)$ are lattices.
- For all $f \in \mathcal{C}$, there is a bound r on the length of a composition $f = s_1 \cdots s_r$ with $s_i \neq 1$ in $\mathcal{C}$.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta)$ is finite and generates $\mathcal{C}$.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.
- For all $u \in \mathrm{Ob}(\mathcal{C})$, $(\mathcal{C}(u, -), \preccurlyeq)$ and $(\mathcal{C}(-, u), \succcurlyeq)$ are lattices.
- For all $f \in \mathcal{C}$, there is a bound r on the length of a composition $f = s_1 \cdots s_r$ with $s_i \neq 1$ in $\mathcal{C}$.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta)$ is finite and generates $\mathcal{C}$.

Δ is called a **Garside map**. $(\mathcal{G}, \mathcal{C}, \Delta)$ is called a **Garside groupoid**.

Garside groupoids

Definition

Garside structure on a groupoid $\mathcal{G}$: A category $\mathcal{C} \subset \mathcal{G}$ and $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ such that

- For all $u \in \mathrm{Ob}(\mathcal{C})$, $\Delta(u) \in \mathcal{C}(u, -)$.
- $\mathcal{C}$ generates $\mathcal{G}$ as a groupoid.
- For all $u \in \mathrm{Ob}(\mathcal{C})$, $(\mathcal{C}(u, -), \preccurlyeq)$ and $(\mathcal{C}(-, u), \succcurlyeq)$ are lattices.
- For all $f \in \mathcal{C}$, there is a bound r on the length of a composition $f = s_1 \cdots s_r$ with $s_i \neq 1$ in $\mathcal{C}$.
- $\mathrm{Div}(\Delta) = \mathrm{Div}_R(\Delta)$ is finite and generates $\mathcal{C}$.

Δ is called a **Garside map**. $(\mathcal{G}, \mathcal{C}, \Delta)$ is called a **Garside groupoid**.

For $u \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u, u)$ is a **weak Garside group**.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W. $p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W. $p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Theorem (Bessis '15)

There is a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ *with* $\mathcal{G} \simeq \mathcal{B}_p^q$ *(isomorphism).*
The groupoid $\mathcal{G}$ *is the* **Springer Groupoid** *attached to* (B, d).

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W. $p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Theorem (Bessis '15)

There is a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ with $\mathcal{G} \simeq \mathcal{B}_p^q$ (isomorphism).
The groupoid $\mathcal{G}$ is the **Springer Groupoid** *attached to (B, d).*

The Springer groupoid attached to (B, d) is equivalent to B^{μ_d}.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W. $p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Theorem (Bessis '15)

There is a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ with $\mathcal{G} \simeq \mathcal{B}_p^q$ (isomorphism).
The groupoid $\mathcal{G}$ is the **Springer Groupoid** *attached to (B, d).*

The Springer groupoid attached to (B, d) is equivalent to B^{μ_d}.

Corollary

The group B^{μ_d} is a weak Garside group.
In particular, $B_{31} \simeq (B_{37})^{\mu_4}$ is a weak Garside group.

Bessis' work for regular centralizers

W well-generated with highest degree h.
d: regular number for W. $p := \frac{d}{d \wedge h}$, $q := \frac{h}{d \wedge h}$.

Theorem (Bessis '15)

There is a Garside groupoid $(\mathcal{G}, \mathcal{C}, \Delta)$ with $\mathcal{G} \simeq \mathcal{B}_p^q$ (isomorphism).
The groupoid $\mathcal{G}$ is the **Springer Groupoid** *attached to (B, d).*

The Springer groupoid attached to (B, d) is equivalent to B^{μ_d}.

Corollary

The group B^{μ_d} is a weak Garside group.
In particular, $B_{31} \simeq (B_{37})^{\mu_4}$ is a weak Garside group.

Corollary (Bessis '15)

The space $(X/W)^{\mu_d}$ is a $K(\pi, 1)$ space.

Lifting regular elements

Broué, Michel 96: Possibility of “lifting” the theory of regular elements to braid groups.

Lifting regular elements

Broué, Michel 96: Possibility of "lifting" the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t}*$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Lifting regular elements

Broué, Michel 96: Possibility of “lifting” the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t}*$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Theorem (Bessis '15)

Assume that W is irreducible and well-generated.

Lifting regular elements

Broué, Michel 96: Possibility of “lifting” the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t} *$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Theorem (Bessis '15)

Assume that W is irreducible and well-generated.

- *d is regular for W if and only if d-regular braids exist in B.*

Lifting regular elements

Broué, Michel 96: Possibility of “lifting” the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t}*$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Theorem (Bessis '15)

Assume that W is irreducible and well-generated.

- *d is regular for W if and only if d-regular braids exist in B.*
- *If so, then all d-regular braids are conjugate in B. The image in W of a d-regular braid is a d-regular element.*

Lifting regular elements

Broué, Michel 96: Possibility of "lifting" the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t}*$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Theorem (Bessis '15)

Assume that W is irreducible and well-generated.

- *d is regular for W if and only if d-regular braids exist in B.*
- *If so, then all d-regular braids are conjugate in B. The image in W of a d-regular braid is a d-regular element.*
- *If $\rho \in B$ is d-regular, then $C_B(\rho) \simeq B^{\mu_d}$.*

Lifting regular elements

Broué, Michel 96: Possibility of "lifting" the theory of regular elements to braid groups.

Definition

The **full twist** is the homotopy class τ in B of $t \mapsto e^{2i\pi t}*$.
An element $b \in B$ is d-**regular** if $b^d = \tau$.

Theorem (Bessis '15, G. '23)

For all W.

- *d is regular for W if and only if d-regular braids exist in B.*
- *If so, then all d-regular braids are conjugate in B. The image in W of a d-regular braid is a d-regular element.*
- *If $\rho \in B$ is d-regular, then $C_B(\rho) \simeq B^{\mu_d}$.*

Reidemeister-Schreier method for groupoids

S : oriented graph. R : set of couples of paths in S.

Reidemeister-Schreier method for groupoids

S : oriented graph. R : set of couples of paths in S.
$\mathcal{G} := \langle S \mid R \rangle$ presented groupoid. For $u \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u,u)$ is a group.

Reidemeister-Schreier method for groupoids

S : oriented graph. R : set of couples of paths in S.
$\mathcal{G} := \langle S \mid R \rangle$ presented groupoid. For $u \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u, u)$ is a group.

Definition

A **Schreier transversal** is a family $T = \{t_v\}_{v \in \mathrm{Ob}(\mathcal{S})}$ stable under prefix and such that for all $v \in \mathrm{Ob}(S)$, $t_v : u \to v$.

Reidemeister-Schreier method for groupoids

S : oriented graph. R : set of couples of paths in S.
$\mathcal{G} := \langle S \mid R \rangle$ presented groupoid. For $u \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u,u)$ is a group.

Definition

A **Schreier transversal** is a family $T = \{t_v\}_{v \in \mathrm{Ob}(\mathcal{S})}$ stable under prefix and such that for all $v \in \mathrm{Ob}(S)$, $t_v : u \to v$.

For such a transversal, and $s \in S(v, v')$, $\gamma(s) = t_v s (t_{v'})^{-1} \in \mathcal{G}(u,u)$.

Reidemeister-Schreier method for groupoids

S : oriented graph. R : set of couples of paths in S.
$\mathcal{G} := \langle S \mid R \rangle$ presented groupoid. For $u \in \mathrm{Ob}(\mathcal{G})$, $\mathcal{G}(u,u)$ is a group.

Definition

A **Schreier transversal** is a family $T = \{t_v\}_{v \in \mathrm{Ob}(\mathcal{S})}$ stable under prefix and such that for all $v \in \mathrm{Ob}(S)$, $t_v : u \to v$.

For such a transversal, and $s \in S(v,v')$, $\gamma(s) = t_v s (t_{v'})^{-1} \in \mathcal{G}(u,u)$.

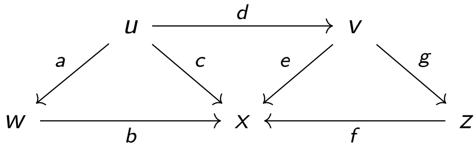
Proposition (Reidemeister-Schreier method for groupoids, G. '21)

The group $\mathcal{G}(u,u)$ is generated by $\gamma(s)$ for $s \in S$, with the relations

$$\gamma(s_1)\cdots\gamma(s_p) = \gamma(t_1)\cdots\gamma(t_q)$$

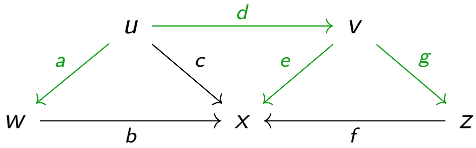
for $s_1 \cdots s_p = t_1 \cdots t_q \in R$.

Reidemeister-Schreier method for groupoids, example



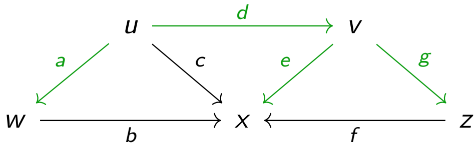
with $ab = c$

Reidemeister-Schreier method for groupoids, example



with $ab = c$

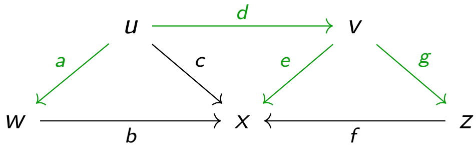
Reidemeister-Schreier method for groupoids, example



with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.

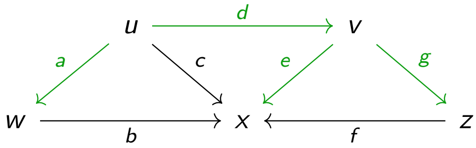
Reidemeister-Schreier method for groupoids, example



with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,

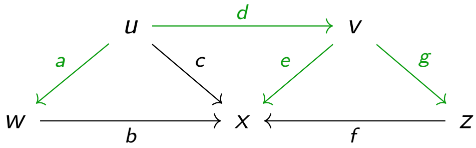
Reidemeister-Schreier method for groupoids, example



with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,
- $\gamma(c) = ce^{-1}d^{-1}$,

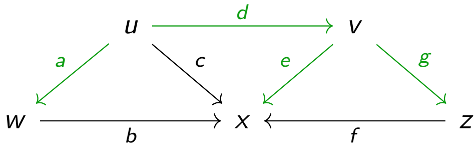
Reidemeister-Schreier method for groupoids, example



with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,
- $\gamma(c) = ce^{-1}d^{-1}$,
- $\gamma(f) = dgfe^{-1}d^{-1}$.

Reidemeister-Schreier method for groupoids, example

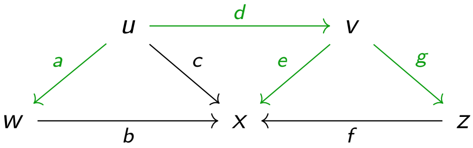


with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,
- $\gamma(c) = ce^{-1}d^{-1}$,
- $\gamma(f) = dgfe^{-1}d^{-1}$.

$ab = c$ induces $\gamma(b) = \gamma(c)$ (which we can check directly).

Reidemeister-Schreier method for groupoids, example



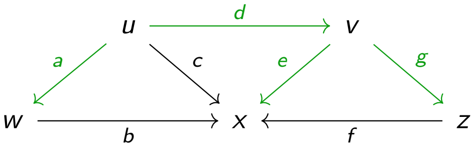
with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,
- $\gamma(c) = ce^{-1}d^{-1}$,
- $\gamma(f) = dgfe^{-1}d^{-1}$.

$ab = c$ induces $\gamma(b) = \gamma(c)$ (which we can check directly).

$$\mathcal{G}(u, u) = \langle \gamma(b), \gamma(c), \gamma(f) \rangle \simeq \langle X, Y, Z \mid X = Y \rangle = \langle X, Z \mid \varnothing \rangle.$$

Reidemeister-Schreier method for groupoids, example



with $ab = c$

- $\gamma(a) = \gamma(d) = \gamma(e) = \gamma(g) = 1_u$.
- $\gamma(b) = abe^{-1}d^{-1}$,
- $\gamma(c) = ce^{-1}d^{-1}$,
- $\gamma(f) = dgfe^{-1}d^{-1}$.

$ab = c$ induces $\gamma(b) = \gamma(c)$ (which we can check directly).

$$\mathcal{G}(u, u) = \langle \gamma(b), \gamma(c), \gamma(f) \rangle \simeq \langle X, Y, Z \mid X = Y \rangle = \langle X, Z \mid \varnothing \rangle.$$

The method requires heavy simplifications.

Presentation of B_{31}

The Springer groupoid attached to $(G_{37}, 4)$ has a presentation

$$\mathcal{B}_{31} = \langle \text{atoms} \mid \text{commutative squares of atoms} \rangle.$$

Presentation of B_{31}

The Springer groupoid attached to $(G_{37}, 4)$ has a presentation

$$\mathcal{B}_{31} = \langle \text{atoms (660)} \mid \text{commutative squares of atoms (4230)} \rangle.$$

Presentation of B_{31}

The Springer groupoid attached to $(G_{37}, 4)$ has a presentation

$$\mathcal{B}_{31} = \langle \text{atoms (660)} \mid \text{commutative squares of atoms (4230)} \rangle.$$

By applying the Reidemeister-Schreier method, we obtain

Theorem (G. '21)

The complex braid group B_{31} admits the following presentation

$$\left\langle s, t, u, v, w \;\middle|\; \begin{array}{l} st = ts,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw \end{array} \right\rangle.$$

Presentation of B_{31}

The Springer groupoid attached to $(G_{37}, 4)$ has a presentation

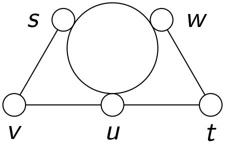
$$\mathcal{B}_{31} = \langle \text{atoms (660)} \mid \text{commutative squares of atoms (4230)} \rangle.$$

By applying the Reidemeister-Schreier method, we obtain

Theorem (G. '21)

The complex braid group B_{31} admits the following presentation

$$\left\langle s, t, u, v, w \;\middle|\; \begin{array}{l} st = ts,\ vt = tv,\ wv = vw, \\ suw = uws = wsu, \\ svs = vsv,\ vuv = uvu,\ utu = tut,\ twt = wtw \end{array} \right\rangle.$$



Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.
Main argument:

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.
Main argument:

- B has a Garside structure (G, M, Δ).

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.
Main argument:

- B has a Garside structure (G, M, Δ).
- The atoms of M represent all braided reflections up to conjugacy.

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.
Main argument:

- B has a Garside structure (G, M, Δ).
- The atoms of M represent all braided reflections up to conjugacy.
- Show that for s atom and $x \in G$, then $xs^n = s^n x$ implies $xs = sx$.

Center of $Z(P_{31})$

Theorem (Digne, Marin, Michel '11)

Let W be irreducible, and let $U \subset B$ have finite index. We have $Z(U) \subset Z(B)$.

The proof is Garside theoretic **except for** G_{31}.
Main argument:

- B has a Garside structure (G, M, Δ).
- The atoms of M represent all braided reflections up to conjugacy.
- Show that for s atom and $x \in G$, then $xs^n = s^n x$ implies $xs = sx$.

Problem: how to recognize braided reflections in Springer groupoids ?

Center of $Z(P_{31})$

W well-generated irreducible CRG. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: Springer groupoid attached to (B, d). $u \in \mathrm{Ob}(\mathcal{G})$.

Center of $Z(P_{31})$

W well-generated irreducible CRG. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: Springer groupoid attached to (B, d). $u \in \mathrm{Ob}(\mathcal{G})$.
The monoid $\mathcal{C}(u, u)$ contains a finite set of well-defined **atomic loops**.

Center of $Z(P_{31})$

W well-generated irreducible CRG. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: Springer groupoid attached to (B, d). $u \in \mathrm{Ob}(\mathcal{G})$.
The monoid $\mathcal{C}(u, u)$ contains a finite set of well-defined **atomic loops**.

Proposition (G. '23)

The distinguished braided reflections of $B^{\mu_d} \simeq \mathcal{G}(u, u)$ are exactly the conjugates in $\mathcal{G}$ of the atomic loops. If s is an atomic loop in $\mathcal{C}$, and $x \in \mathcal{G}$ is such that $xs^n = s^n x$, then we have $xs = sx$.

Center of $Z(P_{31})$

W well-generated irreducible CRG. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: Springer groupoid attached to (B, d). $u \in \mathrm{Ob}(\mathcal{G})$.
The monoid $\mathcal{C}(u, u)$ contains a finite set of well-defined **atomic loops**.

Proposition (G. '23)

The distinguished braided reflections of $B^{\mu_d} \simeq \mathcal{G}(u, u)$ are exactly the conjugates in $\mathcal{G}$ of the atomic loops. If s is an atomic loop in $\mathcal{C}$, and $x \in \mathcal{G}$ is such that $xs^n = s^n x$, then we have $xs = sx$.

Corollary

Let $\sigma \in B_{31}$ be a braided reflection. If $x\sigma^n = \sigma^n x$, then $x\sigma = \sigma x$.

Center of $Z(P_{31})$

W well-generated irreducible CRG. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: Springer groupoid attached to (B, d). $u \in \mathrm{Ob}(\mathcal{G})$.
The monoid $\mathcal{C}(u, u)$ contains a finite set of well-defined **atomic loops**.

Proposition (G. '23)

The distinguished braided reflections of $B^{\mu_d} \simeq \mathcal{G}(u, u)$ are exactly the conjugates in $\mathcal{G}$ of the atomic loops. If s is an atomic loop in $\mathcal{C}$, and $x \in \mathcal{G}$ is such that $xs^n = s^n x$, then we have $xs = sx$.

Corollary

Let $\sigma \in B_{31}$ be a braided reflection. If $x\sigma^n = \sigma^n x$, then $x\sigma = \sigma x$.

Note: For all $u \in \mathrm{Ob}(\mathcal{B}_{31})$, atomic loops of $\mathcal{C}_{31}(u, u)$ generate $\mathcal{B}_{31}(u, u)$.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).
- If $\gamma : * \to x$ is a **normal ray/capillary path**, we obtain a morphism $\pi_1(U \cap X/W, \gamma(t)) \to \pi_1(X/W, \gamma(t)) \to B$ for some $t < 1$.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).
- If $\gamma : * \to x$ is a **normal ray/capillary path**, we obtain a morphism $\pi_1(U \cap X/W, \gamma(t)) \to \pi_1(X/W, \gamma(t)) \to B$ for some $t < 1$.
- The image in B of such a morphism is called a **parabolic subgroup**.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).
- If $\gamma : * \to x$ is a **normal ray/capillary path**, we obtain a morphism $\pi_1(U \cap X/W, \gamma(t)) \to \pi_1(X/W, \gamma(t)) \to B$ for some $t < 1$.
- The image in B of such a morphism is called a **parabolic subgroup**.
- Replacing γ with another path η yields a conjugate parabolic subgroup of B.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).
- If $\gamma : * \to x$ is a **normal ray/capillary path**, we obtain a morphism $\pi_1(U \cap X/W, \gamma(t)) \to \pi_1(X/W, \gamma(t)) \to B$ for some $t < 1$.
- The image in B of such a morphism is called a **parabolic subgroup**.
- Replacing γ with another path η yields a conjugate parabolic subgroup of B.

The definition is purely topological.

Parabolic subgroups of complex braid groups, definition

W arbitrary.
$\mathcal{H}$: complement of X/W in V/W.

- For $x \in \mathcal{H}$, $\pi_1(U \cap (X/W))$ does not depend on $U \ni x$ small enough. We call it a **local fundamental group** (at x).
- If $\gamma : * \to x$ is a **normal ray/capillary path**, we obtain a morphism $\pi_1(U \cap X/W, \gamma(t)) \to \pi_1(X/W, \gamma(t)) \to B$ for some $t < 1$.
- The image in B of such a morphism is called a **parabolic subgroup**.
- Replacing γ with another path η yields a conjugate parabolic subgroup of B.

The definition is purely topological.
$B(2,1,n)$ and $B(d,1,n)$ are isomorphic and have the same collection of parabolic subgroups.

Parabolic subgroups of complex braid groups, first results

Lemma

Parabolic subgroups of B of rank 1 are exactly the subgroups generated by distinguished braided reflections.

Parabolic subgroups of complex braid groups, first results

Lemma

Parabolic subgroups of B of rank 1 are exactly the subgroups generated by distinguished braided reflections.

Proposition

Let $B_0 \subset B$ a parabolic subgroup. The image W_0 of B_0 in W is a parabolic subgroup of W, and $B_0 \simeq B(W_0)$.

Parabolic subgroups of complex braid groups, first results

Lemma

Parabolic subgroups of B of rank 1 are exactly the subgroups generated by distinguished braided reflections.

Proposition

Let $B_0 \subset B$ a parabolic subgroup. The image W_0 of B_0 in W is a parabolic subgroup of W, and $B_0 \simeq B(W_0)$.

Proposition

B_1, B_2 parabolic subgroups of B. W_1, W_2 their images in W.

- W_1 and W_2 are conjugate if and only if B_1 and B_2 are conjugate.
- $W_1 = W_2$ if and only if B_1 and B_2 are conjugate by some $x \in P$.

Main results of González-Meneses and Marin

W: irreducible CRG distinct from G_{31}.

Main results of González-Meneses and Marin

W: irreducible CRG distinct from G_{31}.

Theorem (González-Meneses, Marin '22)

Every $x \in B$ is contained in a smallest parabolic subgroup $\mathrm{PC}(x)$ of B. We have $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ for all $m \neq 0$.

Main results of González-Meneses and Marin

W: irreducible CRG distinct from G_{31}.

Theorem (González-Meneses, Marin '22)

Every $x \in B$ is contained in a smallest parabolic subgroup $\mathrm{PC}(x)$ of B. We have $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ for all $m \neq 0$.

Theorem (González-Meneses, Marin '22)

Parabolic subgroups of B are stable under intersection.

Main results of González-Meneses and Marin

W: irreducible CRG distinct from G_{31}.

Theorem (González-Meneses, Marin '22)

Every $x \in B$ is contained in a smallest parabolic subgroup $\mathrm{PC}(x)$ *of B. We have* $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ *for all $m \neq 0$.*

Theorem (González-Meneses, Marin '22)

Parabolic subgroups of B are stable under intersection.

Irreducible parabolic subgroups of B make up the vertices of a graph Γ. In Γ, B_1, B_2 are adjacent if $B_1 \neq B_2$ and either $B_1 \subset B_2, B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = \{1\}$.

Main results of González-Meneses and Marin

W: irreducible CRG distinct from G_{31}.

Theorem (González-Meneses, Marin '22)

Every $x \in B$ is contained in a smallest parabolic subgroup $\mathrm{PC}(x)$ of B. We have $\mathrm{PC}(x^m) = \mathrm{PC}(x)$ for all $m \neq 0$.

Theorem (González-Meneses, Marin '22)

Parabolic subgroups of B are stable under intersection.

Irreducible parabolic subgroups of B make up the vertices of a graph Γ. In Γ, B_1, B_2 are adjacent if $B_1 \neq B_2$ and either $B_1 \subset B_2, B_2 \subset B_1$, or $B_1 \cap B_2 = [B_1, B_2] = \{1\}$.

Theorem (González-Meneses, Marin '22)

B_1, B_2 are adjacent if and only if $z_{B_1} z_{B_2} = z_{B_2} z_{B_1}$, where $\langle z_{B_i} \rangle = Z(B_i)$.

Parabolic subgroups of Garside groups

(G, M, Δ): Garside group.

Parabolic subgroups of Garside groups

(G, M, Δ): Garside group.
For $\delta \in M$, $M_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $G_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Parabolic subgroups of Garside groups

(G, M, Δ): Garside group.
For $\delta \in M$, $M_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $G_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '07)

If δ is a **parabolic Garside element**, then $(G_\delta, M_\delta, \delta)$ is a Garside group, called a **standard parabolic subgroup** of (G, M, Δ).

Parabolic subgroups of Garside groups

(G, M, Δ): Garside group.
For $\delta \in M$, $M_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $G_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '07)

If δ is a **parabolic Garside element**, then $(G_\delta, M_\delta, \delta)$ is a Garside group, called a **standard parabolic subgroup** of (G, M, Δ).

Definition (Godelle '07)

A group of the form $gG_\delta g^{-1}$ is called a **parabolic subgroup**.

Parabolic subgroups of Garside groups

(G, M, Δ): Garside group.
For $\delta \in M$, $M_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $G_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '07)

If δ is a **parabolic Garside element**, then $(G_\delta, M_\delta, \delta)$ is a Garside group, called a **standard parabolic subgroup** of (G, M, Δ).

Definition (Godelle '07)

A group of the form $gG_\delta g^{-1}$ is called a **parabolic subgroup**.

Note that parabolic subgroups depend on (G, M, Δ), not only on G.
Example: $\langle a, b \mid aba = bab \rangle^+$ and $\langle x, y \mid x^2 = y^3 \rangle^+$.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.
- Show that Garside and topological parabolic subgroups are the same.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.
- Show that Garside and topological parabolic subgroups are the same.
- By a general Garside argument, show that parabolic closures exist.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.
- Show that Garside and topological parabolic subgroups are the same.
- By a general Garside argument, show that parabolic closures exist.

Step 1 of the general Garside argument is easy:

Proposition (Godelle '07)

Standard parabolic subgroups of G are stable under intersection.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.
- Show that Garside and topological parabolic subgroups are the same.
- By a general Garside argument, show that parabolic closures exist.

Step 1 of the general Garside argument is easy:

Proposition (Godelle '07)

Standard parabolic subgroups of G are stable under intersection.

For $x \in G$, we can define $\mathrm{SPC}(x)$ the **standard parabolic closure** of x.

Parabolic subgroups of Garside groups

Strategy to prove the first two theorems:

- Have a Garside structure (G, M, Δ) on B.
- Show that Garside and topological parabolic subgroups are the same.
- By a general Garside argument, show that parabolic closures exist.

Step 1 of the general Garside argument is easy:

Proposition (Godelle '07)

Standard parabolic subgroups of G are stable under intersection.

For $x \in G$, we can define $\mathrm{SPC}(x)$ the **standard parabolic closure** of x.

Step 2: Construct parabolic closure (PC) using SPC.

Support-preservingness

(G, M, Δ): Garside group.
Guess: for good x (at least $x \in M$), we should have $\mathrm{SPC}(x) = \mathrm{PC}(x)$.

Support-preservingness

(G, M, Δ): Garside group.
Guess: for good x (at least $x \in M$), we should have $\mathrm{SPC}(x) = \mathrm{PC}(x)$.

Definition

(G, M, Δ) is **support-preserving** if for all $x, y \in M$, $\alpha \in G$, we have

$$x^\alpha = y \Rightarrow \mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y).$$

Support-preservingness

(G, M, Δ): Garside group.
Guess: for good x (at least $x \in M$), we should have $\mathrm{SPC}(x) = \mathrm{PC}(x)$.

Definition

(G, M, Δ) is **support-preserving** if for all $x, y \in M$, $\alpha \in G$, we have

$$x^{\alpha} = y \Rightarrow \mathrm{SPC}(x)^{\alpha} = \mathrm{SPC}(y).$$

Theorem (González-Meneses, Marin '22)

If (G, M, Δ) is support-preserving, then parabolic closures exist in G and $\mathrm{SPC}(x) = \mathrm{PC}(x)$ for $x \in M$.
Furthermore, if G is homogeneous, then parabolic subgroups of G are stable under intersection.

Support-preservingness

(G, M, Δ): Garside group.
Guess: for good x (at least $x \in M$), we should have $\mathrm{SPC}(x) = \mathrm{PC}(x)$.

Definition

(G, M, Δ) is **support-preserving** if for all $x, y \in M$, $\alpha \in G$, we have

$$x^\alpha = y \Rightarrow \mathrm{SPC}(x)^\alpha = \mathrm{SPC}(y).$$

Theorem (González-Meneses, Marin '22)

If (G, M, Δ) is support-preserving, then parabolic closures exist in G and $\mathrm{SPC}(x) = \mathrm{PC}(x)$ for $x \in M$.
Furthermore, if G is homogeneous, then parabolic subgroups of G are stable under intersection.

Question: Is there a non support-preserving Garside group ?

Support-preservingness

The following Garside monoids are known to be support preserving:

Support-preservingness

The following Garside monoids are known to be support preserving:

- Artin-Tits monoids of spherical type
  (Cumplido, Gebhardt, González-Meneses, Wiest '19).

Support-preservingness

The following Garside monoids are known to be support preserving:

- Artin-Tits monoids of spherical type
  (Cumplido, Gebhardt, González-Meneses, Wiest '19).
- The "parachute monoid" for $G(e, e, n)$
  (González-Meneses, Marin '22).

Support-preservingness

The following Garside monoids are known to be support preserving:

- Artin-Tits monoids of spherical type (Cumplido, Gebhardt, González-Meneses, Wiest '19).
- The "parachute monoid" for $G(e, e, n)$ (González-Meneses, Marin '22).
- The dual braid monoids of type $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$ (González-Meneses, Marin '22).

Support-preservingness

The following Garside monoids are known to be support preserving:

- Artin-Tits monoids of spherical type (Cumplido, Gebhardt, González-Meneses, Wiest '19).
- The "parachute monoid" for $G(e, e, n)$ (González-Meneses, Marin '22).
- The dual braid monoids of type $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$ (González-Meneses, Marin '22).

Theorem (G. '24)

Let W be a well-generated irreducible CRG. The dual braid monoid $(G(W), M(W), \Delta)$ is support-preserving.

Support-preservingness

The following Garside monoids are known to be support preserving:

- Artin-Tits monoids of spherical type (Cumplido, Gebhardt, González-Meneses, Wiest '19).
- The "parachute monoid" for $G(e, e, n)$ (González-Meneses, Marin '22).
- The dual braid monoids of type $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$ (González-Meneses, Marin '22).

Theorem (G. '24)

Let W be a well-generated irreducible CRG. The dual braid monoid $(G(W), M(W), \Delta)$ is support-preserving.

Note: if (G, M, Δ) and (G, M', Δ') are two Garside groups, then support-preservingness of the first does not imply support preservingness of the second.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Parabolic Garside elements are replaced by maps.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Parabolic Garside elements are replaced by maps.

For $\delta : E \subset \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$, $\mathcal{C}_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $\mathcal{G}_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.
Parabolic Garside elements are replaced by maps.
For $\delta : E \subset \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$, $\mathcal{C}_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $\mathcal{G}_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '10)

If δ is a **parabolic Garside map**, then $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ is a Garside groupoid, called a **standard parabolic subgroupoid** of $(\mathcal{G}, \mathcal{C}, \Delta)$.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.
Parabolic Garside elements are replaced by maps.
For $\delta : E \subset \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$, $\mathcal{C}_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $\mathcal{G}_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '10)

If δ is a **parabolic Garside map**, then $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ is a Garside groupoid, called a **standard parabolic subgroupoid** of $(\mathcal{G}, \mathcal{C}, \Delta)$.

Definition (Godelle '10)

A group of the form $G_\delta(u, u)$ is called a **standard parabolic subgroup** of $\mathcal{G}(u, u)$. For $g \in \mathcal{G}(v, u)$, a group of the form $gG_\delta(u, u)g^{-1}$ is called a **parabolic subgroup** of $\mathcal{G}(v, v)$.

Parabolic subgroupoids of Garside groupoids

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.
Parabolic Garside elements are replaced by maps.
For $\delta : E \subset \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$, $\mathcal{C}_\delta := \langle \mathrm{Div}(\delta) \rangle^+$ and $\mathcal{G}_\delta := \langle \mathrm{Div}(\delta) \rangle$.

Proposition (Godelle '10)

If δ is a **parabolic Garside map**, then $(\mathcal{G}_\delta, \mathcal{C}_\delta, \delta)$ is a Garside groupoid, called a **standard parabolic subgroupoid** of $(\mathcal{G}, \mathcal{C}, \Delta)$.

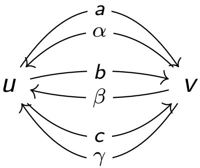
Definition (Godelle '10)

A group of the form $G_\delta(u, u)$ is called a **standard parabolic subgroup** of $\mathcal{G}(u, u)$. For $g \in \mathcal{G}(v, u)$, a group of the form $gG_\delta(u, u)g^{-1}$ is called a **parabolic subgroup** of $\mathcal{G}(v, v)$.

We can now ask about the existence of (standard) parabolic closures in weak Garside groups.

Intersection of standard parabolic subgroupoids

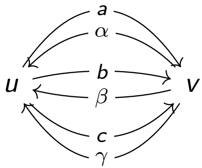
Consider the Garside groupoid presented by



with $\begin{cases} a\alpha a = b\beta b = c\gamma c = \Delta(u), \\ \alpha a\alpha = \beta b\beta = \gamma c\gamma = \Delta(v). \end{cases}$

Intersection of standard parabolic subgroupoids

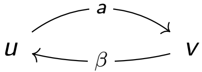
Consider the Garside groupoid presented by

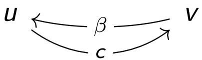


with $\begin{cases} a\alpha a = b\beta b = c\gamma c = \Delta(u), \\ \alpha a\alpha = \beta b\beta = \gamma c\gamma = \Delta(v). \end{cases}$

We have two parabolic Garside maps

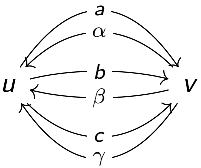
$$\delta_1(u) = a, \delta_1(v) = \beta \quad \delta_2(u) = c, \delta_2(v) = \beta$$





Intersection of standard parabolic subgroupoids

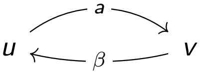
Consider the Garside groupoid presented by

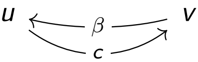


with $\begin{cases} a\alpha a = b\beta b = c\gamma c = \Delta(u), \\ \alpha a\alpha = \beta b\beta = \gamma c\gamma = \Delta(v). \end{cases}$

We have two parabolic Garside maps

$$\delta_1(u) = a, \delta_1(v) = \beta \quad \delta_2(u) = c, \delta_2(v) = \beta$$





The intersection $\mathcal{G}_{\delta_1} \cap \mathcal{G}_{\delta_2}$ is $u \xleftarrow{\beta} v$, which has no Garside map.

Shoals

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Shoals

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Definition (G. '24)

Shoal: family $\mathcal{T}$ of standard parabolic subgroupoids of $\mathcal{G}$ such that

- $\mathcal{G} \in \mathcal{T}$ and $\{1_u\}_{u \in \mathrm{Ob}(\mathcal{G})} \in \mathcal{T}$.
- $\mathcal{T}$ is stable under the Garside automorphism (conjugation by Δ).
- The intersection of two elements of $\mathcal{T}$, if nonempty, lies in $\mathcal{T}$.

Shoals

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid.

Definition (G. '24)

Shoal: family $\mathcal{T}$ of standard parabolic subgroupoids of $\mathcal{G}$ such that

- $\mathcal{G} \in \mathcal{T}$ and $\{1_u\}_{u \in \mathrm{Ob}(\mathcal{G})} \in \mathcal{T}$.
- $\mathcal{T}$ is stable under the Garside automorphism (conjugation by Δ).
- The intersection of two elements of $\mathcal{T}$, if nonempty, lies in $\mathcal{T}$.

Definition

For $\mathcal{G}_\delta \in \mathcal{T}$ and $u \in \mathrm{Ob}(\mathcal{G}_\delta)$, the group $\mathcal{G}_\delta(u, u) \subset \mathcal{G}(u, u)$ is called a $\mathcal{T}$-**standard parabolic subgroup**. $\mathcal{T}$-**Parabolic subgroups** of $\mathcal{G}(u, u)$ are defined as Groupoid conjugates of $\mathcal{T}$-standard parabolic subgroups.

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.
By construction of a shoal, every $x \in \mathcal{G}(u, u)$ admits a $\mathcal{T}$-standard parabolic closure $\mathrm{SPC}_{\mathcal{T}}(x)$.

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.
By construction of a shoal, every $x \in \mathcal{G}(u, u)$ admits a $\mathcal{T}$-standard parabolic closure $\mathrm{SPC}_{\mathcal{T}}(x)$.

Definition

$\mathcal{T}$ is **support-preserving** if for all endomorphisms $x, y \in \mathcal{C}$, $\alpha \in \mathcal{G}$

$$x^\alpha = y \Rightarrow \mathrm{SPC}_{\mathcal{T}}(x)^\alpha = \mathrm{SPC}_{\mathcal{T}}(y)$$

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.
By construction of a shoal, every $x \in \mathcal{G}(u, u)$ admits a $\mathcal{T}$-standard parabolic closure $\mathrm{SPC}_{\mathcal{T}}(x)$.

Definition

$\mathcal{T}$ is **support-preserving** if for all endomorphisms $x, y \in \mathcal{C}$, $\alpha \in \mathcal{G}$

$$x^\alpha = y \Rightarrow \mathrm{SPC}_{\mathcal{T}}(x)^\alpha = \mathrm{SPC}_{\mathcal{T}}(y)$$

Theorem (G. '24)

If $\mathcal{T}$ is support-preserving, then $\mathcal{T}$-parabolic closures exist in $\mathcal{G}$. For $x \in \mathcal{C}$ an endomorphism, $\mathrm{SPC}_{\mathcal{T}}(x) = \mathrm{PC}_{\mathcal{T}}(x)$.

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.
By construction of a shoal, every $x \in \mathcal{G}(u, u)$ admits a $\mathcal{T}$-standard parabolic closure $\mathrm{SPC}_{\mathcal{T}}(x)$.

Definition

$\mathcal{T}$ is **support-preserving** if for all endomorphisms $x, y \in \mathcal{C}$, $\alpha \in \mathcal{G}$

$$x^{\alpha} = y \Rightarrow \mathrm{SPC}_{\mathcal{T}}(x)^{\alpha} = \mathrm{SPC}_{\mathcal{T}}(y)$$

Theorem (G. '24)

If $\mathcal{T}$ is support-preserving, then $\mathcal{T}$-parabolic closures exist in $\mathcal{G}$. For $x \in \mathcal{C}$ an endomorphism, $\mathrm{SPC}_{\mathcal{T}}(x) = \mathrm{PC}_{\mathcal{T}}(x)$.

- No general argument for intersections of $\mathcal{T}$-parabolic sugroups.

$\mathcal{T}$-parabolic closure

$(\mathcal{G}, \mathcal{C}, \Delta)$: Garside groupoid, $u \in \mathrm{Ob}(\mathcal{G})$.
By construction of a shoal, every $x \in \mathcal{G}(u, u)$ admits a $\mathcal{T}$-standard parabolic closure $\mathrm{SPC}_{\mathcal{T}}(x)$.

Definition

$\mathcal{T}$ is **support-preserving** if for all endomorphisms $x, y \in \mathcal{C}$, $\alpha \in \mathcal{G}$

$$x^\alpha = y \Rightarrow \mathrm{SPC}_{\mathcal{T}}(x)^\alpha = \mathrm{SPC}_{\mathcal{T}}(y)$$

Theorem (G. '24)

If $\mathcal{T}$ is support-preserving, then $\mathcal{T}$-parabolic closures exist in $\mathcal{G}$. For $x \in \mathcal{C}$ an endomorphism, $\mathrm{SPC}_{\mathcal{T}}(x) = \mathrm{PC}_{\mathcal{T}}(x)$.

- No general argument for intersections of $\mathcal{T}$-parabolic sugroups.
- How to construct shoals with interesting parabolic subgroups ?

Shoal for Springer groupoids

W irreducible well-generated. d: regular integer for W.

Shoal for Springer groupoids

W irreducible well-generated. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: the associated Springer groupoid.

Shoal for Springer groupoids

W irreducible well-generated. d: regular integer for W.
$(\mathcal{G}, \mathcal{C}, \Delta)$: the associated Springer groupoid.

Theorem (G. '24)

There is a support-preserving shoal $\mathcal{T}$ for $\mathcal{G}$ such that the $\mathcal{T}$-parabolic subgroups of $\mathcal{G}(u,u)$ coincide with the topological parabolic subgroups of $B^{\mu_d} \simeq \mathcal{G}(u,u)$.

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.

ρ : d-regular braid in B.

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.
ρ : d-regular braid in B.

Theorem (G. '24)

The isomorphism $B^{\mu_d} \simeq C_B(\rho)$ induces a bijection

$$\{B_0 \subset B^{\mu_d} \text{ para.}\} \simeq \{B_1 \cap C_B(\rho) \mid B_1 \subset B \text{ para. and } B_1^{\rho} = B_1\}.$$

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.
ρ : d-regular braid in B.

Theorem (G. '24)

The isomorphism $B^{\mu_d} \simeq C_B(\rho)$ induces a bijection

$$\{B_0 \subset B^{\mu_d} \text{ para.}\} \simeq \{B_1 \cap C_B(\rho) \mid B_1 \subset B \text{ para. and } B_1^{\rho} = B_1\}.$$

Furthermore, two parabolic subgroups of B which are normalized by ρ are equal if and only if their intersections with $C_B(\rho)$ are equal.

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.
ρ : d-regular braid in B.

Theorem (G. '24)

The isomorphism $B^{\mu_d} \simeq C_B(\rho)$ induces a bijection

$$\{B_0 \subset B^{\mu_d} \text{ para.}\} \simeq \{B_1 \cap C_B(\rho) \mid B_1 \subset B \text{ para. and } B_1^\rho = B_1\}.$$

Furthermore, two parabolic subgroups of B which are normalized by ρ are equal if and only if their intersections with $C_B(\rho)$ are equal.

The proof is a case-by case analysis on well-generated groups, groups of the infinite series, and regular centralizers in well-generated groups.

Parabolic subgroups of regular centralizers

W irreducible. d: regular integer for W.
ρ : d-regular braid in B.

Theorem (G. '24)

The isomorphism $B^{\mu_d} \simeq C_B(\rho)$ induces a bijection

$$\{B_0 \subset B^{\mu_d} \text{ para.}\} \simeq \{B_1 \cap C_B(\rho) \mid B_1 \subset B \text{ para. and } B_1^{\rho} = B_1\}.$$

Furthermore, two parabolic subgroups of B which are normalized by ρ are equal if and only if their intersections with $C_B(\rho)$ are equal.

The proof is a case-by case analysis on well-generated groups, groups of the infinite series, and regular centralizers in well-generated groups.

Corollary

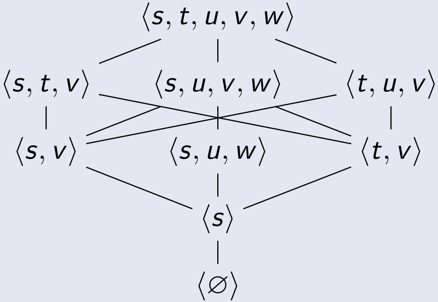
Parabolic subgroups of B_{31} are stable under intersection.

Theorem (G. '24)

The lattice of parabolic subgroups of B_{31} up to conjugacy is given by

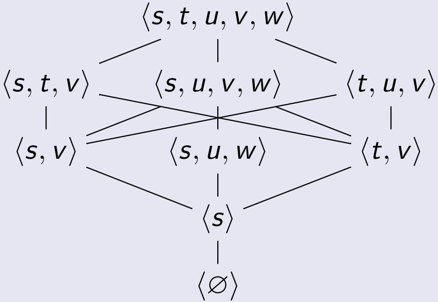
Theorem (G. '24)

The lattice of parabolic subgroups of B_{31} up to conjugacy is given by



Theorem (G. '24)

The lattice of parabolic subgroups of B_{31} up to conjugacy is given by

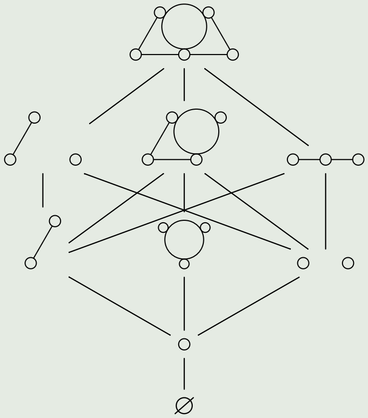


Corollary (G. '24)

The third main theorem of González-Meneses, Marin holds for G_{31}.

Corollary

The BMR diagram of G_{31} gives presentations of the parabolic subgroups.



Thank you !