20/09/23: A Stroll in group theory with circular groups.
PhD student's seminar.

0 Introduction, presentation

E a set of letters (eg $E=\{s,t\}$)
Word: finite sequence of letters (eg ststssts)
+ empty word ()

Concatenation of words: $sts.tt=ststt$

Def: The free monoid $F(E)$ (on E) is the monoid of words over E, endowed with concatenation.

ex: $\mathbb{N}=F(\{1\})$.

Relation = pair of words (that we want to be equal) (eg $\{a^n,()\}$)
↳ equivalence relation $\equiv$ on words (congruence)

$F(E)/_{\equiv}$:= presented monoid. (notation $\langle S \mid R\rangle^+$)

ex: $\langle a \mid a^n=()\rangle^+ \simeq \mathbb{Z}/n\mathbb{Z}$

Presented groups: same, but add a formal copy of each letter + relations $a\bar{a}=()$; $()=\bar{a}a$. to get inverses.

Presented groups are sometimes hard to work with
↳ "nice presentations".

I. Circular groups

1) Defs, examples

Let $\{a_0, \dots, a_{m-1}\}$ an alphabet (index in $\mathbb{Z}/m\mathbb{Z}$: $a_m = a_0$).

$S(i,p) := a_i \cdots a_{p+i-1}$ for $i \in [\![0, m-1]\!]$, $p \in \mathbb{N}$.

Def: Let m, ℓ be positive integers. The circular group $G(m\ \ell)$ is def by $G(m,\ell) = \langle a_0 \dots a_{m-1} \mid S(i,\ell) = S(i+1,\ell) \ \forall i \in [\![0, m-1]\!] \rangle$.
+ $\Delta := S(0,\ell)$ $(= S(1,\ell) = S(2,\ell) \dots)$

Eg: $G(3,3) = \langle a,b,c \mid abc = bca = cab \rangle$.
$\mathbb{Z}^2 \simeq G(2,2) = \langle a,b \mid ab = ba \rangle$
$\mathbb{Z} \simeq G(1,1) \simeq G(1,\ell) \simeq G(m,1)$.

Rq: $G(m\ m) \simeq$ fund. grp of $\mathbb{C}^2 \setminus m$ lines through the origin.
Rq: All complex braid groups of rank 2 are isom. to circular.

Lem: $G(m,m) \simeq \mathbb{Z} \times F_{m-1}$ ↳ free group with $m-1$ gen.

dem: By def, $\mathbb{Z} \times F_{m-1} = \langle z, \alpha_1 \dots \alpha_{m-1} \mid z\alpha_i = \alpha_i z \ \forall i \in [\![1, m-1]\!] \rangle$.

Morphism $f: \mathbb{Z} \times F_{m-1} \longrightarrow G(m\,m)$
$$z \longmapsto \Delta$$
$$\alpha_i \longmapsto a_i \qquad i \in [\![1, m-1]\!]$$

Since Δ is central (finite check), this is a morphism. We have
$$a_0 = a_0 a_1 \dots a_{m-1} (a_1 \dots a_{m-1})^{-1} = \Delta s(1, m-1)^{-1}.$$

Morphism $g: G(m\,m) \longrightarrow \mathbb{Z} \times F_{m-1}$
$$a_0 \longmapsto z(\alpha_1 \dots \alpha_{m-1})^{-1}$$
$$a_i \longmapsto \alpha_i \qquad i \in [\![1, m-1]\!]$$

We have $f \circ g = Id_{G\,m\,m}$ $\quad g \circ f = Id_{\mathbb{Z} \times F_{m-1}}$. $\square$

We see that $G(m\,m) \not\cong G(m', m')$ when $m \neq m'$, can we generalize?

Question: What are the pairs $\{(m, \ell), (m', \ell')\}$ such that
$$G(m\,\ell) \simeq G(m', \ell').$$

2) Description of elements

Def: A simple element of $G(m\,\ell)$ is a (positive) product of at most ℓ consecutive letters. i.e some $s(i\,p)$, $i \in [\![0, m-1]\!]$, $p \in [\![0, \ell]\!]$.

Since $a_i = s(i, 1)$, simples generate $G(m, \ell)$.

↳ find a canonical expression

Consider a product of two simples $s(i,p)\, s(i',p')$.

* The last letter of $s(i,p)$ is a_{i+p-1}
* the first letter of $s(i',p')$ is $a_{i'}$.

<u>Def</u>: The product $s(i,p)s(i',p')$ is <u>normal</u> if $a_{i+p-1}, a_{i'}$ are not consecutive. ie $i+p \not\equiv i' [m]$, or if $s(i,p)=\Delta$.

<u>Prop</u>: Any non normal product of two simples can be canonically rewritten as a normal product.

$i+p \equiv i' [m]$

$$s(i,p)\, s(i',p') = s(i,p+p') = \begin{cases} s(i,p+p') & p+p'<\ell \\ \Delta & p+p'=\ell \\ \Delta s(i+\ell, p+p'-\ell) & p+p'>\ell \end{cases}$$

An induction later, we get

<u>Théo</u>: Every $x \in G(m,\ell)$ can be written uniquely as

$$x = \Delta^k s(i_1,p_1) \cdots s(i_r,p_r)$$

with $k \in \mathbb{Z}$, $\Delta \neq s(i_1,p_1)$ and each $s(i_a,p_a)s(i_{a+1},p_{a+1})$ is normal

<u>Eg</u>: in $G(3,3)$. $s(1,1)\, s(0,1)^{-1} s(1,2)\, s(1,3) = ba^{-1} bc \cdot bca$
$= b \cdot a^{-1} \cdot bca \cdot bc$
$= b \cdot a^{-1} \cdot abc \cdot bc$
$= b \cdot bc \cdot bc = s(1,1)\, s(1,2)\, s(1,2)$.

<u>Lemma</u>: $s(i,p)\Delta = \Delta s(i+\ell,p)$. (conjugation by Δ)

→ it preserves normality

→ Some power of Δ is central ($\Delta^{\frac{m}{m \wedge \ell}}$ in fact)

II : Periodic elements, center

$x \in G(m\ell)$ is periodic $\iff \langle x \rangle \cap \langle \Delta \rangle \neq \{1\}$

$\iff x^a = \Delta^b$ for some int. a and b.

(a,b)-periodic.

eg: In $G(3,4)$, $x = s(0,3)$ is $(4,3)$ periodic.

Prop: (1) An elmt $\Delta^k s(i,p)$ is periodic iff $p + k\ell \equiv 0\ [m]$. In this case all $\Delta^k s(j,p)$ are periodic too.

(2) Any periodic element is conjugate to some $\Delta^k s(j,p)$.

(3) All $\Delta^k s(j,p)$ ($j \in [0, m-1]$) are conjugate if $p + k\ell \equiv 0\ [m]$

(4) $C_{G(m\ell)}(\Delta^k s(0,p)) = \langle s(p,m) \rangle$.

proof (outline). (1) $(\Delta^k s(i,p))^2 = \Delta^k s(i,p) \Delta^k s(i,p) = \Delta^{2k} \underbrace{s(i+k\ell,p)\, s(i,p)}_{\text{normal?}}$

(2) Garside theory.

(3),(4). We have the conjugacy graph.

$$\begin{tikzcd} & \Delta^k s(1,p) \arrow[r, "s(p+1,1)"] & \Delta^k s(2,p) \arrow[d, dashed] \\ \Delta^k s(0,p) \arrow[ur, "s(p,1)"] & & \Delta^k s(m-1,p) \arrow[ll, "s(p+m-1,1)"] \end{tikzcd}$$

Prop: Periodic elements are exactly conjugates of powers of either $s(0,m)$ or $s(0,\ell) = \Delta$.

pf
+ if $m | \ell$, $s(0,m)$ is the only (up to conj) periodic elmt with no root
+ if $\ell | m$, Δ
+ $\begin{cases} m \nmid \ell \\ \ell \nmid m \end{cases}$ Δ and $s(0,m)$ give 2 classes of periodic elements with no roots

+ having roots is a group theoretic property.

Théo: If $G(m\ell)$ is not abelian, then
$$Z(G(m\ell)) = \langle \Delta^{\frac{m}{m\wedge\ell}} \rangle$$
(abelian: $G(1,\ell) = G(m,1) = \mathbb{Z}$, $G(2,2) = \mathbb{Z}^2$).

proof (outline)

- $m \neq \ell$, $Z(G(m\ell)) \subseteq C_{G(m\ell)}(\Delta) = \langle \Delta \rangle$. and $\Delta^{\frac{m}{m\wedge\ell}}$ is the smallest central power of Δ.
- $m = \ell$ $G(m\,m) \simeq \mathbb{Z} \times F_{m-1} \Rightarrow Z(G(m\,m)) = \langle \Delta \rangle$.
- $\ell \neq m$ $Z(G(m\ell)) \subseteq C_{G(m\ell)}(s(\ell,m)) = \langle s(\ell,m) \rangle$, and $\Delta^{\frac{m}{m\wedge\ell}}$ is the smallest central power of $s(\ell,m)$.

Cor: $x \in G(m\ell)$ is periodic iff $\langle x \rangle \cap Z(G(m\ell)) \neq \{1\}$, i.e iff x has a central power.

This is group theoretic

↳ an isomorphism $G(m\ell) \simeq G(m'\ell')$ must preserve periodic elements and periodic elements with no roots.

III. Abelianization

Abelianization of G = "biggest abelian quotient of G".
$= G/D(G) \to$ commutator subgroups.

$G \simeq H \Rightarrow G^{ab} \simeq H^{ab}$.

G finitely gen $\Rightarrow G^{ab}$ finitely gen abelian group $\Rightarrow$ those are classified!

Prop: If $G = \langle S \mid R \rangle$, then $G^{ab} = \langle S \mid R \cup \{ab = ba, \forall a, b \in S\} \rangle$.

Theo: $G(m,\ell)^{ab} \simeq \mathbb{Z}^{m \wedge \ell}$.

dem (outline): $G(m,\ell)^{ab} = G(m,\ell) / (a_i = a_{i+\ell} \ \forall i \in [\![0, m-1]\!]) = \mathbb{Z}^{m\wedge \ell}$.

thus $G(m,\ell) \simeq G(m',\ell') \Rightarrow m \wedge \ell = m' \wedge \ell'$.

IV. Complete classification

Theo: If $G(m,\ell)$, $G(m',\ell')$ non abelian, then
$$G(m,\ell) \simeq G(m',\ell') \iff (m,\ell) = (m',\ell') \text{ or } (m,\ell) = (\ell',m')$$

+ abelian case is easy.

proof: (outline) ($\Rightarrow$). Set $d = m \wedge \ell = m' \wedge \ell' = d'$. In $G(m,\ell)$, the smallest central power of $s(0,m)$ (resp Δ) is $\frac{\ell}{d}$ (resp $\frac{m}{d}$)

- if $m \mid \ell$, 1 class of irred perio element $s(0,m)$
  $\hookrightarrow$ 1 class of irred perio element in $G(m',\ell')$: $m' \mid \ell'$ or $\ell' \mid m'$.
  $\hookrightarrow$ $d = m = d' = m'$ or $d = m = d' = \ell'$.
  $$\frac{\ell}{d} = \frac{\ell'}{d'} \text{ and } \ell = \ell' \qquad \frac{\ell}{d} = \frac{m'}{d'} \Rightarrow \ell = m'.$$
- if $\ell \mid m$, same reasoning
- if $m \nmid \ell$ and $\ell \nmid m$, 2 classes of irred perio element $\Rightarrow m' \nmid \ell'$ and $\ell' \nmid m'$.
  We either have $\left(\frac{\ell}{d}, \frac{m}{d}\right) = \left(\frac{\ell'}{d'}, \frac{m'}{d'}\right)$ or $\left(\frac{\ell}{d}, \frac{m}{d}\right) = \left(\frac{m'}{d'}, \frac{\ell'}{d'}\right)$

in both cases, the result holds.

($\Leftarrow$) Let $a_0 \dots a_{m-1}$ be the generators of $G(m\ell)$
$b_0 \dots b_{m-1}$ ——— $G(\ell, m)$.

the correspondance

$$\begin{cases} f(a_0) := b_{m-1} \\ f(a_1) := b_{m-2}^{\,b_{m-1}} \\ f(a_2) := b_{m-3}^{\,b_{m-1} b_{m-2}} \\ \quad\vdots \\ f(a_{m-1}) := b_0^{\,b_1 \cdots b_{m-1}} \end{cases}$$

induces an isomorphism
$G(m\ell) \simeq G(\ell\, m)$.

Rq: for $m=2$, classical and dual presentation of dihedral Artin groups